v-i curve is represented by $i = v^3 - 2$ and suppose a constant voltage $v = 1$ volt is applied. The instantaneous and average power, respectively, are given by

$$p_R(t) = 1(1 - 2) = -1 \qquad \text{and} \qquad p_{R_{av}} = \frac{(-1)(T)}{T} = -1 \quad (1\text{-}50)$$

Since the average power is negative, energy is being supplied (instead of being absorbed) by the nonlinear resistor to the external circuit. Since energy cannot be created, this nonlinear resistor must have an external power source (e.g., a battery) associated with it, and is therefore called an *active resistor*. Without an external power source, a nonlinear resistor can only absorb power; namely, $p_R(t) \geq 0$. Such a resistor is said to be *passive*. It is easy to see that *a nonlinear resistor is passive if, and only if, its v-i curve lies entirely in the first and the third quadrants*. This follows from the fact that the instantaneous power is always nonnegative; namely,

$$p_R(t) = v(t)i(t) \geq 0 \tag{1-51}$$

Clearly, in its original form, a physical resistor must necessarily be passive. This is true, for example, with the commercial resistors listed in Table 1-1. Any of these resistors can, of course, be transformed into an active resistor by connecting a battery in series with it.

Exercise: The v-i curve of a certain nonlinear resistor is given by $i = 10(v^3 - 3v)$ma, and the voltage excitation is given by $v(t) = 10 \sin t$ volts. (*a*) Find the instantaneous power $p_R(t)$. (*b*) Find the energy flow $w_R(0,t_1)$ for all $t_1 > 0$. (*c*) Find the average power by using Eq. (1-48) and check by using Eq. (1-45). (*d*) Is this nonlinear resistor passive or active? Explain why.

Case 2: Two-terminal nonlinear capacitor Consider the nonlinear capacitor shown in Fig. 1-22*a* and the three typical types of v-q curves shown in Fig. 1-22*b* to *d*. The v-q curve can be described in the functional form by $v = v(q)$ if it is charge-controlled or by $q = q(v)$ if it is voltage-controlled. A strictly monotonically increasing v-q curve can obviously be described by either $v = v(q)$ or $q = q(v)$. The energy flow $w_C(t_0,t_1)$ into a capacitor during the time interval (t_0,t_1) is given by

$$w_C(t_0,t_1) = \int_{t_0}^{t_1} v(t) \frac{dq(t)}{dt}\, dt \tag{1-52}$$

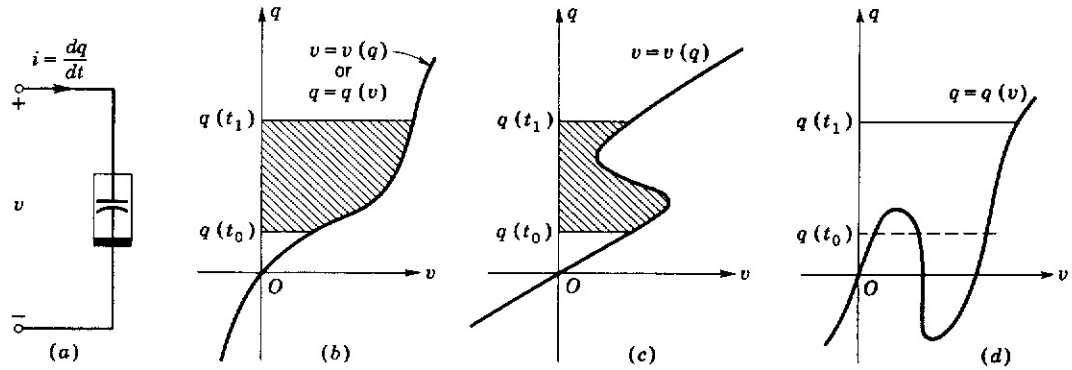

Fig. 1-22. The energy flow $w_C(t_0,t_1)$ from t_0 to t_1 into a nonlinear capacitor is equal numerically to the shaded area

