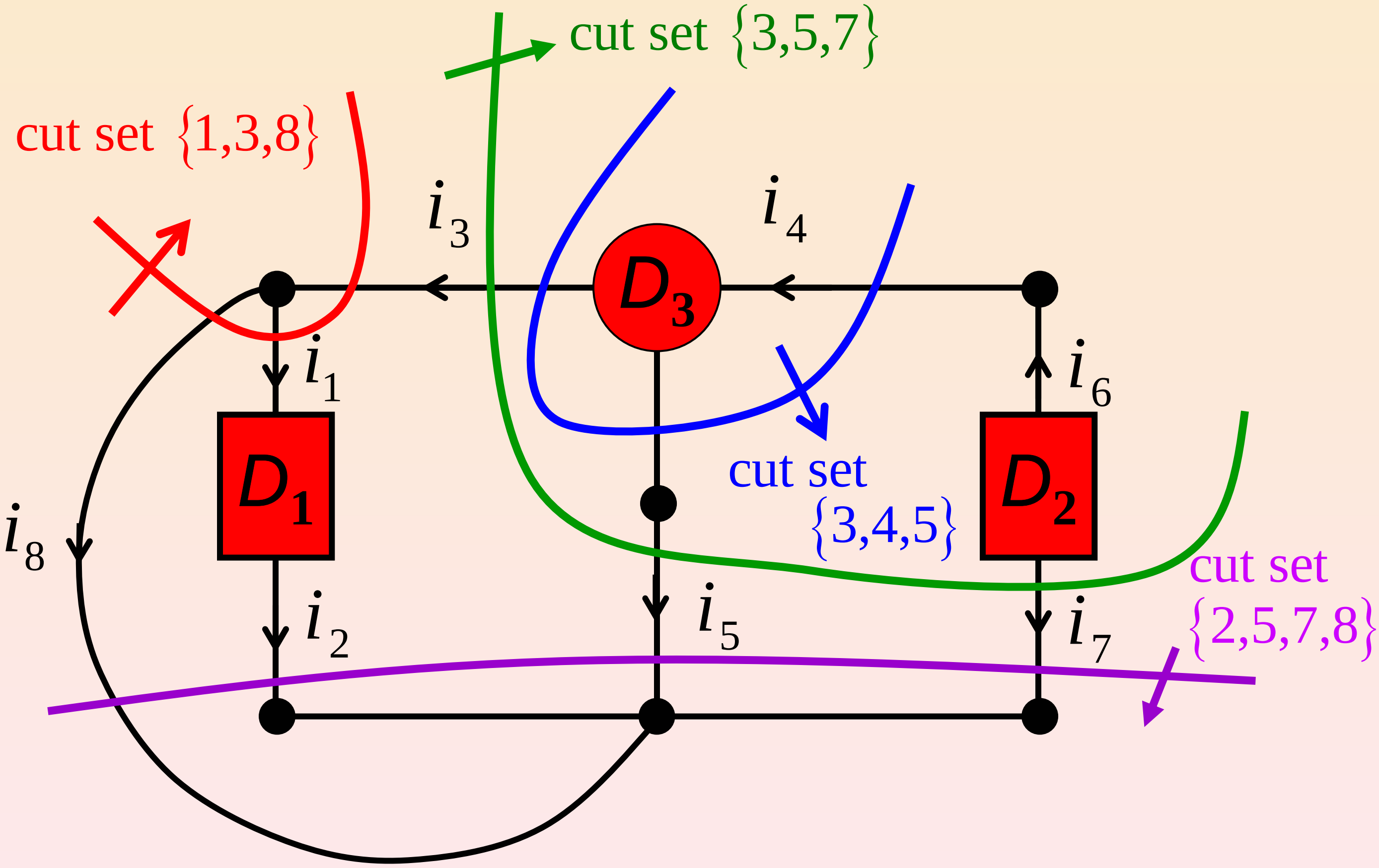
KCL

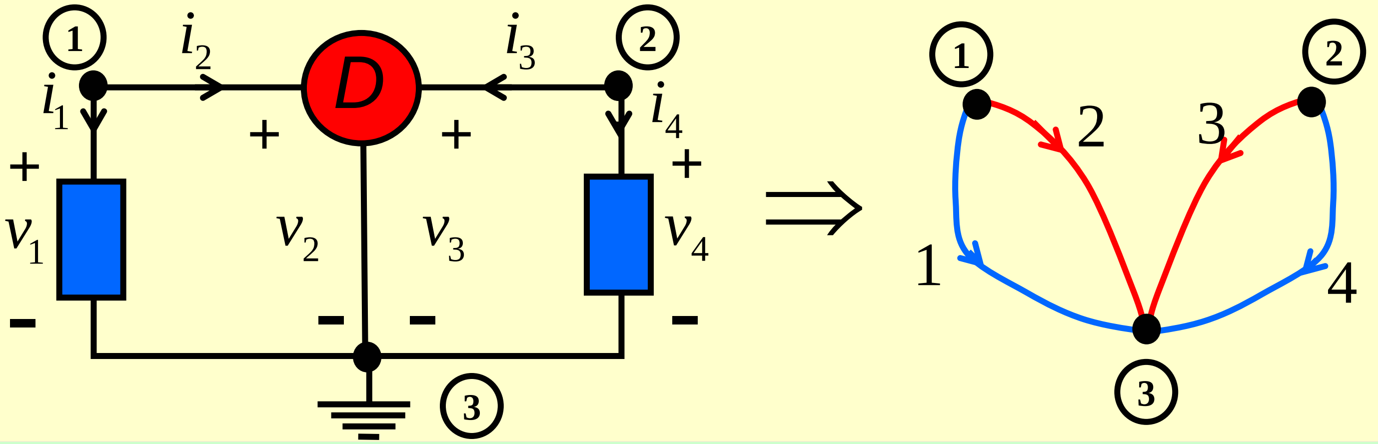
Before writing KCL on a cut set, we assign ***arbitrarily*** a positive reference direction by an arrowhead.



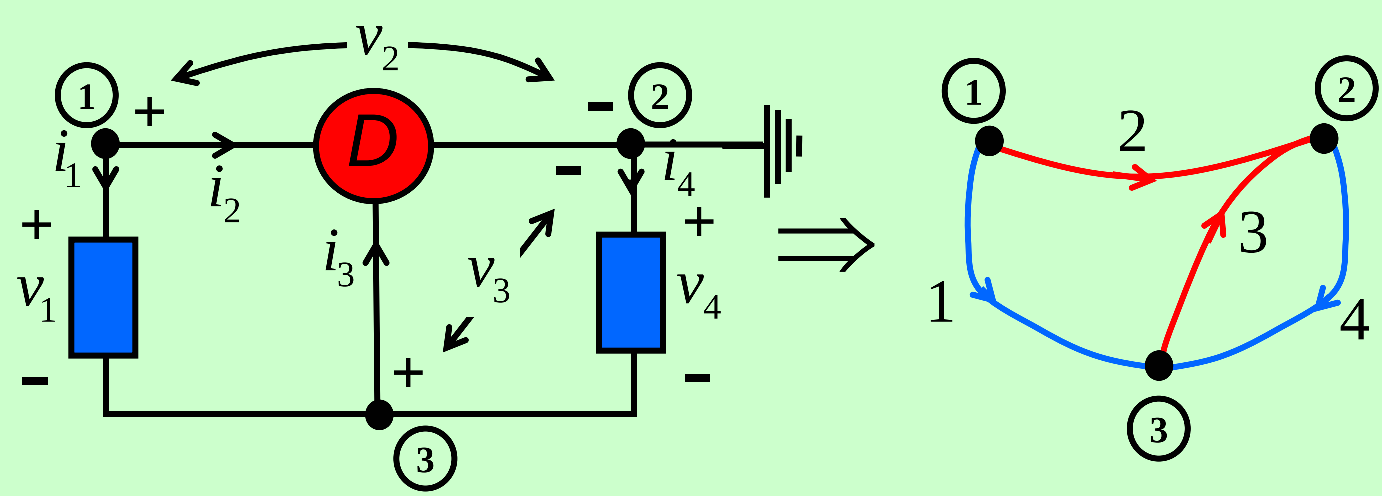
Some Examples of cut sets		KCL cut set equations
$\{3,5,7\}$	$\Rightarrow$	$-i_3 - i_5 - i_7 = 0$
$\{1,3,8\}$	$\Rightarrow$	$i_3 - i_1 - i_8 = 0$
$\{2,5,7,8\}$	$\Rightarrow$	$i_2 + i_5 + i_7 + i_8 = 0$
$\{3,4,5\}$	$\Rightarrow$	$i_3 - i_4 + i_5 = 0$

