Exercise 1: Use the Euler algorithm to calculate the approximate solution of $dx/dt = x$ over the time interval $0 < t < 1$. Assume an initial condition $x(0) = 10$ and a step size $h = 0.2$. Compare the solution with the exact answer $x(t) = 10e^t$.

Exercise 2: Specify the Euler algorithm for a system of three differential equations in the normal form.

4-8 PRINCIPLES OF DUALITY

There are many physical phenomena or systems in nature which have occurred in *dual* forms. Generally speaking, we say two systems or phenomena are *duals* of each other if we can exhibit some kind of one-to-one correspondence between various quantities or attributes of the two systems. For example, in mathematics, two equations which differ only in symbols but are otherwise identical in form are said to be dual equations. In physics, for each translational system or problem there exists a corresponding rotational system or problem, and they are usually referred to as dual systems or problems. The recognition of dual quantities, attributes, phenomena, properties, or concepts often leads to the discovery and invention of new ideas. In electrical engineering, the application of the principle of duality has often led not only to a simplification of solutions but also to the discovery and invention of new useful networks.

Before we render the concepts of duality more precise, it is instructive to consider first the two nonlinear networks shown in Fig. 4-17*a* and *b*. The equations of motion of these two networks are readily obtained and are tabulated in Table 4-5. A careful comparison of the expressions in the two columns of this table re-

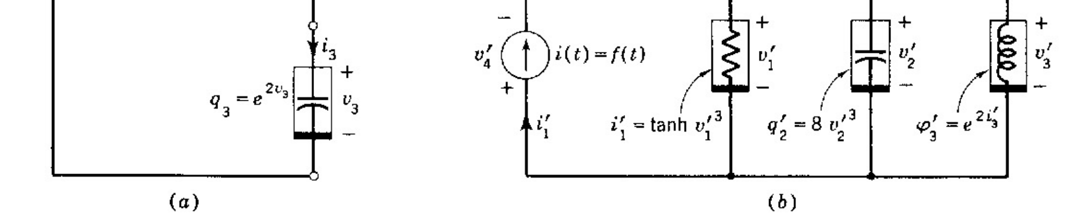


Fig. 4-17. The dual of a series nonlinear network is a parallel nonlinear network.

TABLE 4-5 *Equations of motion for the networks in Fig. 4-17a and b.*

Network of Fig. 4-17*a*	Network of Fig. 4.17*b*
Laws of Elements	Laws of Elements
$v_1 = \tanh i_1{}^3$	$i'_1 = \tanh v'_1{}^3$
$v_2 = \frac{d\varphi_2}{dt} = \frac{d\varphi_2}{di_2}\frac{di_2}{dt} = 24i_2{}^2\frac{di_2}{dt}$	$i'_2 = \frac{dq'_2}{dt} = \frac{dq'_2}{dv'_2}\frac{dv'_2}{dt} = 24v'_2{}^2\frac{dv'_2}{dt}$
$i_3 = \frac{dq_3}{dt} = \frac{dq_3}{dv_3}\frac{dv_3}{dt} = 2e^{2v_3}\frac{dv_3}{dt}$	$v'_3 = \frac{d\varphi'_3}{dt} = \frac{d\varphi'_3}{di'_3}\frac{di'_3}{dt} = 2e^{2i'_3}\frac{di'_3}{dt}$
$v_4 = f(t)$	$i'_4 = f(t)$
Laws of Interconnection	Laws of Interconnection
KVL: $v_1 + v_2 + v_3 - v_4 = 0$	KCL: $i'_1 + i'_2 + i'_3 - i'_4 = 0$
KCL: $i_1 + i_4 = 0$	KVL: $v'_1 + v'_4 = 0$
$i_1 - i_2 = 0$	$v'_1 - v'_2 = 0$
$i_2 - i_3 = 0$	$v'_2 - v'_3 = 0$

veals a one-to-one correspondence between the equations. As a matter of fact, except for the symbols, the equations in the two columns are identical in form. Observe that, had we replaced v_j by i'_j, i_j by v'_j, φ_j by q'_j, and q_j by φ'_j for the variables in the left column, the result would be identical with that in the right column, and therefore the two networks are said to be dual networks. One of the significant facts about dual networks is that once we know the solution of one network, the solution of the dual network can be obtained immediately by simply interchanging the symbols. This means that as soon as we know the behavior and properties of one network, we immediately know the behavior and properties of the dual network. Hence a lot of redundancy is avoided if we can recognize dual networks. In this book, there will be many occasions when we shall take advantage of the principle of duality. In view of its importance, we shall now precisely define the concept of duality.

