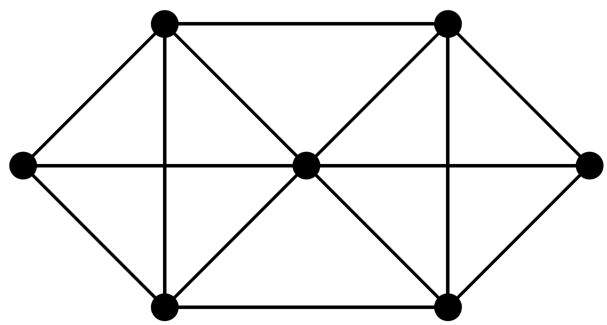
1. Degree

Prove that for any graph of $n \geq 2$ vertices, there are at least two vertices which have the same degree.

We will use contrapositive proof here. Assume that n vertices all have different degrees. Because we are talking about a simple graph, the degrees that a vertex can possible have are $0, 1, 2, \ldots, n-1$. And since all vertices have different degrees, there are exactly one vertex with degree 0 and one vertex with degree $n-1$. But the fact that the vertex of degree $n-1$ must connect to all other vertices, contradicts with the assumption that there is one vertex with degree 0.

2. Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above?

No. Two vertices have odd degree.

(b) Is there an Eulerian walk in the graph above?

Yes. One of the two vertices with odd degree must be the first vertex, and the other one must be the last vertex.

(c) What is the condition that there is an Eulerian walk in an undirected graph?

An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and at most two vertices have odd degree.

Note: There is no graph with only one odd degree vertex.

3. Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$, where recall $\{0,1\}^n$ denotes the set of all n-bit strings. There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Depict 1-, 2-, and 3-dimensional hypercubes.

Check the graphs on Page 9, Lecture Note 18.

(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

Consider each edge that changes the i^{th} bit for some $i \leq n$. Every vertex touches exactly one of these edges, because there is exactly one way to change the i^{th} bit in any bitstring. Coloring each of these edges color i ensures that each vertex will then be adjacent to n differently colored edges, since there are n different bits to change, and no two edges representing bit changes on different bits have the same color.

(c) Show that the vertices of an n-dimensional hypercube can be colored using 2 colors so that no pair of adjacent vertices have the same color. (This is equivalent to showing that a hypercube is *bipartite*: the vertices can be partitioned into two groups (according to color) so that every edge goes between the two groups.)

Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). By coloring the vertices with an even number of 0 bits color 0 and vertices with an odd number of 0 bits color 1, no two adjacent vertices will share a color.

4. Bipartite Graph

Consider an undirected bipartite graph with two disjoint sets L, R. Prove that a bipartite graph has no cycles of odd length.

Let us start traveling the cycle from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the set connects our start node n_0 to the a node n_1 in R. The 2nd edge in the cycle must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our cycle in L, the cycle must edge with an even number of edges.