A Circuit with 3 different digraphs

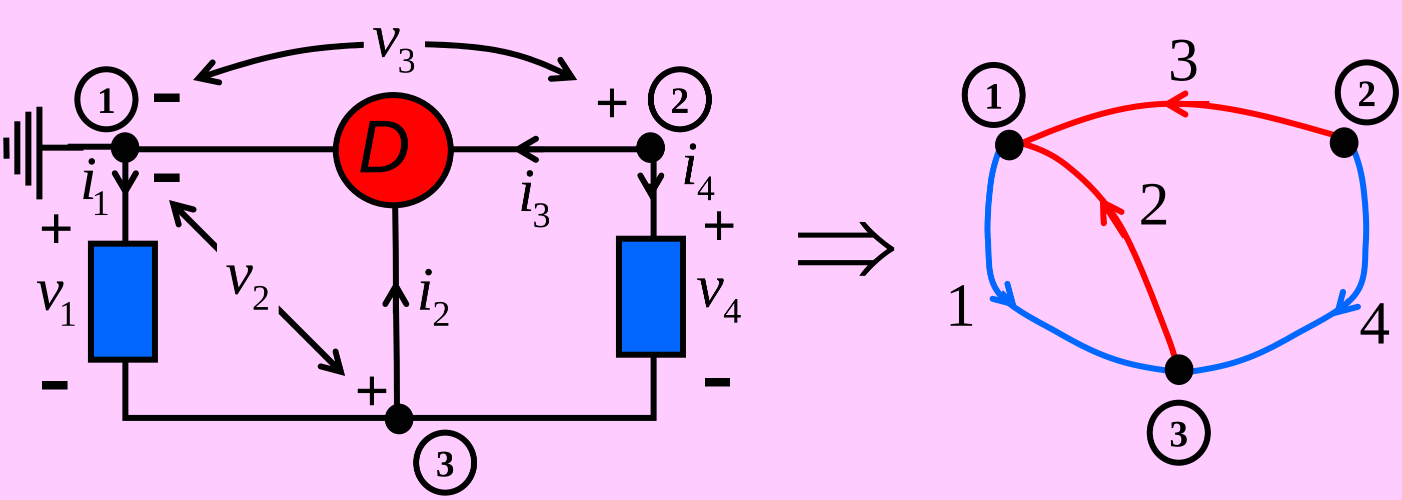
1. Choose 3 as datum for *D*



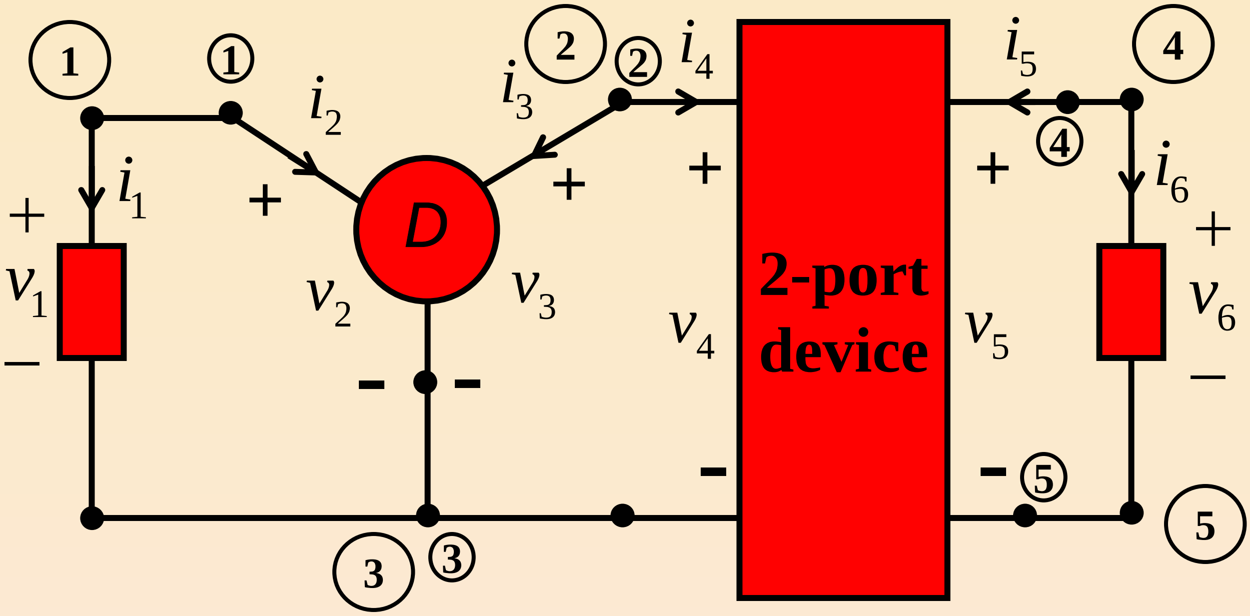
2. Choose 2 as datum for *D*

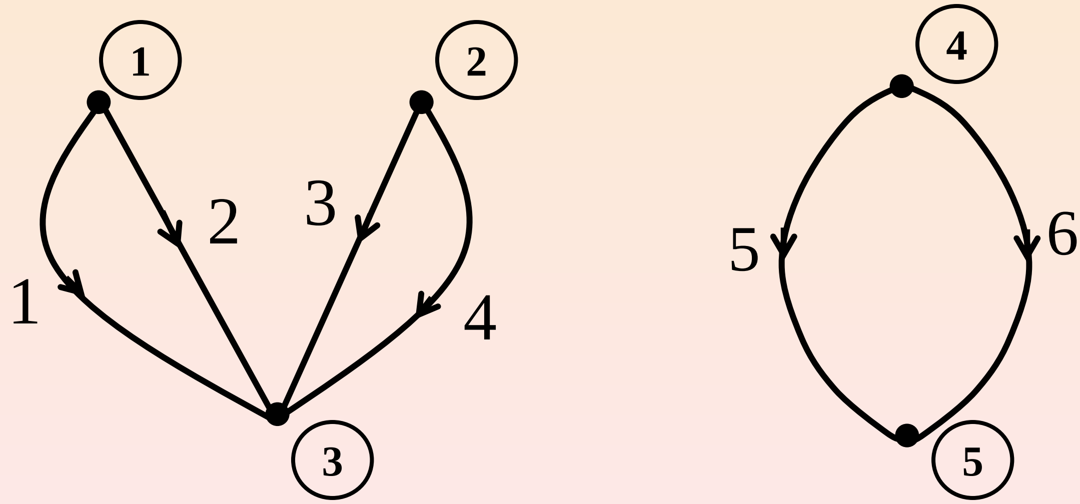


3. Choose 1 as datum for *D*



- Circuits containing *n*-terminal devices can have many distinct digraphs, due to different (arbitrary) choices of the datum terminal for each *n*-terminal device.
- Although the KCL and KVL equations associated with 2 different digraphs of a given circuit are different, they contain the same information because each set of equations can be derived from the other.