In the case where the capacitor v-q curve is either strictly monotonically increasing or charge-controlled, Eq. (1-52) can be written as

$$w_C(t_0,t_1) = \int_{t_0}^{t_1} v(q(t)) \frac{dq(t)}{dt}\, dt \tag{1-53}$$

By a standard change of variable, Eq. (1-53) becomes

$$w_C(t_0,t_1) = \int_{q(t_0)}^{q(t_1)} v(q)\, dq \tag{1-54}$$

In the special, but very important, case of a *linear* capacitor [$q = Cv$ or $v = (1/C)q$], Eq. (1-54) can be reduced to

$$w_C(t_0,t_1) = \int_{q(t_0)}^{q(t_1)} \frac{1}{C} q\, dq = \frac{1}{2C} \int_{q(t_0)}^{q(t_1)} d(q^2)$$

or

$$w_C(t_0,t_1) = \frac{1}{2C}[q^2(t_1) - q^2(t_0)] \tag{1-55}$$

Equation (1-55) can also be expressed in terms of v by substituting $q = Cv$ for q:

$$w_C(t_0,t_1) = \frac{C}{2}[v^2(t_1) - v^2(t_0)] \tag{1-56}$$

Referring to Fig. 1-22, Eq. (1-54) can be interpreted as follows: The energy flow $w_C(t_0,t_1)$ from t_0 to t_1 into a charge-controlled

nonlinear capacitor is equal numerically to the area under the v-q curve (bounded by the q axis and the lines $q = q(t_0)$ and $q = q(t_1)$. This interpretation is significant because it shows that only three pieces of information are needed to determine $w_C(t_0,t_1)$, namely

1. The v-q curve
2. The initial value of the charge at $t = t_0$
3. The final value of the charge at $t = t_1$

Since no information is required of the waveforms of $q(t)$ and $v(t)$, the energy $w_C(t_0,t_1)$ is said to be independent of the excitation waveforms. This property is very different from the resistor case where the complete voltage and current waveforms are required to compute $w_R(t_0,t_1)$. Observe further from Fig. 1-22 that whenever the waveform $v(t)$ returns to the same initial point, i.e., when $q(t_1) = q(t_0)$, the energy $w_C(t_0,t_1) = 0$. For example, Eqs. (1-55) and (1-56) are both equal to zero under this condition. Hence, unlike the resistor case, there must be some form of "energy-swapping" mechanism between a capacitor and the external circuit connected across it. To investigate this mechanism, let us calculate the average power using Eq. (1-45); thus

$$P_{C_{\text{av}}} = \lim_{t_1 \to \infty} \frac{1}{t_1} \int_{q(0)}^{q(t_1)} v(q)\, dq \tag{1-57}$$

Now observe that except when $q(t)$ goes to infinity, a case that cannot occur in practice, the value of $q(t_1)$ will always be a finite number. This means that the area under the curve representing the integral in Eq. (1-57) will always be a finite number. But the value of t_1 in Eq. (1-57) must tend to infinity, therefore

$$P_{C_{\text{av}}} = 0 \tag{1-58}$$

Since this equation is derived only under the assumption that the v-q curve be charge-controlled (this includes clearly the special case of a monotonically increasing curve), it is a very general result. We can, therefore, conclude that *the average power entering a charge-controlled nonlinear capacitor is zero.* This condition is true for any capacitor current and voltage waveforms. In the special case where $q(t)$ and $v(t)$ are periodic, Eq. (1-57) can be simplified to

$$P_{C_{av}} = \frac{1}{T}\int_{q(0)}^{q(T)} v(q)\,dq \tag{1-59}$$

But $q(T) = q(0)$ for a periodic waveform of period T; therefore, Eq. (1-59) will integrate to zero, as it should.

From the preceding discussion, we can now conclude that a charge-controlled capacitor does not dissipate energy. Any energy entering it must be stored inside the capacitor and *may* eventually be returned. Because of this interpretation, a capacitor is often referred to as an *energy-storage element.* In the case of parallel-plate capacitors, it is possible to show, by electromagnetic field theory, that the energy is stored in the electric field between the plates. In view of this observation, the energy $w_C(t_0,t_1)$ in a capacitor is usually called the *electric stored energy.*

What happens if the v-q curve is neither monotonically increasing nor charge-controlled? In this case, it is no longer possible to describe the v-q curve by a function of q. It is not possible, therefore, to specify the area representing $v\,dq$ uniquely. To investigate this more general case, a new approach is required.[1]

Exercise 1: The v-q curve of a certain nonlinear capacitor is given by $q = \frac{1}{3}v^3$. Let the terminal voltage be given by $v(t) = e^{-t}$. (*a*) Find $w_C(0,t_1)$ for all $t_1 > 0$ by determining first $i(t) = (dq/dv)(dv/dt)$ and then using Eq. (1-43). (*b*) Repeat (*a*) by determining first $q(t)$ and then using Eq. (1-53). (*c*) Repeat (*a*) by using Eq. (1-54). (*d*) Let $v(t) = E \sin \omega t$ and verify that $P_{C_{av}} = 0$.

