EE 100/42 Spring 2009
Solutions to Homework 5

A.1 Complex number arithmetic:

a)

$$\begin{aligned} Z_1 + Z_2 &= (2 + j3) + (4 - j3) \\ &= 6 \end{aligned}$$

b)

$$\begin{aligned} Z_1 - Z_2 &= (2 + j3) - (4 - j3) \\ &= -2 + j6 \end{aligned}$$

c)

$$\begin{aligned} Z_1 Z_2 &= (2 + j3)(4 - j3) \\ &= (8 + 9) + j(-6 + 12) \\ &= 17 + j6 \end{aligned}$$

d)

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{2 + j3}{4 - j3}\left(\frac{4 + j3}{4 + j3}\right) \\ &= \frac{-1 + j18}{25} \end{aligned}$$

A.7 More complex number arithmetic. The fact that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ will be used throughout this exercise.

a)

$$\begin{aligned} Z_a &= 5 + j5 + 10\angle 30^\circ \\ &= 5 + j5 + 10\left(\cos(30^\circ) + j\sin(30^\circ)\right) \\ &= 5 + j5 + 5\sqrt{3} + j5 \\ &= (5 + 5\sqrt{3}) + j10 \end{aligned}$$

b)

$$\begin{aligned} Z_b &= 5\angle 45^\circ - j10 \\ &= 5\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) - j10 \\ &= \frac{5}{\sqrt{2}} + j\left(\frac{5}{\sqrt{2}} - 10\right) \end{aligned}$$

c)

$$\begin{aligned} Z_c &= \frac{10\angle 45^\circ}{3+j4} \\ &= \frac{\left(\frac{10}{\sqrt{2}} + j\frac{10}{\sqrt{2}}\right)(3-j4)}{(3+j4)(3-j4)} \\ &= \frac{70}{25\sqrt{2}} - j\frac{10}{25\sqrt{2}} \end{aligned}$$

d)

$$\begin{aligned} Z_d &= \frac{15}{5\angle 90^\circ} \\ &= 3\angle -90^\circ \\ &= -j3 \end{aligned}$$

5.13 The rms value of the current is given by:

$$i_{rms}^2 = \frac{1}{T}\int_0^T i(t)^2 dt$$

The integral may be easily calculated for the particular current values given in the graph. Notice that $T = 4$.

$$\begin{aligned} i_{rms}^2 &= \frac{1}{4}\left[\int_0^2 (2)^2 dt + \int_2^4 (-1)^2 dt\right] \\ &= \frac{1}{4}(10) \\ &= \frac{10}{4} \end{aligned}$$

Taking the square root gives the answer:

$$i_{rms} = \frac{\sqrt{10}}{2} = 1.581 \text{ A}$$

5.22 The angular frequency is $\omega = 2\pi f = 400\pi$ rad/sec. The time-domain relationship may be recovered from phasor notation by the following formula:

$$v(t) = \text{Re}\left\{Me^{j(\omega t+\phi)}\right\}$$

where the phasor is $\mathbf{V} = Me^{j\phi}$. This allows the easy recovery from the phasors given in the phasor diagram (note that the phase must be expressed in radians).

$$\begin{aligned} v_1(t) &= 10\cos\left(400\pi t + \frac{\pi}{6}\right) \text{ V} \\ v_2(t) &= 5\cos\left(400\pi t + \frac{5\pi}{6}\right) \text{ V} \\ v_3(t) &= 10\cos\left(400\pi t + \frac{\pi}{2}\right) \text{ V} \end{aligned}$$

The phase relationships are as follows:

- $v_1(t)$ lags $v_2(t)$ by $\frac{2\pi}{3}$ rad.
- $v_1(t)$ lags $v_3(t)$ by $\frac{\pi}{3}$ rad.
- $v_3(t)$ lags $v_2(t)$ by $\frac{\pi}{3}$ rad.

5.26 From the figure, the following information can be found:

- Period $T = 0.5$ sec. $\Rightarrow$ $\omega = 4\pi$ rad/sec.
- Amplitude $V_m = 3$ V.
- Phase $\phi = \frac{-0.0625}{T}(2\pi) = -\frac{\pi}{4}$ rad.