DEFINITION OF DUAL NETWORKS

Let N and N' be a pair of networks each containing b two-terminal network elements which are not controlled sources.[1] Then N and N' are dual networks if the elements in N and N' can be labeled, respectively, as $b_1, b_2, \ldots, b_b$ and $b'_1, b'_2, \ldots, b'_b$ such that the equations of motion of the two networks are identical in form.

[1] It is possible to generalize the definition of dual networks to include controlled sources. However, the procedure for constructing such networks is more complicated.

That is, if we replace v_j by i'_j, i_j by v'_j, φ_j by q'_j, and q_j by φ'_j in the equations of motion for network N, we obtain the equations of motion for network N', and vice versa.

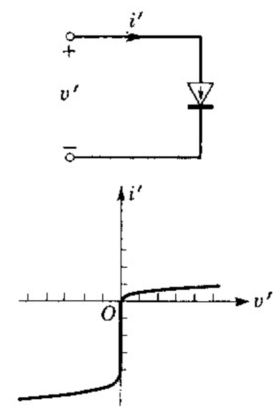
In view of the above definition, it is clear that the two networks N and N' are dual networks if the solution v_j, i_j, φ_j, or q_j for element b_j in network N is equal to the solution i'_j, v'_j, q'_j, or φ'_j in element b'_j of N', and vice versa. Our next task will be to investigate what classes of networks would satisfy the above definition. This question is clearly equivalent to the following problem: Given a network N containing two-terminal elements, does there exist another network N' which is the dual of N, and if so, how can we find N'? To answer this question, it is necessary to uncover the duality relationships that must be satisfied by the laws of elements and the laws of interconnection of the two dual networks.

4-8-1 DUALITY RELATIONSHIPS FROM THE LAWS OF ELEMENTS

The definition for dual networks tells us that the equation of motion for element b_j is the dual of the equation of motion for element b'_j if the following one-to-one correspondence exists between the elements in N and N'.

Resistor If element b_j is a two-terminal resistor in N characterized by a curve Γ in the v-i plane, then the corresponding dual element b'_j in N' must be also a two-terminal resistor characterized by the same curve Γ in the i'-v' plane, i.e., with the i axis replaced by the v' axis and the v axis replaced by the i' axis. For example, if element b_j of N is a resistor characterized by $i_j = v_j^3 - 3v_j$, then the dual resistor in N' is a resistor characterized by $v'_j = i'^3_j - 3i'_j$. In particular, if element b_j in N is a voltage source with $v_s(t) = K \sin t$, then the dual element b'_j in N' must be a current source with $i'_s(t) = K \sin t$. Observe that the dual of a given resistor is a new resistor, which may need a new name and a new symbol. For example, the dual of a zener diode with a v-i curve as defined in Table 1-1 is a new resistor whose new symbol and v'-i' curve are shown in Fig. 4-18. However, there are some two-terminal elements which have the interesting property that the dual of the element is the same element with its two terminals interchanged. For such elements, a new symbol is not needed for we only have to interchange the terminals of the given element to obtain its dual. The simplest example of this type of element is the

Fig. 4-18. The symbol and the v-i curve of the dual of a zener diode.



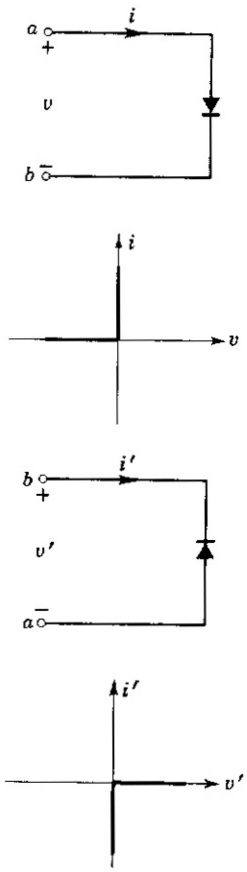


Fig. 4-19. The dual of an ideal diode is also an ideal diode with its terminals interchanged.

ideal diode. Clearly, the dual of an ideal diode is obtained simply by reversing the two terminals of the diode, as shown in Fig. 4-19. Observe that in contrast with the new symbol introduced in Fig. 4-18, no new symbol is needed to specify unambiguously the dual of an ideal diode.

Inductor If element b_j in N is a two-terminal inductor characterized by a curve Γ in the i-φ plane, then the corresponding dual element b_j' in N' must be a capacitor characterized by the same curve Γ in the v'-q' plane; that is, simply change the variables φ and i to q' and v' while retaining the curve Γ. For example, the dual of an inductor characterized by $\varphi = \log i$ is a capacitor characterized by $q' = \log v'$.