Exercise 2: Prove that the electric stored energy in a voltage-controlled capacitor is given by

$$w_C(t_0,t) = q(t_1)v(t_1) - q(t_0)v(t_0) - \int_{v(t_0)}^{v(t_1)} q(v)\,dv$$

HINT: Apply the integration-by-part theorem.

Case 3: Two-terminal nonlinear inductor Consider the nonlinear inductor shown in Fig. 1-23*a* and the three typical types of i-φ curves shown in Fig. 1-23*b* to *d*. The i-φ curve can be described in the functional form by $i = i(\varphi)$ if it is flux-controlled or by $\varphi = \varphi(i)$ if it is current-controlled. A strictly monotonically increasing i-φ curve can obviously be described by either $i = i(\varphi)$ or $\varphi = \varphi(i)$. The energy flow $w_L(t_0,t_1)$ into an inductor during the time interval (t_0,t_1) is given by

$$w_L(t_0,t_1) = \int_{t_0}^{t_1} i(t)\frac{d\varphi(t)}{dt}\,dt \tag{1-60}$$

[1] This approach is called the *parametric approach* and is discussed in Appendix A. See also L. O. Chua and R. A. Rohrer, On the Dynamic Equations of a Class of Nonlinear *RLC* Networks, *IEEE Trans. Circuit Theory,* vol. CT-12, no. 4, pp. 475–489, December, 1965.

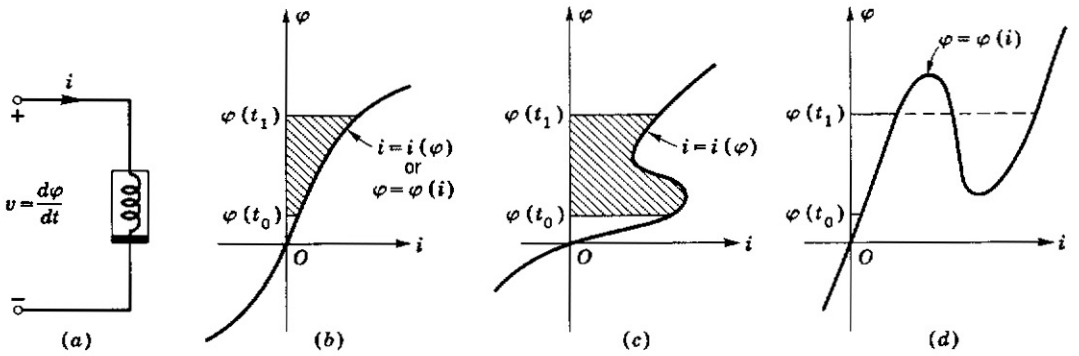

Fig. 1-23. The energy flow $w_L(t_0,t_1)$ from t_0 to t_1 into a nonlinear inductor is equal numerically to the shaded area.

Applying analogous procedure as in the capacitor case, we find that when the i-φ curve is either strictly monotonically increasing or flux-controlled, Eq. (1-60) can be written as

$$w_L(t_0,t_1) = \int_{\varphi(t_0)}^{\varphi(t_1)} i(\varphi)\, d\varphi \tag{1-61}$$

In the special case where the inductor is linear ($\varphi = Li$), Eq. (1-61) can be simplified further to

$$w_L(t_0,t_1) = \frac{1}{2L}[\varphi^2(t_1) - \varphi^2(t_0)] \tag{1-62}$$

or

$$w_L(t_0,t_1) = \frac{L}{2}[i^2(t_1) - i^2(t_0)] \tag{1-63}$$

Referring to Fig. 1-23, Eq. (1-61) can be interpreted as follows: The energy flow $w_L(t_0,t_1)$ from t_0 to t_1 into a flux-controlled nonlinear inductor is equal numerically to the area under the i-φ curve [bounded by the φ axis and the lines $\varphi = \varphi(t_0)$ and $\varphi = \varphi(t_1)$]. This interpretation has the same significance as for the capacitor; namely, only three pieces of information are needed to determine $w_L(t_0,t_1)$:

1. The i-φ curve
2. The initial value of the flux linkage at $t = t_0$
3. The final value of the flux linkage at $t = t_1$

By a similar procedure, we found *the average power in any flux-controlled inductor is zero;* thus

$$P_{L_{av}} = 0 \tag{1-64}$$

This means that a flux-controlled inductor cannot dissipate energy. In view of this observation, the inductor is also called an *energy-storage element.* In the case where the inductor is made of coils around an iron core, the energy can be shown, by electromagnetic principles, to be stored in the magnetic field around the coil. Hence, the energy stored in an inductor is usually called *magnetic stored energy.*

Exercise 1: Prove that Eq. (1-64) holds for a flux-controlled inductor. Verify this with $i(t) = I \cos \omega t$ and $\varphi = i^3$.

Exercise 2: Prove that the energy stored in a current-controlled inductor is given by

$$w_L(t_0,t_1) = \varphi(t_1)i(t_1) - \varphi(t_0)i(t_0) - \int_{i(t_0)}^{i(t_1)} \varphi(i)\, di$$

1-10 TIME-VARYING ELEMENTS

So far, the v-i, v-q, and i-φ curves characterizing a two-terminal resistor, capacitor, and inductor are assumed to remain unchanged for all times. These elements are said to be *time-invariant.* There exist some practical elements, however, whose v-i, v-q, or i-φ curves vary as functions of time. Such elements are said to be *time-varying resistors, capacitors,* or *inductors,* respectively.

Time-varying resistor The simplest example of a time-varying resistor is a potentiometer whose arm is being rotated by a motor as shown in Fig. 1-24*a*. At any time t, the potentiometer is simply a linear resistor with a straight-line v-i characteristic as shown in Fig. 1-24*b*. Hence, a time-varying linear resistor can be characterized by

$$v = R(t)i$$

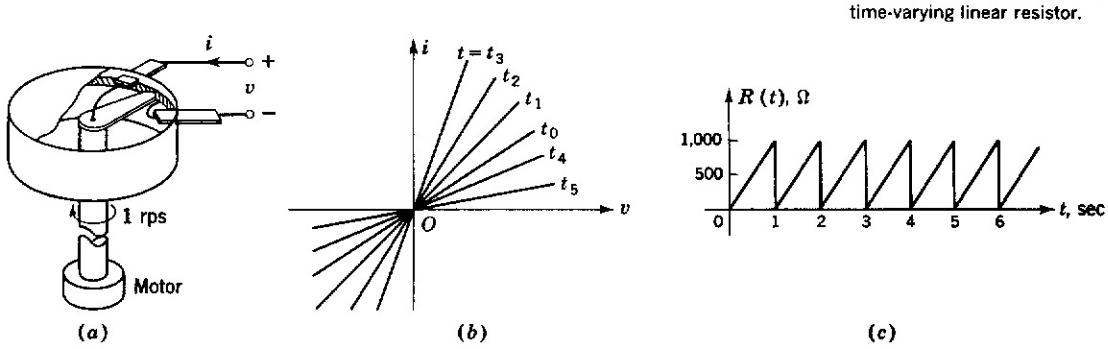

Fig. 1-24. An example of a time-varying linear resistor.

where $R(t)$ is the time-varying resistance representing the reciprocal of the slope of the straight line at any time t. For example, if the potentiometer has a resistance range of 0 to 1,000 Ω uniformly distributed around its rim and if the arm rotates at a speed of 1 rps, then the time-varying resistance is as shown in Fig. 1-24*c*.

A time-varying resistor need not be linear. For example, consider a resistor characterized by

$$i = v^3 + \sin t$$

The v-i curve of this time-varying nonlinear resistor is shown in Fig. 1-25*a* as a function of time. Observe that this resistor can be constructed in practice by connecting a sinusoidal current source in parallel with a time-invariant resistor (Fig. 1-25*b*) with the v'-i' curve shown in Fig. 1-25*c*. In general, a time-varying nonlinear resistor can be characterized by a relationship $i = i(v,t)$ if it is voltage-controlled, or $v = v(i,t)$ if it is current-controlled. A review of the power and energy expressions derived in the preceding section would show that these expressions remain valid for the time-varying case.