Thus, the time-domain signal is

$$v(t) = 3\cos\left(4\pi t - \frac{\pi}{4}\right) \text{ V}$$

The phasor for this signal can be constructed from the information above as:

$$\mathbf{V} = 3e^{-j\frac{\pi}{4}} \text{ V}$$

Finally, the rms value of $v(t)$ may be computed from the formula given earlier. Recall that, for a sinusoid of period T and angular frequency $\frac{2\pi}{T}$, $\int_0^T \cos^2(\omega t)dt = \frac{T}{2}$.

$$\begin{aligned} v_{rms} &= \sqrt{\frac{1}{0.5}\int_0^{0.5} v(t)^2 dt} \\ &= \sqrt{\frac{1}{0.5}\frac{0.5}{2}(9)} \\ &= \frac{3}{\sqrt{2}} \text{ V} \\ &= 2.121 \text{ V} \end{aligned}$$

5.40 For each frequency, $Z = Z_L + Z_R + Z_C$, where each subscript refers to the inductor, resistor, and capacitor, respectively. The expressions for each impedance are:

$$\begin{aligned} Z_L &= j\omega L \\ Z_R &= R \\ Z_C &= \frac{1}{j\omega C} \end{aligned}$$

a) $\omega = 500$:

$$\begin{aligned} Z &= j50 + 50 - j200 = 50 - j150 \\ &= 158.1e^{-j1.249} \ \Omega \end{aligned}$$

b) $\omega = 1000$:

$$\begin{aligned} Z &= j100 + 50 - j100 \\ &= 50 \ \Omega \end{aligned}$$

c) $\omega = 2000$:

$$\begin{aligned} Z &= j200 + 50 - j50 = 50 + j150 \\ &= 158.1e^{j1.249}\ \Omega \end{aligned}$$

5.48 The phasor for the source current $i_S(t)$ can be found by inspection:

$$\mathbf{I}_S = 0.1\ \text{A}$$

By applying the governing equations (in phasor form) for the resistor $\mathbf{V} = \mathbf{I}_R R$ and the capacitor $\mathbf{I}_C = j\omega\mathbf{V}$, we can arrive at the following equation (using KCL):

$$\begin{aligned} \mathbf{I}_S &= \mathbf{I}_R + \mathbf{I}_C \\ &= \frac{\mathbf{V}}{R} + j\omega C\mathbf{V} \\ &= \left(\frac{1}{R} + j\omega C\right)\mathbf{V} \\ \Rightarrow \mathbf{V} &= \left(\frac{1}{R} + j\omega C\right)^{-1}\mathbf{I}_S \end{aligned}$$

Plugging in the values for the variables in the equation above yields the expression for the voltage phasor:

$$\begin{aligned} \mathbf{V} &= \left(\frac{1}{100} + j0.5 \times 10^{-2}\right)^{-1} 0.1 \\ &= 8.944e^{-j0.4636}\ \text{V} \end{aligned}$$

Now, the phasors for the current through the resistor and the capacitor may be found. Note that $j = e^{j\frac{\pi}{2}}$.

$$\begin{aligned} \mathbf{I}_R &= \frac{\mathbf{V}}{R} = \frac{8.944e^{-j0.4636}}{100} \\ &= 0.0894e^{-j0.4636}\ \text{A} \\ \mathbf{I}_C &= j\omega C\mathbf{V} = 10^4(0.5 \times 10^{-6})e^{j\left(0.4636 + \frac{\pi}{2}\right)} \\ &= 44.72e^{j1.1072}\ \text{A} \end{aligned}$$

The phasor diagram is given in Figure 1.[1]

The phase relationship between $\mathbf{V}$ and $\mathbf{I}_S$ is that $\mathbf{V}$ lags $\mathbf{I}_S$ by 0.4636 rad.

5.53 The nodes are already labeled in the problem diagram. Since we are looking for node voltages, the nodal method will give us the answer right away. Since there is a current source, define the current flowing through the source as $\mathbf{I}_S$, in the direction of increasing potential according to the diagram drawing.

[1] Image taken from the companion website for the book: http://wps.prenhall.com/wps/media/objects/4045/4142137/Chapter_05/ICE_Chapter_05.pdf.

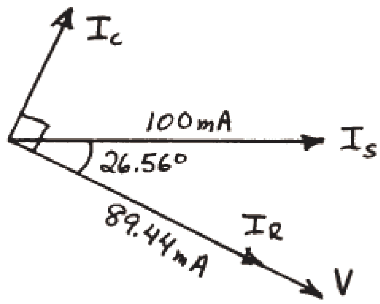

Figure 1: Phasor diagram for Problem 5.48. It is not drawn to scale.