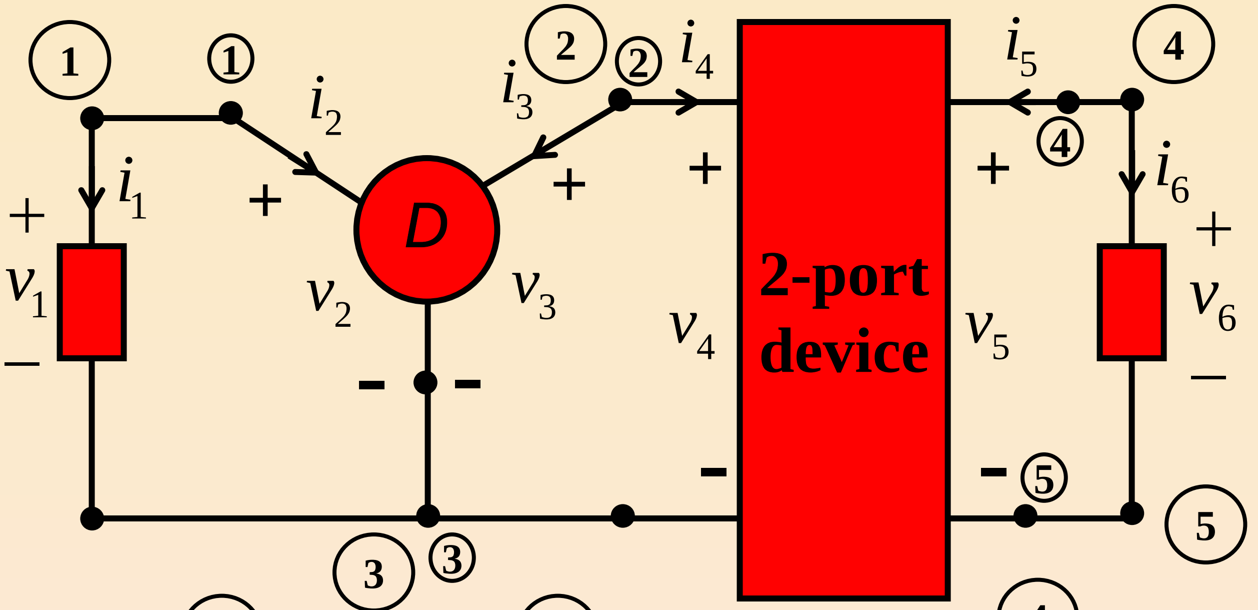
disconnected digraph

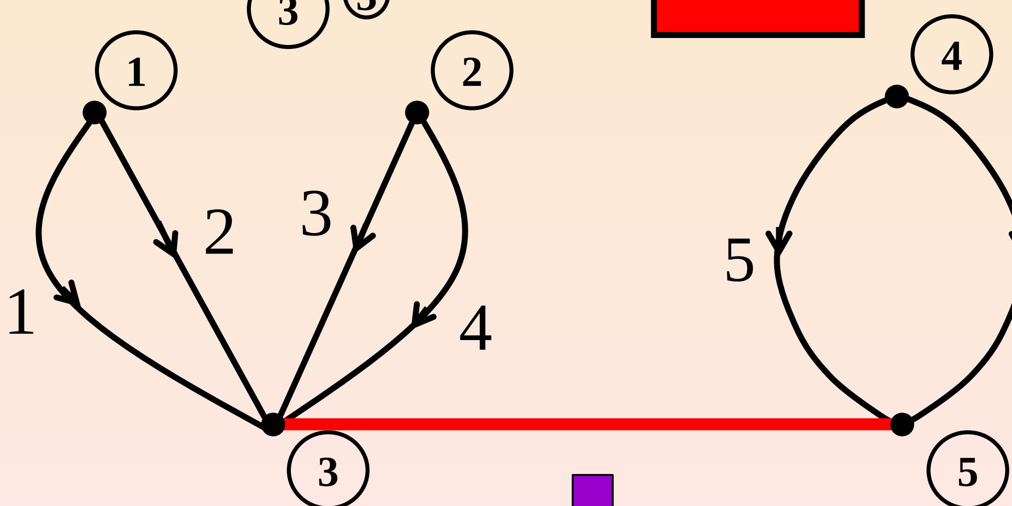
KCL at ② : $i_3 + i_4 = 0$

KCL at ④ : $i_5 + i_6 = 0$

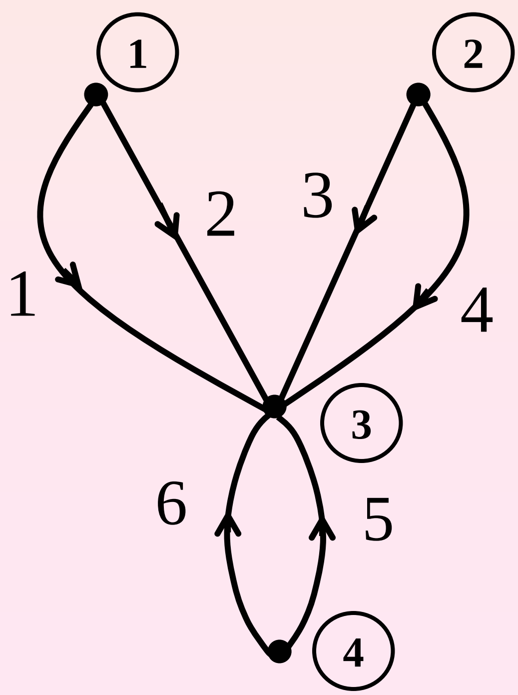
KVL around ②–③–② : $v_4 - v_3 = 0$

KVL around ④–⑤–④ : $v_6 - v_5 = 0$

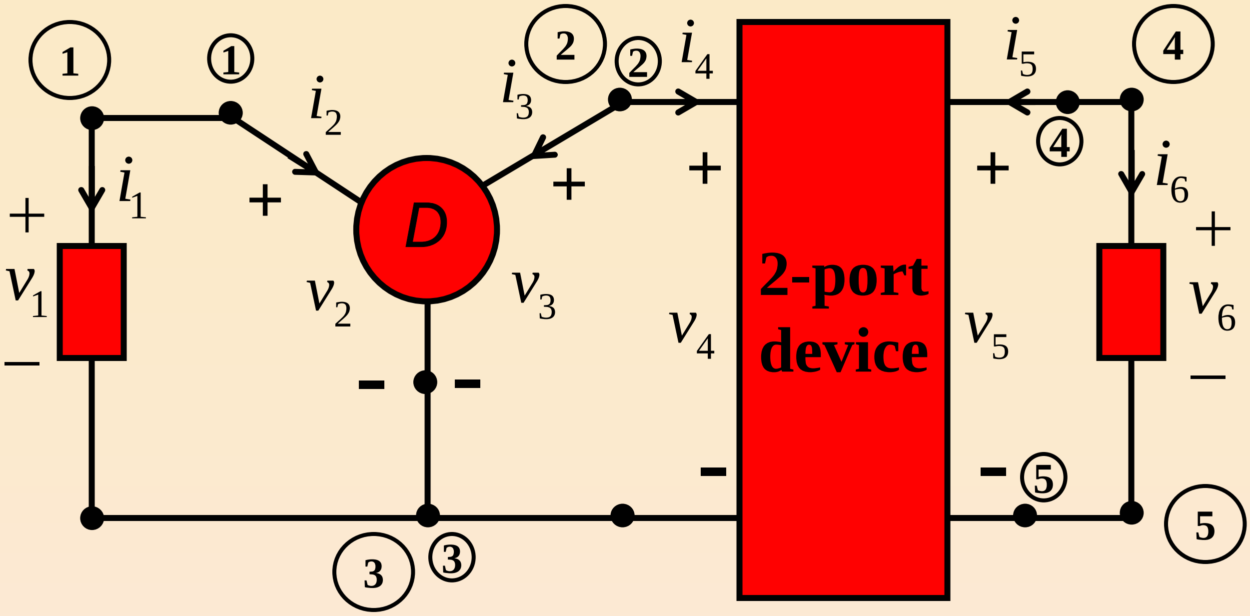


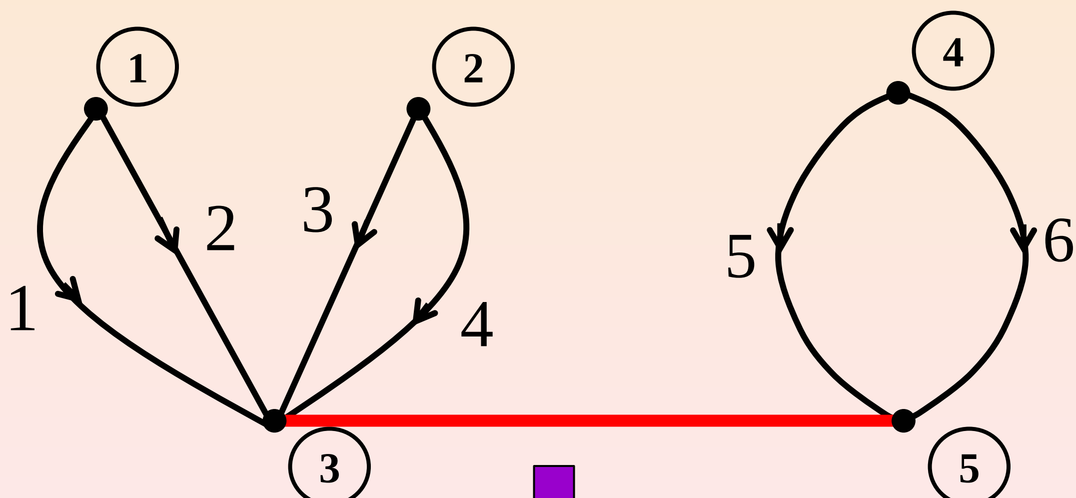


HINGED DIGRAPH