Capacitor If element b_j in N is a two-terminal capacitor characterized by a curve Γ in the v-q plane, then the corresponding dual element b_j' in N' must be an inductor characterized by the same curve Γ in the i'-φ' plane; that is, simply change the variables q and v to φ' and i' while retaining the curve Γ. For example, the dual of a capacitor characterized by $q = \tanh v$ is an inductor characterized by $\varphi' = \tanh i'$.

4-8-2 DUALITY RELATIONSHIPS FROM THE LAWS OF INTERCONNECTION

Since the equations of motion from the laws of interconnection are independent of the network elements and can be written directly from the network graph, it is clear that the duality relationships for these equations can be found directly from the network graphs of the dual networks. Now, from the definition of dual networks, each branch voltage v_j in N becomes a branch current i_j' in N', and each branch current i_j in N becomes a branch voltage v_j' in N'. It is clear that the equations of motion due to KVL in N must correspond to the equations of motion due to KCL in N', and vice versa. This means that if the network N has n nodes and b branches, resulting in $(n - 1)$ independent KCL equations and $b - (n - 1)$ independent KVL equations,[1] then the dual network N' must contain $(n - 1)$ *fundamental loops*, or meshes, and $b - (n - 1) + 1$ nodes in order to have correspondingly $(n - 1)$ independent KVL equations and $b - (n - 1)$ KCL equations. For example, the network N shown in Fig. 4-17*a* has four nodes and four branches, and the dual network shown in Fig. 4-17*b* has, correspondingly, three meshes, as expected. Given a network graph N with n nodes and

[1] Review the independent KVL and KCL equations criteria in Sec. 4-3.

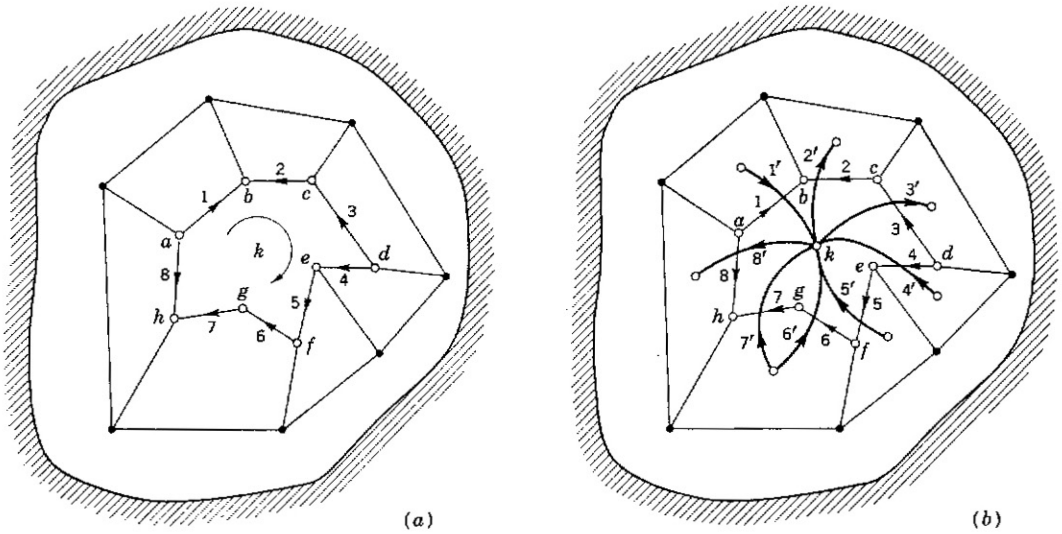


Fig. 4-20. A typical mesh k of a planar network and its dual branches.

b branches, we shall now attempt to find a procedure (if it exists) to construct a dual network graph N' which we know must consist of $(n - 1)$ fundamental loops, or meshes, and b branches.

If the network is planar, we know from the KVL mesh equation criteria (Sec. 4-3) that the number of independent KVL equations is equal to the number of meshes, namely, $b - (n - 1)$. Hence, the dual network N' must have $b - (n - 1)$ independent KCL equations. This means that N' must contain $b - (n - 1) + 1$ nodes. Let us consider the typical planar graph shown in Fig. 4-20a. In particular, consider a typical mesh such as mesh k formed by the branches 1, 2, 3, 4, 5, 6, 7, and 8. The KVL equation around mesh k is given by[1]

$$v_1 - v_2 - v_3 + v_4 + v_5 + v_6 + v_7 - v_8 = 0 \qquad (4\text{-}70)$$

Now if N' is the dual of N, then the graph of N' must necessarily contain a corresponding node k whose KCL equation is the dual of Eq. (4-70), namely,

$$i'_1 - i'_2 - i'_3 + i'_4 + i'_5 + i'_6 + i'_7 - i'_8 = 0 \qquad (4\text{-}71)$$