What are time-varying resistors good for? To give one simple application, let us consider the current waveform

$$i(t) = [1 + f(t)] \sin \omega t \tag{1-65}$$

Equation (1-65) is called an *amplitude-modulated waveform* because the amplitude of the sine wave varies with time. This is the type

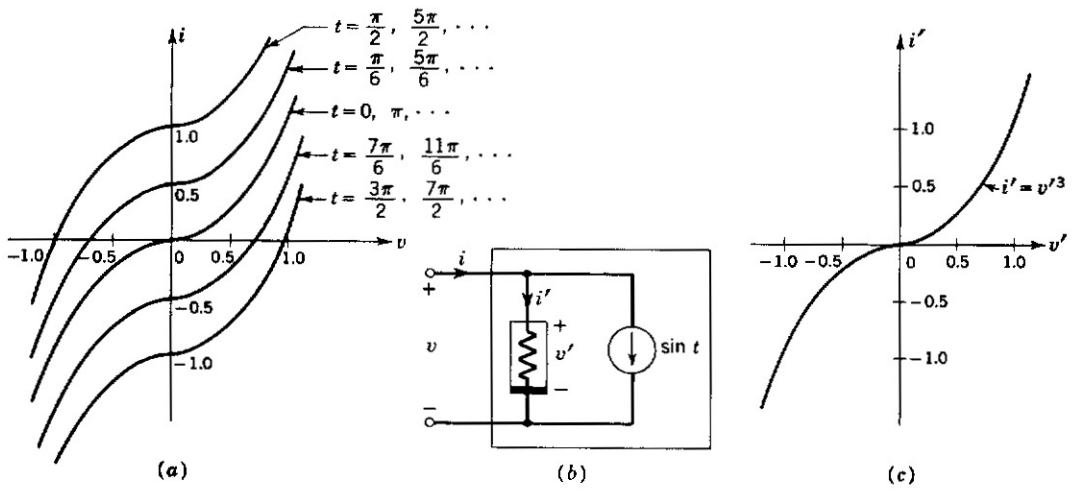

Fig. 1-25. An example of a time-varying nonlinear resistor.

of signal that an AM radio transmitter sends out. In practice, $f(t)$ represents a slowly changing signal and $\sin \omega t$ represents a relatively high-frequency sine wave known as the "carrier." We are not equipped to explain why $f(t)$ cannot be transmitted directly, and why it must be "carried" by the sine wave. Suffice it to say that it takes a high-frequency waveform to traverse a long distance in space. Our objective here is to show how we may recover the signal $f(t)$ from Eq. (1-65). One possible method consists of applying this current to a time-varying linear resistor whose resistance changes at the *same* frequency as the carrier, namely,

$$R(t) = 1 + \sin \omega t$$

The voltage drop across this resistor is given by

$$\begin{aligned} v(t) &= R(t)i(t) \\ &= (1 + \sin \omega t)[1 + f(t)] \sin \omega t \\ &= \tfrac{1}{2} f(t) + \tfrac{1}{2} + [1 + f(t)] \sin \omega t - \tfrac{1}{2}[1 + f(t)] \cos 2\omega t \end{aligned} \tag{1-66}$$

Observe that Eq. (1-66) contains four terms; the first term is the signal that we would like to recover, the second term is a dc voltage, the third term is the carrier-frequency term, and the last term is at twice the carrier frequency. Through the use of a "filter," the last three components can be easily suppressed, thus leaving the desired signal $f(t)$. This recovering process is known as *synchronous detection* because the frequency of the time-varying resistance is synchronized at the same frequency as the carrier.

Exercise 1: Sketch the amplitude-modulated waveform given by Eq. (1-65) with $f(t) = \sin t$ and $\omega = 100$. What can you say about the "envelope" of this waveform?

Exercise 2: It is possible to rectify a sinusoidal current waveform $i(t) = I \sin t$ by applying this current to an appropriate time-varying linear resistance $R(t)$. Find $R(t)$ so that the resistor voltage is a rectified version of the current waveform; that is, $v(t) = i(t)$ whenever $i(t) \geq 0$, and $v(t) = 0$ whenever $i(t) \leq 0$.