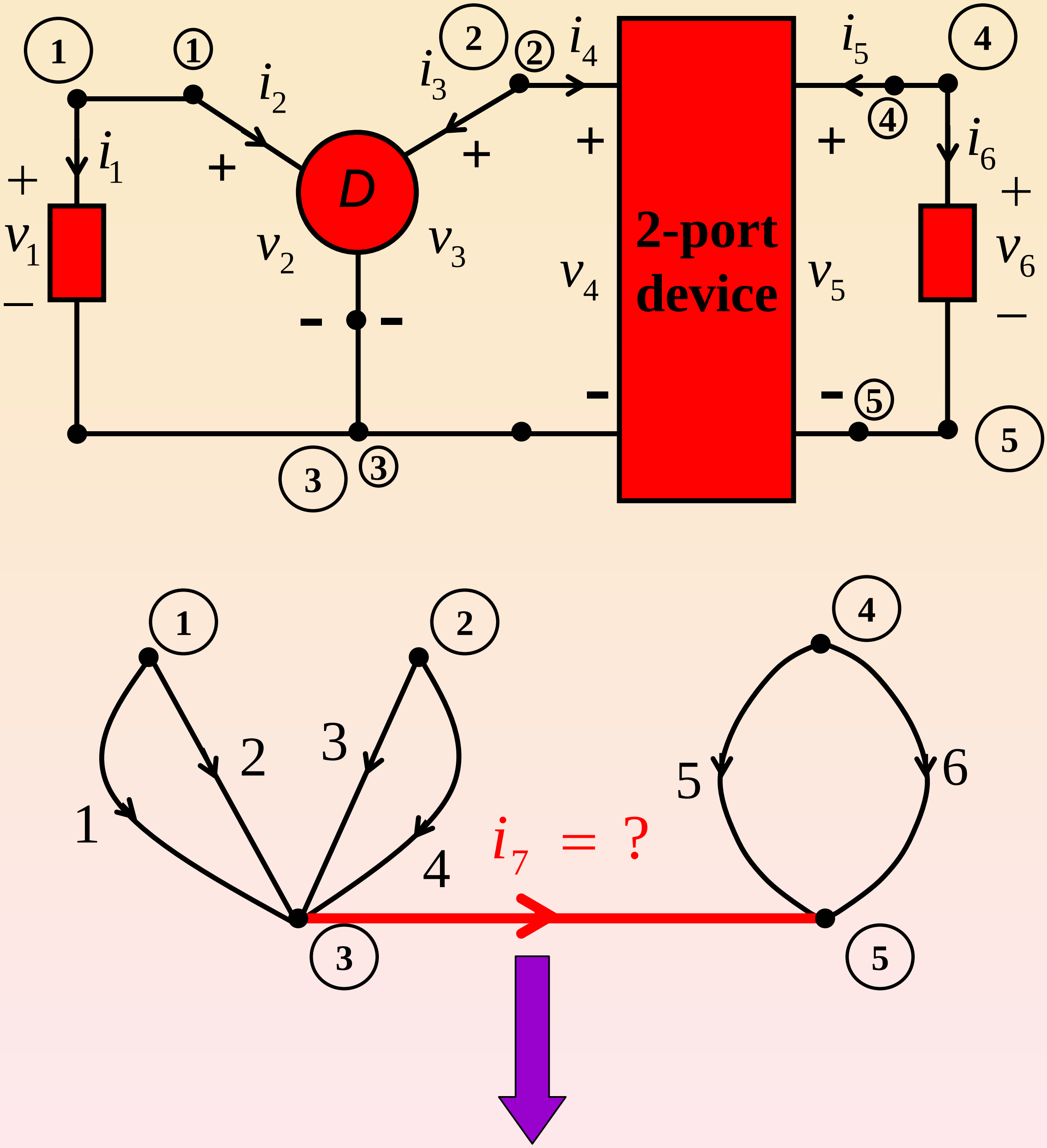


Since nodes 3 and 5 are now the same node, they can be combined into one node, and the redrawn digraph is called a **hinged** graph.

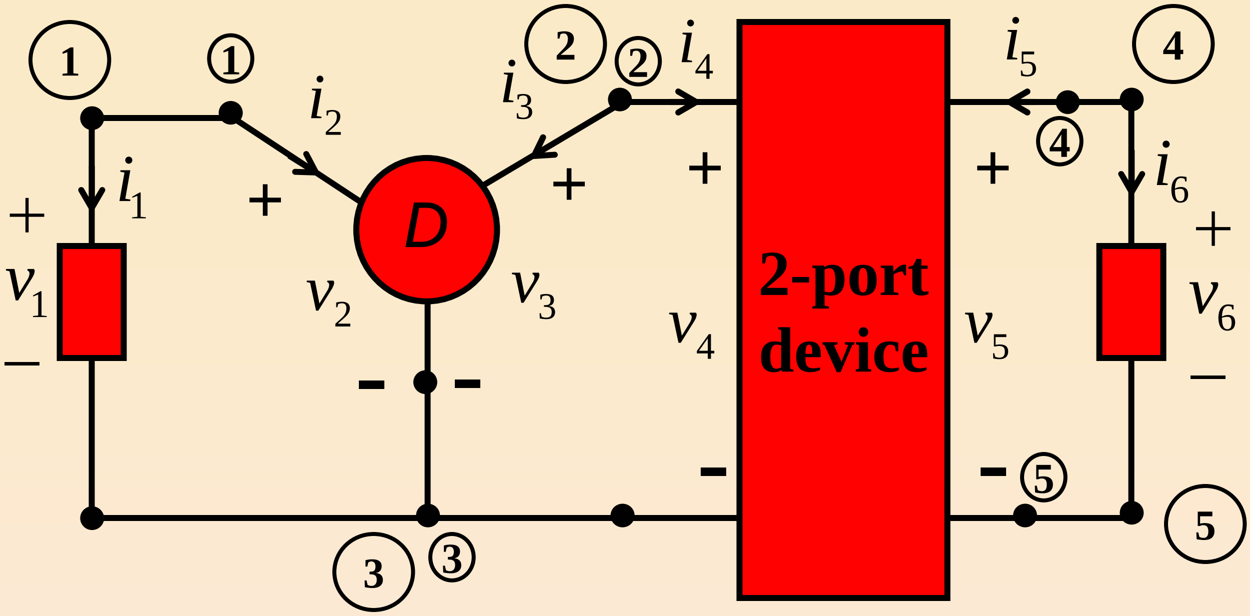


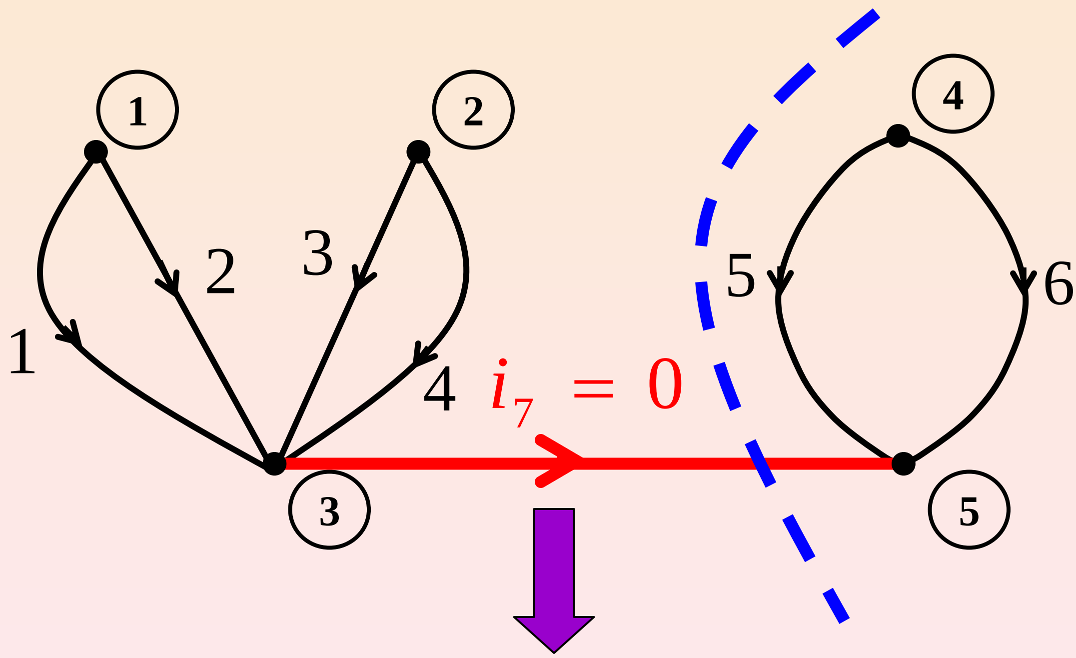


Adding a wire connecting one node from each separate component does **not** change KVL or KCL equations.