We can accomplish the above task if we place node k inside mesh k and, for each branch b_j around mesh k, draw a corresponding branch b'_j from node k to a node placed inside the mesh having branch b_j in common with mesh k, as shown in Fig. 4-20b. In view

[1] Recall that when we draw the network graph, we automatically assume that the positive polarity of each two-terminal element is located at the tail end of the current reference direction. This assumption must always be kept in mind. In particular, before we find the dual of a two-terminal element, we must first see to it that the characteristic curve of the element is specified with a set of references consistent with the above assumption.

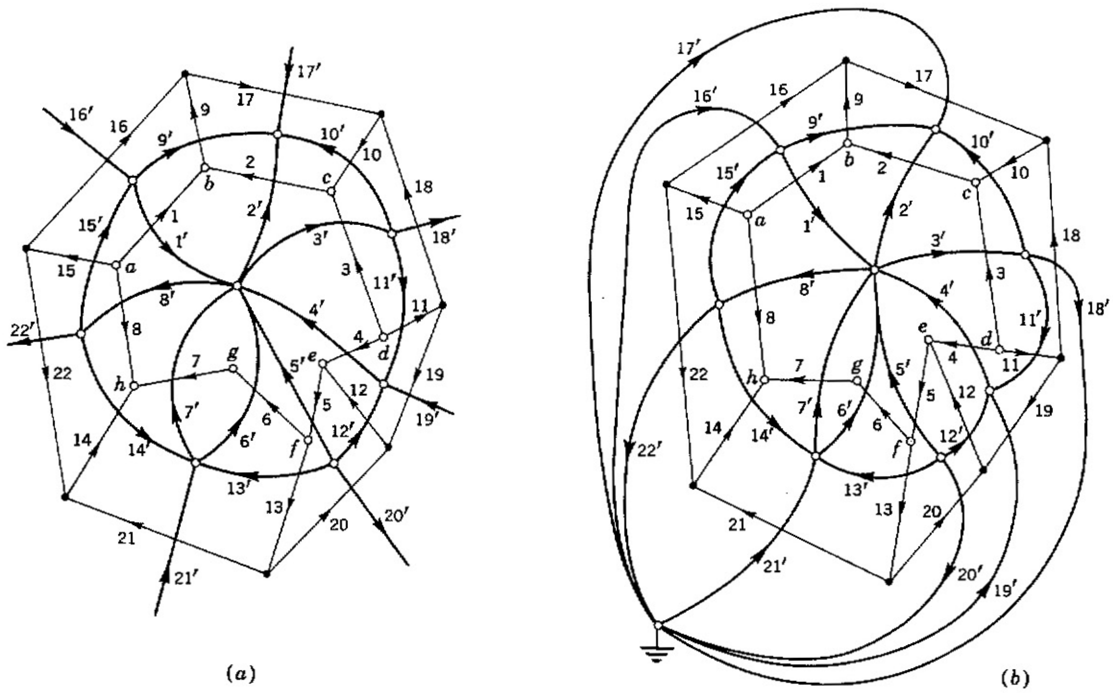


Fig. 4-21. The procedure for constructing a dual graph is as follows: (*a*) put a node inside each mesh and a node outside the network; (*b*) corresponding to each branch b_j common to two meshes, connect a branch b_j' between the two nodes in these meshes such that it cuts the branch b_j; and (*c*) assign a direction for b_j' in accordance with the reference connection.

of the above construction procedure, each branch b_j' in the dual graph must necessarily intersect branch b_j of the given graph. Finally, let us assign reference directions to each branch b_j' in the dual graph according to the following convention.

REFERENCE CONVENTION FOR DUAL GRAPH

Branch b_j' is assigned a direction toward node k if the corresponding branch b_j of the given graph is in a clockwise direction with respect to node k. Otherwise, branch b_j' will be assigned a direction away from node k.