Time-varying capacitor The simplest example of a time-varying linear capacitor is the air capacitor consisting of a fixed set of plates in mesh with a movable set of plates which is being rotated by a motor. A time-varying linear capacitor is therefore characterized by

$$q(t) = C(t)v(t) \tag{1-67}$$

where $C(t)$ is the time-varying capacitance. Unlike the resistor case, the expressions previously derived for the nonlinear capacitors do not apply in the time-varying case because when we differentiate $q(t)$ with respect to time, we obtain an additional term, namely,

$$i(t) = \frac{dq}{dt} = C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt} \tag{1-68}$$

Since $C(t)$ is not a constant, the expressions given by Eqs. (1-55) and (1-56) are no longer applicable. Hence, to calculate the power or energy flow, we must resort to the original definitions.

Just as for the resistor, a time-varying capacitor may be nonlinear; in this case it is characterized by $q = q(v,t)$ if it is voltage-controlled or $v = v(q,t)$ if it is charge-controlled. Time-varying capacitors are useful in the study of parametric amplifiers. They are also useful in the modeling of many time-varying physical and biological systems. For example, the mass of a rocket during lift-off decreases rapidly with time as the rocket fuel is burned. This time-varying mass can be modeled by a time-varying capacitor.

Exercise 1: Find the average power $P_{C_{av}}$ entering a time-varying capacitor $C(t) = 2 - \cos \omega t$ and a terminal voltage $v(t) = E \sin \omega t$. Interpret whether this energy is being absorbed, delivered, or stored.

Exercise 2: Give an example for each of the following: (*a*) A time-varying linear capacitor, (*b*) a time-varying charge-controlled capacitor, and (*c*) a time-varying voltage-controlled capacitor.

Time-varying inductor By exact analogy to the capacitor, a time-varying linear inductor is characterized by

$$\varphi(t) = L(t)i(t) \tag{1-69}$$

where $L(t)$ is the time-varying inductance. Since $L(t)$ is no longer a constant, the expressions derived previously in the preceding sections are no longer valid. In particular, the inductor voltage is now given by

$$v(t) = \frac{d\varphi(t)}{dt} = L(t)\frac{di(t)}{dt} + i(t)\frac{dL(t)}{dt} \tag{1-70}$$

A time-varying inductor may be nonlinear; in this case it is characterized by $\varphi = \varphi(i,t)$ if it is current-controlled and $i = i(\varphi,t)$ if it is flux-controlled.

The analysis of a nonlinear network containing time-varying elements is a very difficult problem requiring advanced mathematics. Hence, we shall not consider any time-varying elements in the rest of this book. The above discussion is included mainly to emphasize the fact that most of the equations we derived in the previous sections are not valid for time-varying elements.

Exercise 1: Give an example for each of the following: (*a*) A time-varying linear inductor, (*b*) a time-varying flux-controlled inductor, and (*c*) a time-varying current-controlled inductor.

Exercise 2: Prove or disprove the assertion that if the current into a time-varying current-controlled inductor is periodic, then the instantaneous power $P_L(t)$ is also periodic.

1-11 CONCEPTS OF MODELING

One of the most basic principles in scientific analysis is that of modeling. Engineers and scientists seldom analyze a physical system in its original form. Instead, they construct a model which approximates the behavior of the system. By analyzing the behavior of the model, they hope to predict the behavior of the actual system. The primary reason for constructing models is that physical systems are usually too complex to be amenable to a practical analysis. In most cases, the complexity of a system is due in part to the presence of many nonessential factors. One basic principle of modeling consists, therefore, of extracting only the essential factors.

To illustrate the process of modeling, let us consider the problem of predicting the trajectory of a ballistic missile. This problem cannot be analyzed exactly because an exact analysis would require inclusion of all possible factors that may affect the trajectory. Some of these factors may be the weight and shape of the missile, the amount of thrust, the atmospheric drag, the deformation of the missile during flight, the distribution of weights of the internal components, the wind velocity, the impurity of the fuel, and the color of the missile. From experience, we know that the first three factors have a more significant influence on the trajectory than the remaining factors. This leads us to replace the missile by a model which includes only the first three factors. Obviously, the predicted trajectory based on this model is not going to be identical with that of the actual system. But as engineers, we are interested only in an "accurate" solution, not the exact solution. Hence, as long as the discrepancy between the predicted and the actual