Adding a wire connecting one node from each separate component does **not** change KVL or KCL equations.

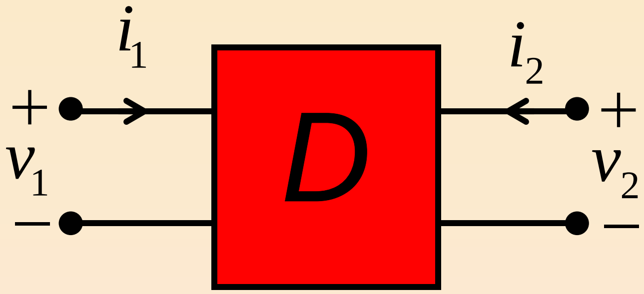




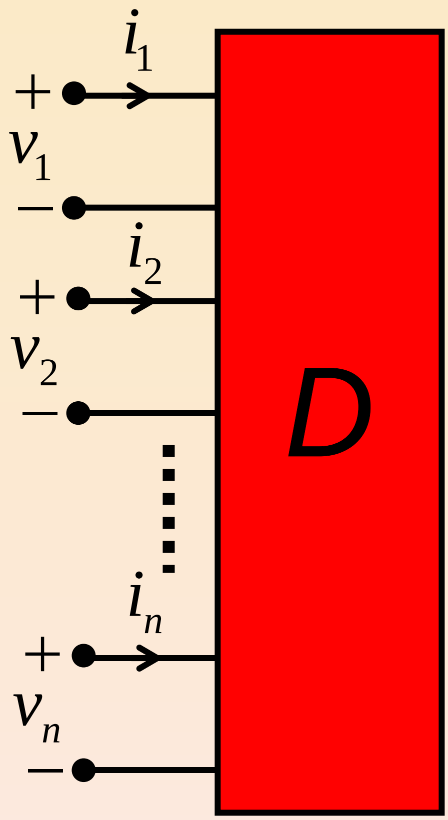
Adding a wire connecting one node from each separate component does **not** change KVL or KCL equations.

$\{7\}$ is a **cut set** $\Rightarrow$ $i_7 = 0$

Associated Reference Convention :