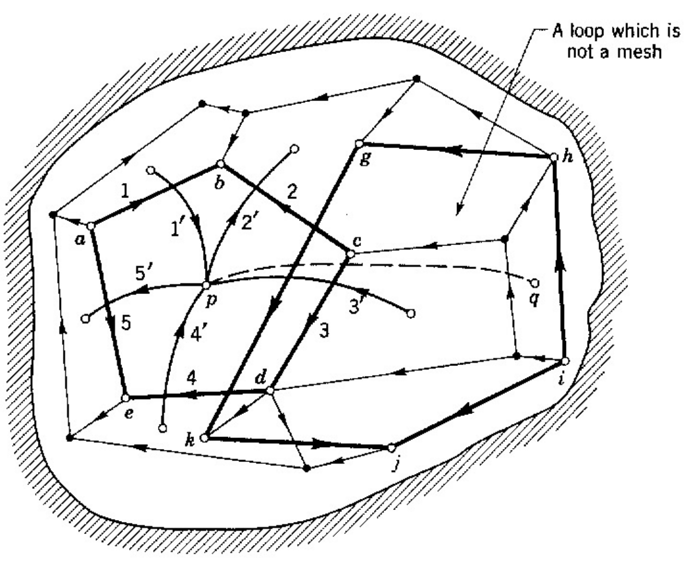
With the branches in Fig. 4-20*b* directed consistently according to the above convention, it is clear that we indeed obtain the desired Eq. (4-71).

If we repeat the above procedure for each of the $b - (n - 1)$ meshes, we would eventually end up with a network graph such as the examples shown in Fig. 4-21*a*, with branches cutting the outer boundary of the given graph N, and so far left unconnected. To complete the dual graph, we observe that KVL must be satisfied around the outer boundary of the given graph N, and similarly, in the dual graph, the branches cutting the outer boundary in Fig. 4-21*a* must satisfy KCL. This condition can be satisfied by placing one node outside the dual graph to terminate all branches of the dual graph which cut the outer boundary of N at the external node. The resulting network graph shown in Fig. 4-21*b* is then the dual of the given network N. This procedure, together with the principle for drawing dual elements, would enable us to find the dual of any planar network.

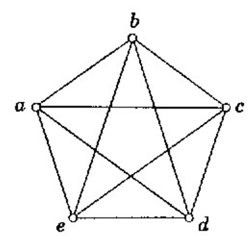
What happens if network N is nonplanar? Since the concept of a mesh is undefined for nonplanar networks, the dual quantities for this class of networks are loops and nodes. Moreover, since the network is nonplanar, there must be at least one closed loop that is not a mesh (remember that a mesh encloses no branches, a closed loop does), such as the typical case shown in Fig. 4-22*a*.

Consider first writing KVL around loop *abcde* consisting of branches 1, 2, 3, 4, and 5. Using the procedure for a planar network, we place a node p inside loop *abcde* with five branches as shown. This would give us the desired KCL equation at node p. However, suppose we repeat the procedure for loop *ghijk* and place a node q inside it. The procedure for drawing a planar network would now require a branch connecting nodes p and q,

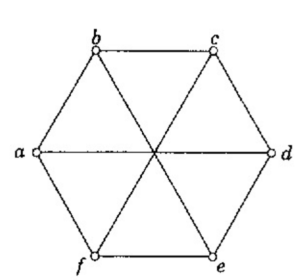
Fig. 4-22. The network shown in (*a*) demonstrates that a nonplanar network cannot possess a dual. A network graph is nonplanar if, and only if, it contains either one of the two Kuratowski subgraphs shown in (*b*) and (*c*).



(*a*)



(*b*) Kuratowski subgraph 1



(*c*) Kuratowski subgraph 2

thus adding one more branch to node p. But this would violate the desired KCL equation at node p. Hence the procedure would not work for a nonplanar network. Since the above argument is valid in general, it is impossible to find a network graph which is the dual of a nonplanar network. This result is important enough to be stated as a theorem.

DUAL NETWORK EXISTENCE THEOREM

A network N has a dual if, and only if, N is a planar network.

It is clear from this theorem that before we attempt to find the dual of a given network, we must determine whether it is planar. For simple networks, this can be readily determined by inspection. For more complicated networks, however, it is sometimes not easy to distinguish a planar from a nonplanar network. Kuratowski's theorem applies to this situation, but because it is extremely difficult to prove, it is only stated here.

KURATOWSKI'S THEOREM

The necessary and sufficient condition that a network N be planar is that the network graph contain neither the subgraph (i.e., part of the graph obtained by removing some branches and nodes) shown in Fig. 4-22*a* nor the subgraph shown in Fig. 4-22*b*.[1]

As a consequence of Kuratowski's theorem, it is clear that any network with less than five nodes is necessarily planar. Hence, without having to redraw the network in Fig. 4-4*a*, we can conclude that it is planar because it contains only four nodes. When a network contains more than five nodes, we can apply Kuratowski's theorem by searching for the existence of a set of five nodes with each node having a branch connected to each of the other nodes, as in Fig. 4-22*b*. Or we can look for the existence of a set of six nodes with branches connected in the form of Fig. 4-22*c*. For example, the network shown earlier in Fig. 4-5 is nonplanar because it contains a Kuratowski subgraph of the form shown in Fig. 4-22*b*.