KCL at node 1:

$$\frac{0-\mathbf{V}_1}{10}+\frac{0-\mathbf{V}_1}{j20}+\mathbf{I}_S=0$$

KCL at node 2:

$$\frac{0-\mathbf{V}_2}{15}+\frac{0-\mathbf{V}_2}{-j5}-\mathbf{I}_S=0$$

Voltage source equation:

$$\mathbf{V}_1-\mathbf{V}_2=10\angle 0$$

There are now three equations and three unknowns. The unknowns may be found by setting up and solving a system of equations.

$$\begin{bmatrix} -\frac{1}{10}-\frac{1}{j20} & 0 & 1 \\ 0 & -\frac{1}{15}+\frac{1}{j5} & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{I}_S \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

If the nodal equations were chosen correctly, then the 3×3 matrix above will be invertible. This allows the solution to be easily computed.

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{I}_S \end{bmatrix}=\begin{bmatrix} 8.1768+j4.6409\ \text{V} \\ -1.8232+j4.6409\ \text{V} \\ 1.0497+j0.0552\ \text{A} \end{bmatrix}$$
$$=\begin{bmatrix} 9.402e^{j0.5162}\ \text{V} \\ 4.9862e^{j1.9451}\ \text{V} \\ 1.0512e^{j0.0526}\ \text{A} \end{bmatrix}$$

6.11 One way to find the value of a transfer function is to find the ratio of the output to the input in phasor form. The input phasor is

$$\mathbf{V}_{in}=2e^{-j\frac{5}{36}}\ \text{V}$$

while the output phasor is

$$\mathbf{V}_{out} = 2e^{j\frac{1}{9}} \text{ V}$$

The transfer function at this particular frequency is

$$\begin{aligned} H(\omega = 10^4\pi) &= \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} \\ &= \frac{2e^{j\frac{1}{9}}}{2e^{-j\frac{5}{36}}} \\ &= 1e^{j\frac{\pi}{4}} \end{aligned}$$

This answer can be found by inspection too. The magnification ratio between the input and output is 1, and the output leads the input by $+45°$.

6.24 The transfer function relating the input $v_{in}(t)$ to the output $v_{out}(t)$ can be found from simple circuit analysis. Defining the current flowing through the single loop as $i(t)$ we get:

$$\mathbf{I} = j\omega C\mathbf{V}_{out} = \frac{\mathbf{V}_{in}}{Z}$$

where Z is the equivalent impedance of the resistor and capacitor in series. Plugging in $Z = R + \frac{1}{j\omega C}$ allows us to solve for the transfer function.

$$\begin{aligned} H(\omega) := \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} &= \frac{1}{j\omega C}\frac{1}{R + \frac{1}{j\omega C}} \\ &= \frac{1}{1 + j\omega RC} \\ &= \frac{1}{1 + j\omega\frac{1\times10^{-3}}{\pi}} \end{aligned}$$

Now, we can find the output signal as from the following:

$$\begin{aligned} v_{out}(t) = {} & 5|H(500\pi)|\cos\left(500\pi t + \angle H(500\pi)\right) \\ & + 5|H(1000\pi)|\cos\left(1000\pi t + \angle H(1000\pi)\right) \\ & + 5|H(2000\pi)|\cos\left(2000\pi t + \angle H(2000\pi)\right) \text{ V} \end{aligned}$$

Evaluating the required values of $H(\omega)$ is straightforward from the transfer function.

$$\begin{aligned} H(500\pi) &= \frac{1}{1 + j0.5} \\ &= 0.8944e^{-j0.4637} \\ H(1000\pi) &= \frac{1}{1 + j} \\ &= 0.7071e^{-j\frac{\pi}{4}} \\ H(2000\pi) &= \frac{1}{1 + j2} \\ &= 0.4472e^{-j1.1071} \end{aligned}$$

The output can now be written as

$$
\begin{aligned}
v_{out}(t) = &\ 4.472\cos(500\pi t - 0.4637) + 3.535\cos\left(1000\pi t - \frac{\pi}{4}\right) \\
&+ 2.236\cos(2000\pi t - 1.1071) \ \text{V}
\end{aligned}
$$

6.40 The definition of decibel is $M_{\text{dB}} = 20\log_{10} M$, with an inverse given by $M = 10^{\frac{M_{\text{dB}}}{20}}$.

a) For $|H(f)|_{\text{dB}} = -10$ dB,

$$
\begin{aligned}
|H(f)| &= 10^{-\frac{10}{20}} \\
&= 0.3162
\end{aligned}
$$

b) For $|H(f)|_{\text{dB}} = 10$ dB,

$$
\begin{aligned}
|H(f)| &= 10^{\frac{10}{20}} \\
&= 3.162
\end{aligned}
$$