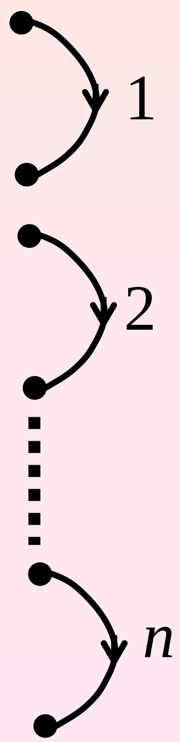


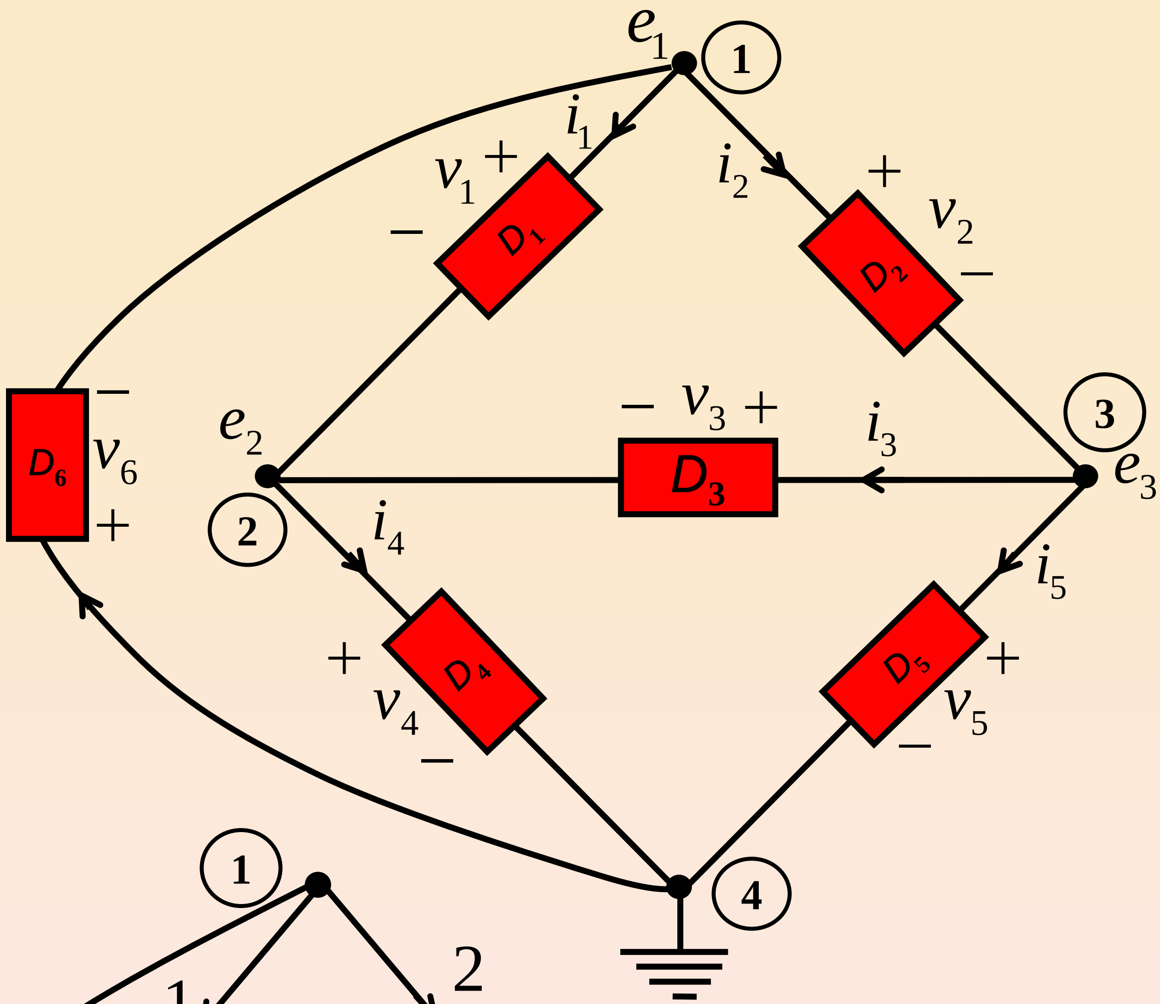
2-port Device

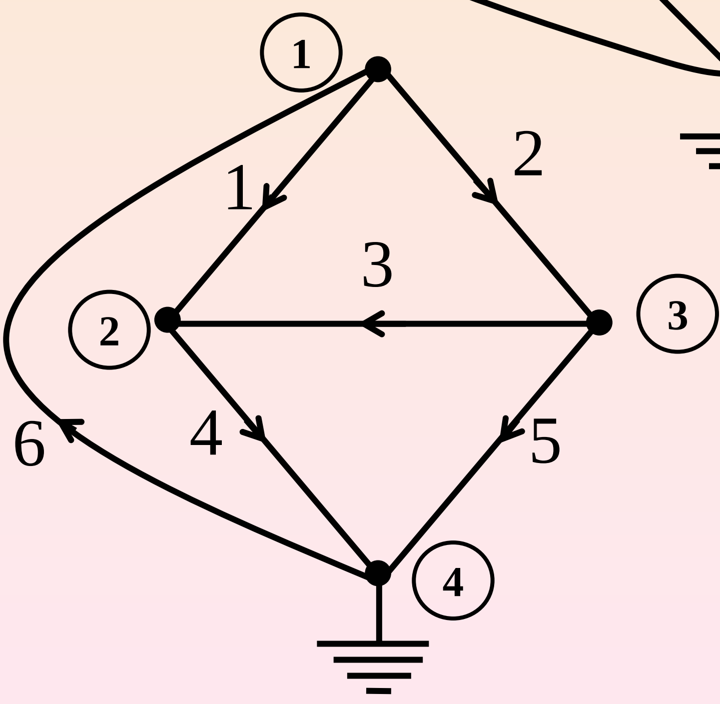


n-port Device

Device Graph



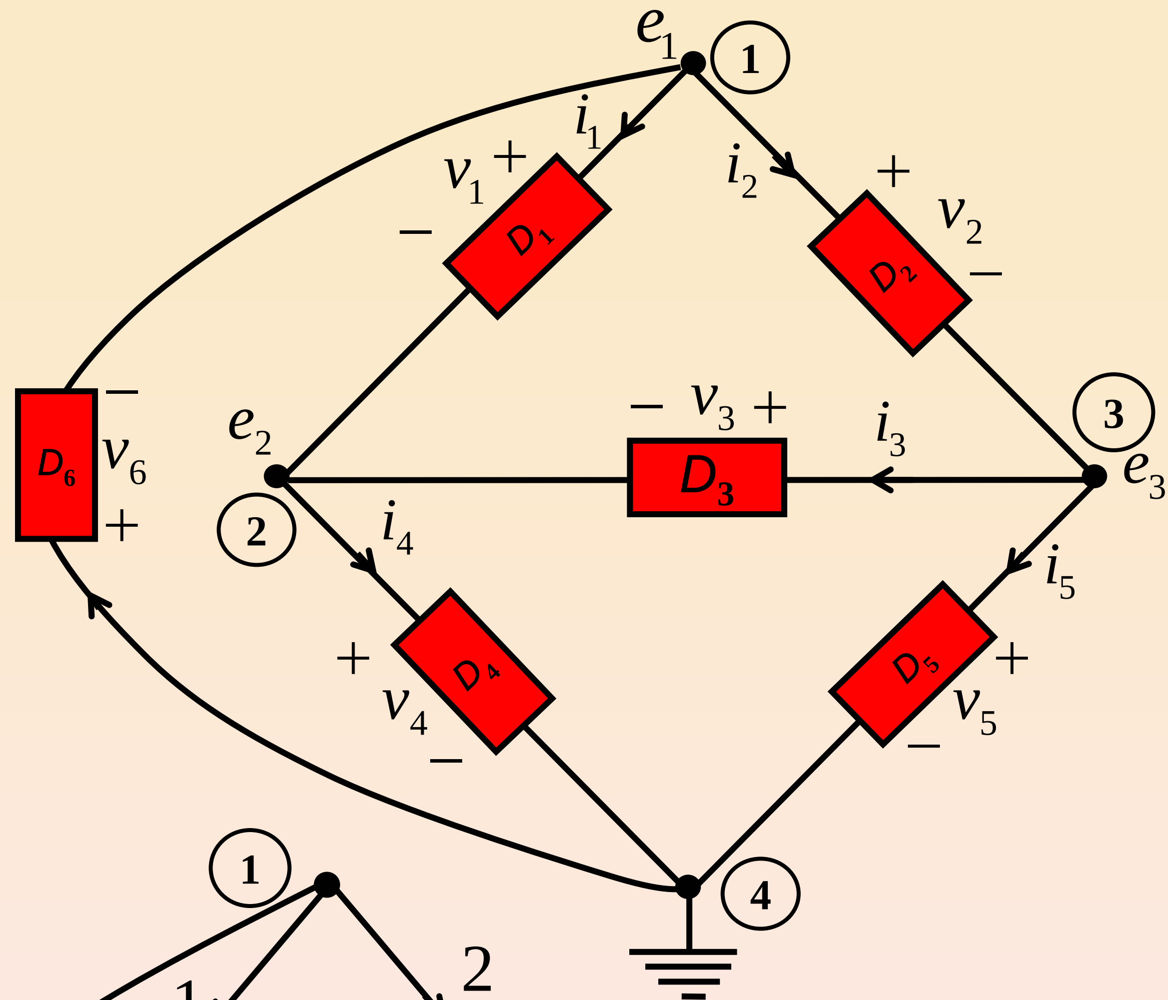


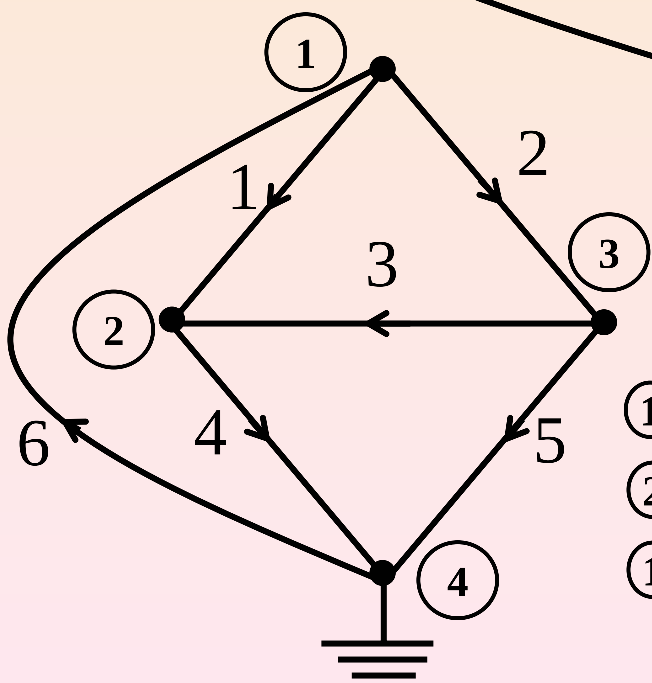


KCL at 1 : $\quad i_1 + i_2 - i_6 = 0$

KVL around 1 − 3 − 4 − 2 − 1 :

$$v_2 + v_5 - v_4 - v_1 = 0$$





KVL around closed node sequence:

1 – 3 – 2 – 1 $: v_2 + v_3 - v_1 = 0$

2 – 3 – 4 – 2 $: -v_3 + v_5 - v_4 = 0$

1 – 3 – 4 – 2 – 1

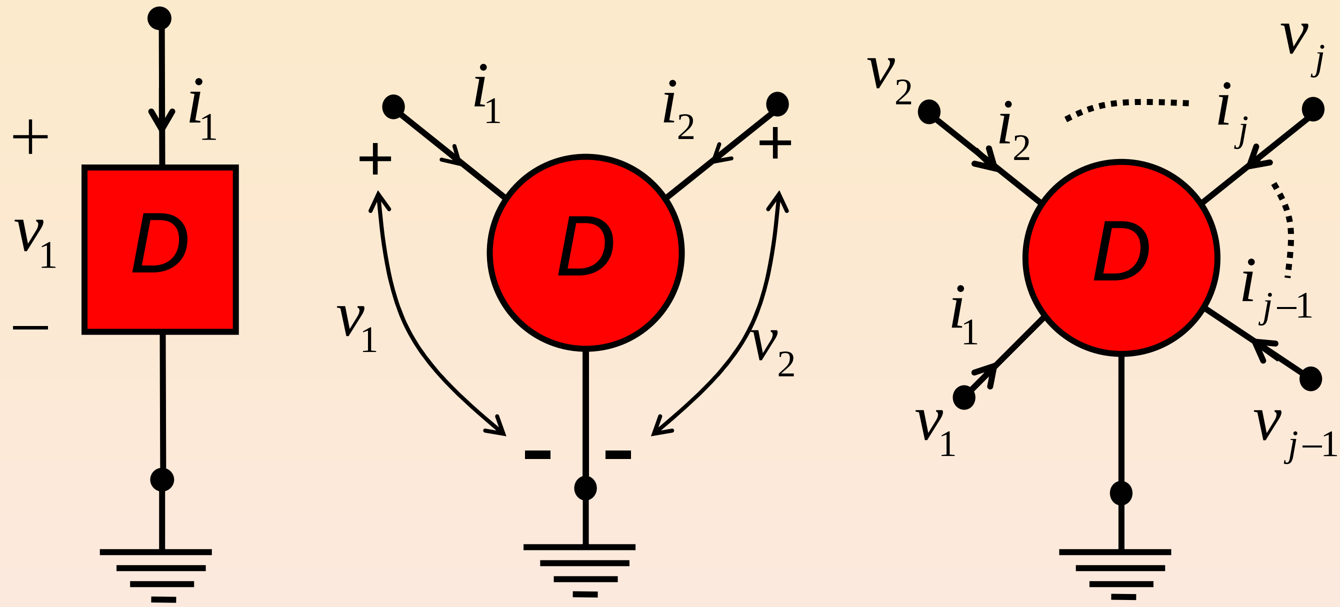
$: v_2 + v_5 - v_4 - v_1 = 0$

These 3 KVL equations are **not** linearly-independent because the 3*rd* equation can be obtained by adding the first 2 equations:

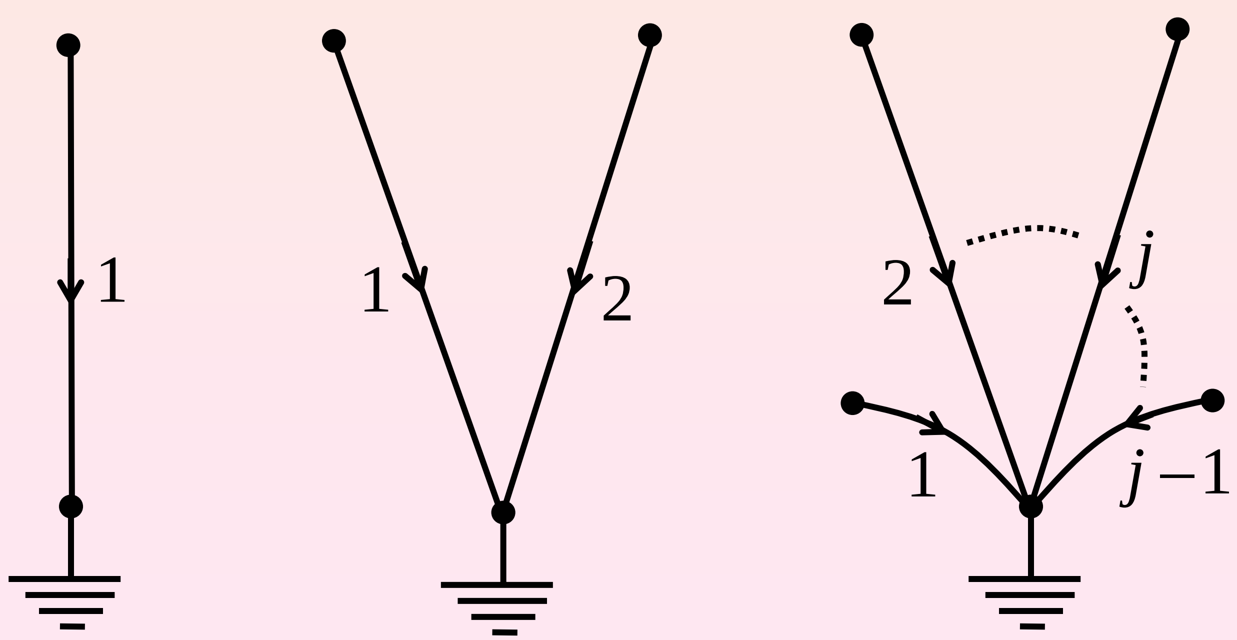
$$(v_2 + v_3 - v_1) + (-v_3 + v_5 - v_4)$$
$$= v_2 - v_1 + v_5 - v_4 = 0$$

Associated Reference Convention :

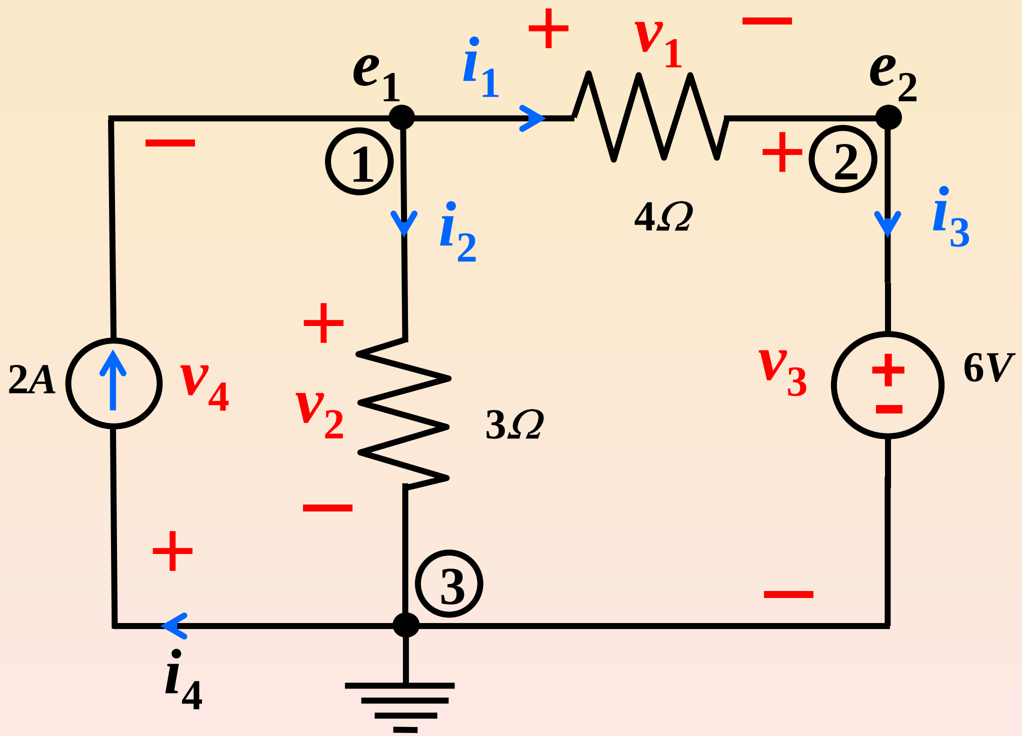
A current direction is chosen
entering each positively-referenced terminal.



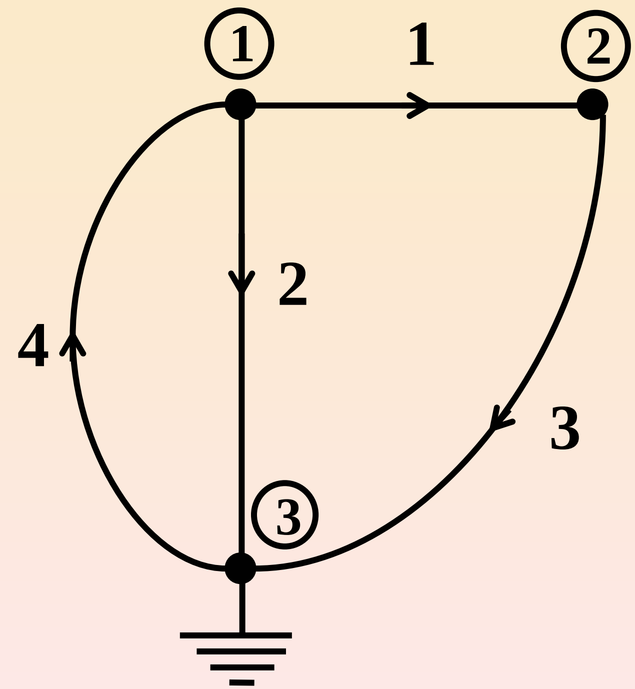
Device Graph : DIGRAPH (Directed Graph)



Circuit N



Digraph G



Reduced Incidence Matrix A

branch number

$$
\begin{array}{c}
\\ \textbf{node number} \\ \\
\end{array}
\quad
\begin{array}{c}
\\ ① \\ ②
\end{array}
\begin{array}{c}
\begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

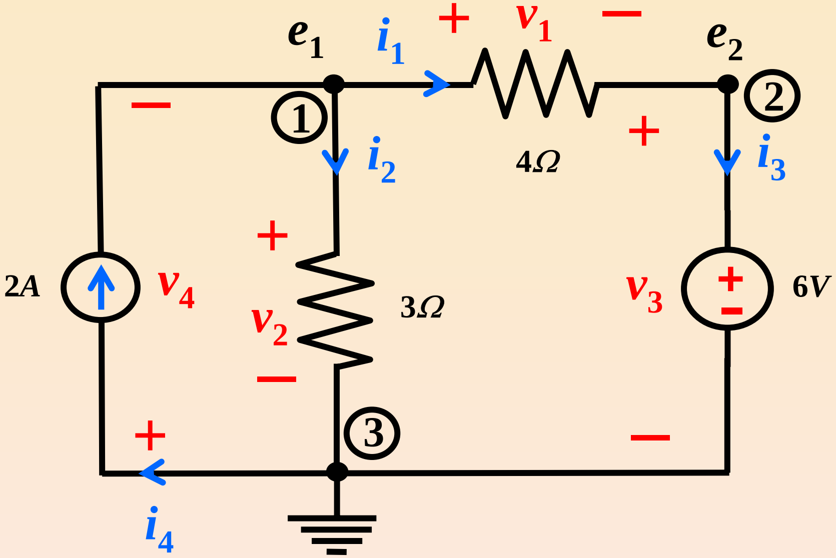
KCL:

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}} \Rightarrow \boxed{\begin{matrix} i_1 + i_2 - i_4 = 0 \\ -i_1 + i_3 = 0 \end{matrix}}$$

KVL:

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\mathbf{A}^{\mathrm{T}}} \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{\mathbf{e}} \Rightarrow \boxed{\begin{matrix} v_1 = e_1 - e_2 \\ v_2 = e_1 \\ v_3 = e_2 \\ v_4 = -e_1 \end{matrix}}$$

Circuit N



Circuit Variables

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Number of nodes: $n = 3$

Number of branches : $b = 4$

Number of circuit variables: $2b+(n-1) = (2\text{x}4)+(3-1) = 10$

Number of Independent KCL Equations : $n-1 = 2$

Number of Independent KVL Equations: $b = 4$

Total number of independent KCL and KVL Equations : $b+(n-1) = 6$

We need "b" **additional** independent equations in order to obtain a system of $2b+(n-1)$ **independent equations** in $2b+(n-1)$ circuit variables.