It is easier to prove that a network is nonplanar because we need only exhibit one of the two Kuratowski subgraphs. To prove that a network is planar, we must examine all possible groups of branches and nodes to ascertain that the network does not contain either of the two Kuratowski subgraphs.

[1] This theorem implicitly assumes that whenever two or more branches are connected in series or in parallel, they are interpreted as only one branch. Hence, before one applies the test, it is convenient to replace all branches which are in series or in parallel by a single branch.

4-8-3 ALGORITHM FOR DRAWING THE DUAL OF A PLANAR NETWORK

Given a planar network N, we can now draw its dual by the following procedure:

1. Check first that all element-characteristic curves are specified with a set of reference directions and polarities consistent with the assumption that the positive terminal of each element is at the tail end of the current reference arrow. If not, redefine the references and the characteristic curves accordingly.
2. Draw the graph of N. Place a node inside each mesh in N and one node outside of N. Corresponding to each branch b_j in N which is common to meshes α and β, draw a branch b_j' from the node inside mesh α to the node inside mesh β. The reference direction for b_j' is then directed toward the node inside mesh α, if the reference direction of branch b_j in mesh α is in the clockwise direction (with respect to the node inside mesh α). Otherwise, the reference direction of b_j' is directed away from the node inside mesh α.
3. The branches b_j' in N' are then replaced by the corresponding network elements dual to branch b_j. For convenience, Table 4-6 tabulates the dual quantities useful for drawing dual networks.

EXAMPLE

Consider the nonlinear bridge network shown in Fig. 4-23a. The elements consist of an ideal diode (element 1), a nonlinear resistor (element 2) whose v_2-i_2 curve is shown in Fig. 4-23b, a current source $i(t) = 5e^{-t}$ (element 3), a nonlinear capacitor (element 4) whose v_4-q_4 curve is shown in Fig. 4-23c, a 5-Ω linear resistor (element 5), a nonlinear inductor (element 6) characterized by $\varphi_6 = 1 + i_6 - 3i_6^2 + 5i_6^3$, a 3-F linear capacitor (element 7), a 2-H linear inductor (element 8), a tunnel diode (element 9) whose v_9-i_9 curve is given in Table 1-1, and a voltage source $v(t) = 2 \cos t$. The units of all voltages, currents, charges, and flux linkages are assumed to be volts, amperes, coulombs, and webers. We draw the dual following the three steps outlined in the above algorithm:

1. A check of the reference directions and polarities of the elements in Fig. 4-23a shows that the references for elements 2, 3, 6, and 10 are not consistent with convention. Hence we must

TABLE 4-6 *Common dual quantities.*

Network N	Network N'
Current i_j	Voltage v'_j
Voltage v_j	Current i'_j
Flux linkage φ_j	Charge q'_j
Charge q_j	Flux linkage φ'_j
Nonlinear resistor (characterized by a curve Γ in the v-i plane)	Nonlinear resistor (characterized by the same curve Γ in the i'-v' plane, i.e., with v and i axes interchanged)
Linear resistor with a resistance of R Ω	Linear resistor with a conductance of R mhos or $1/R$ Ω
Nonlinear inductor (characterized by a curve Γ in the i-φ plane)	Nonlinear capacitor (characterized by a curve Γ in the v'-q' plane)
Linear inductor with an inductance of K H	Linear capacitor with a capacitance of K F
Nonlinear capacitor (characterized by a curve Γ in the v-q plane)	Nonlinear inductor (characterized by a curve Γ in the i'-φ' plane)
Linear capacitor with a "capacitance" of K F	Linear inductor with an "inductance" of K H
Voltage source, $v_j = f(t)$	Current source $i'_j = f(t)$
Current source $i_j = g(t)$	Voltage source $v'_j = g(t)$
Short circuit	Open circuit
Open circuit	Short circuit
Parallel branches	Series branches
Series branches	Parallel branches
Link	Tree branch
Tree branch	Link
Fundamental loop	Fundamental cut set
Fundamental cut set	Fundamental loop
Ideal diode	Ideal diode with its two terminals interchanged

redefine the references of these elements as shown in Fig. 4-24*a*. The v-i curve of element 2 must be changed accordingly, as shown in Fig. 4-24*b*. The i-φ curve of element 6 must also be changed to $\varphi_6 = 1 - i_6 - 3i_6^2 - 5i_6^3$, since i_6 has become $-i_6$.

The voltage source and the current source remain unchanged, since the terminal voltage of a voltage source is independent of its terminal current, and the terminal current of a current source is independent of its terminal voltage.

2. The graph of the modified network in Fig. 4-24*a* is drawn in Fig. 4-24*c*, together with its dual graph.
3. The completed dual graph is redrawn as in Fig. 4-24*d*. It remains to replace each branch by its corresponding dual elements. Therefore, element 1 is an ideal diode with its terminals interchanged; element 2 is a nonlinear resistor with its v_2'-i_2' curve shown in Fig. 4-24*f*; element 3 is a voltage source with terminal voltage $v_3'(t) = 5e^{-t}$; element 4 is a nonlinear inductor with its i_4'-φ_4' curve shown in Fig. 4-24*g*; element 5 is a 1/5-Ω linear resistor; element 6 is a nonlinear capacitor characterized by $q_6' = 1 - v_6' - 3v_6'^2 - 5v_6'^3$; element 7 is a 3-H linear inductor; element 8 is a 2-F linear capacitor; element 9 is a nonlinear resistor with its v_9'-i_9' curve shown in Fig. 4-24*h*; and element 10 is a current source with terminal current $i_{10}(t) = 2 \cos t$.

Observe that the dual of a voltage-controlled *v*-*i* curve is current-controlled, and vice versa. In particular, the dual *v*-*i* curve can be obtained by reflecting the original curve with respect to the 45° line through the origin. In practice, this operation can be simulated by a 45° reflector (more commonly known as a gyrator).

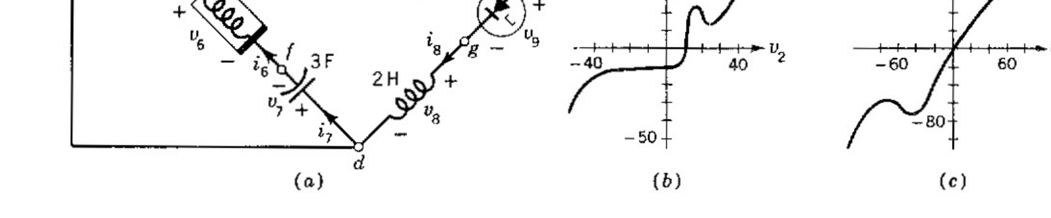


Fig. 4-23. A typical nonlinear bridge network.

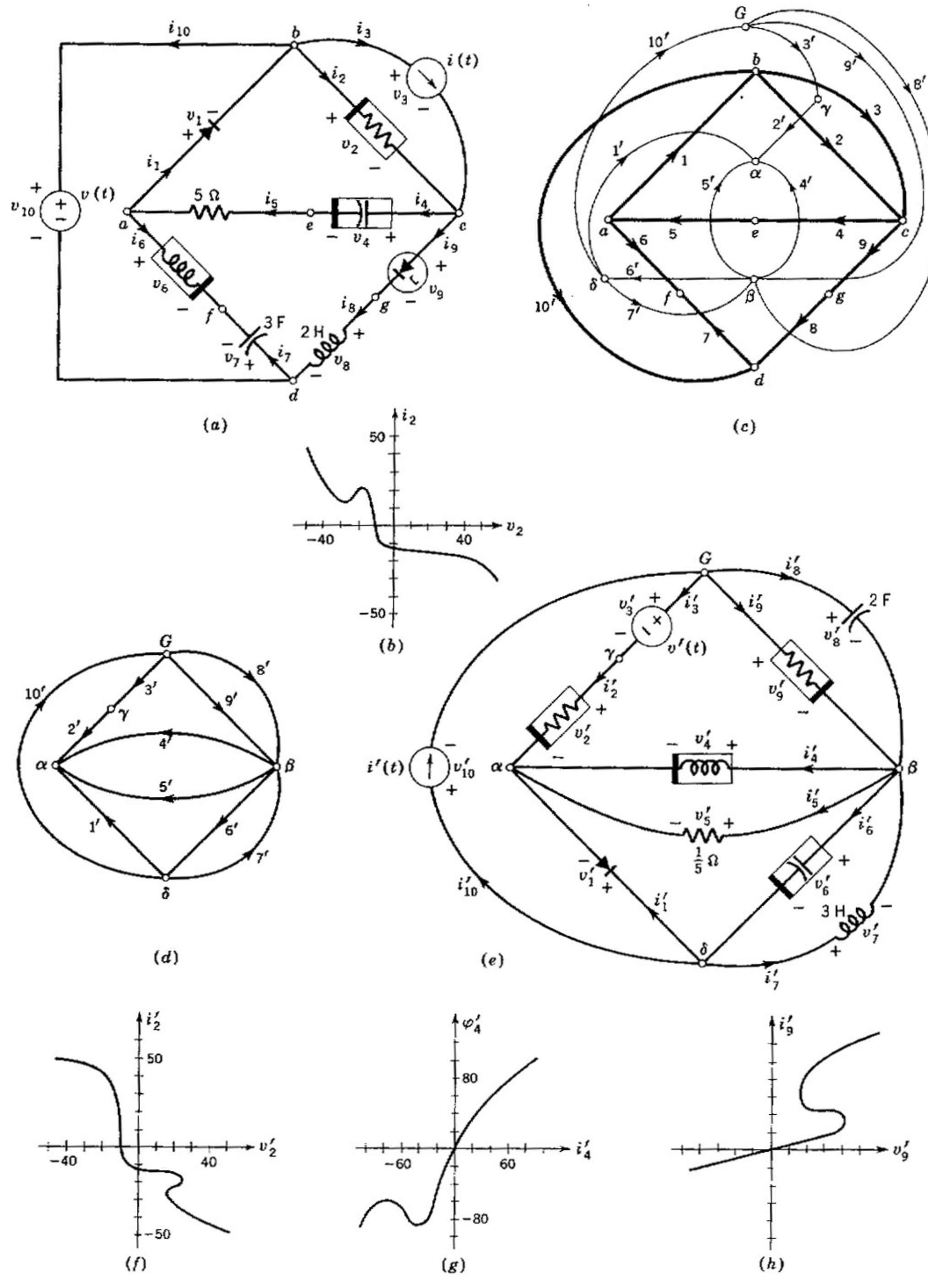


Fig. 4-24. The step-by-step procedure for constructing the dual of the nonlinear bridge network of Fig. 4-23.

Similarly, the type 1 *C-L* mutator presented in Sec. 3-8-3 (also known as a gyrator) can be used to simulate the dual of a nonlinear capacitor and a nonlinear inductor. In other words, the dual of any nonlinear network can be synthesized with the help of a gyrator.

Exercise 1: (*a*) Construct the dual of the network shown in Fig. 4-17*a* by the procedure described in this section. (*b*) Repeat (*a*) for the network shown in Fig. 4-17*b*. (*c*) Show that a planar network can have only one dual.

Exercise 2: Identify the Kuratowski subgraph for the nonplanar network shown in Fig. 4-5.

Exercise 3: Prove that a gyrator can be used to obtain the dual of any nonlinear resistor, inductor, or capacitor.

4-9 SUMMARY

Classification of networks

1. Resistive networks do not contain capacitors or inductors.
   a. Resistive linear networks contain only linear resistors and sources.
   b. Resistive nonlinear networks contain at least one nonlinear resistor.
2. Dynamic networks contain at least one capacitor or inductor.
   a. Dynamic linear networks contain only linear elements and sources.
   b. Dynamic nonlinear networks contain at least one nonlinear element.

Equations of motion

1. Equations from the laws of elements representing the characteristic curves of the elements are always independent.
2. Equations from the laws of interconnection representing equations from KVL and KCL may not be independent.

Network topology

Network topology provides the systematic techniques for obtaining independent equations from the laws of interconnection. Some basic terminologies are as follows:

1. Tree: a set of branches, called tree branches, connecting all nodes but not forming closed loops.

2. Cotree: the set of all branches, called links, not belonging to a tree.
3. Cut set: a set of branches which, if cut, would separate the network into two parts.
4. Fundamental loop: given a tree T, a loop formed by a link of T and one or more tree branches is called a fundamental loop with respect to T.
5. Fundamental cut set: given a tree T, a cut set formed by a tree branch of T and one or more links is called a fundamental cut set with respect to T.

Independent KVL equation criteria The maximum number of independent KVL equations is equal to $b - (n - 1)$, where b is the total number of branches and n is the total number of nodes. These equations can always be written around the $b - (n - 1)$ fundamental loops with respect to a tree T. (These criteria are valid only for networks containing two-terminal elements.)

Generalized KCL The algebraic sum of all branch currents belonging to a cut set is zero.

Independent KCL equation criteria The maximum number of independent KCL equations is equal to $n - 1$ where n is the total number of nodes. These equations can always be written across the fundamental cut sets with respect to a tree T. (These criteria are valid only for networks containing two-terminal elements.)

Ground rule for networks with multiterminal elements If each KVL or KCL equation contains at least one branch variable which did not appear in the preceding equations, then the equations are independent.

Nature of equations of motion

1. Resistive Nonlinear Networks: a system of nonlinear functional equations.
2. Dynamic Nonlinear Networks: a system of nonlinear functional-differential equations. These equations must be recast into the normal form before they are amenable to existing solution techniques. The dependent variables in the normal form are called state variables.