The additional equations must come from the **constitutive relation** which relate the terminal voltages and currents of the circuit elements.

Let us rearrange all 10 independent equations as follow:

$$\left[\begin{array}{cc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ \hline -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} e_1 \\ e_2 \\ \hline v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 6 \\ 2 \end{array}\right]$$

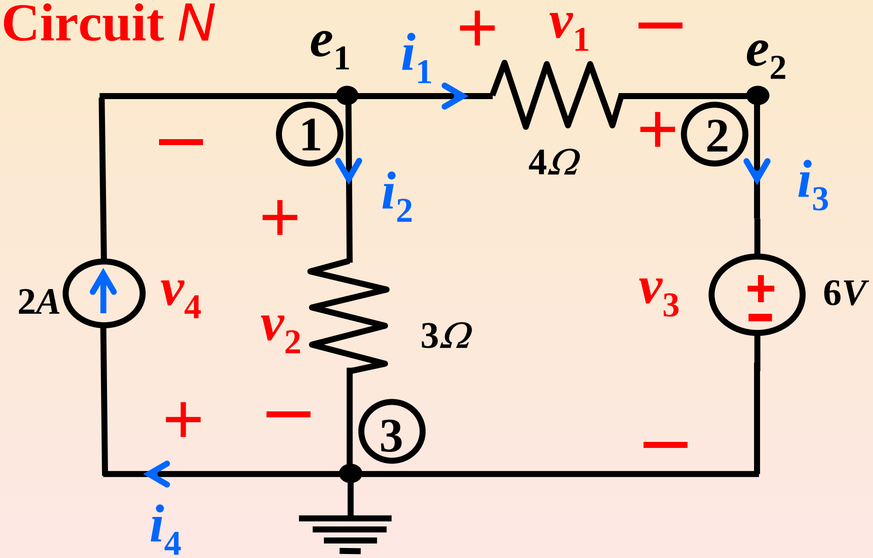
Let us rearrange all 10 independent equations as follow:

$$\underbrace{\left[\begin{array}{c:c:c} \mathbf{0} & \mathbf{0} & \mathbf{A} \\ \hdashline -\mathbf{A}^{\mathbf{T}} & \mathbf{1} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{H}_{\mathbf{v}} & \mathbf{H}_{\mathbf{i}} \end{array}\right]}_{\mathbf{T}} \underbrace{\left[\begin{array}{c} e_1 \\ e_2 \\ \hdashline v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hdashline i_1 \\ i_2 \\ i_3 \\ i_4 \end{array}\right]}_{\mathbf{w}} = \underbrace{\left[\begin{array}{c} 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 6 \\ 2 \end{array}\right]}_{\mathbf{u}}$$

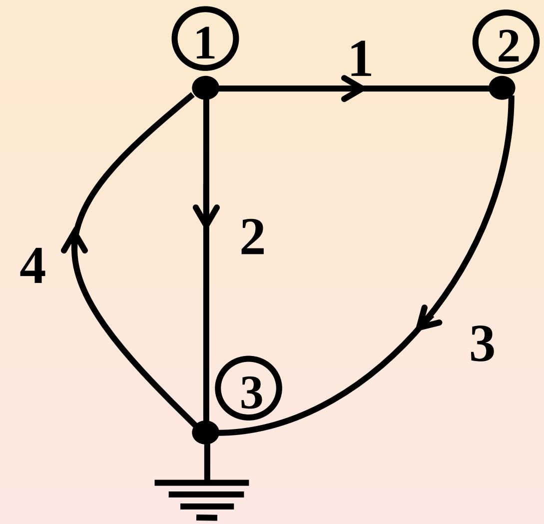
How Many Circuit Variables?

Answer: **Total Number of Circuit Variables = 2*b* + *n* - 1**

Circuit *N*



Digraph *G*



Number of Nodes: $n = 3$

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

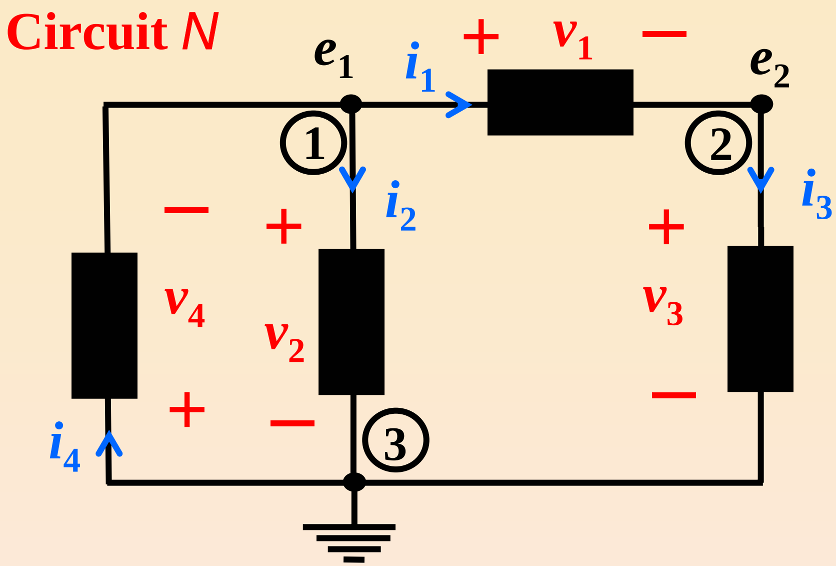
node-to-datum voltages: n-1 = 2

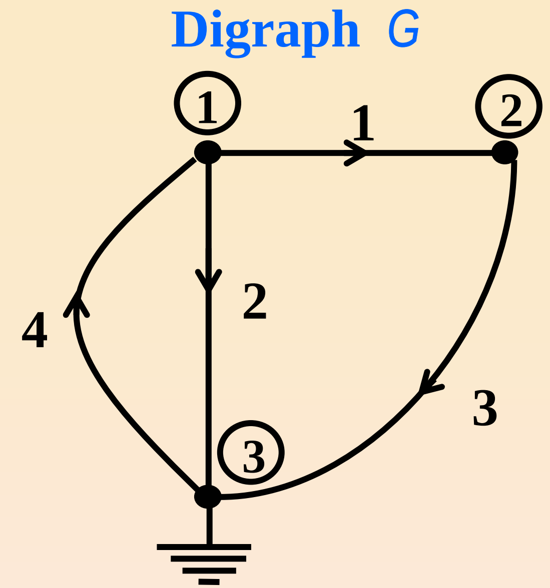
Number of branches: $b = 4$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \qquad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

branch voltages: $b = 4$ branch currents: $b = 4$

$$\therefore \text{ There are } \underbrace{(n-1)+b+b}_{2b+n-1} = 10 \text{ circuit variables; namely, } \{e_1, e_2, v_1, v_2, v_3, v_4, i_1, i_2, i_3, i_4\}.$$





There are infinitely many sets of branch voltages (v_1, v_2, v_3, v_4) which satisfy KVL for G.

2 Examples satisfying KVL:

KVL solution 1: $v_1 = -3V, v_2 = 2V, v_3 = 5V, v_4 = -2V$

KVL solution 2: $\hat{v}_1 = 2V, \hat{v}_2 = 4V, \hat{v}_3 = 2V, \hat{v}_4 = -4V$

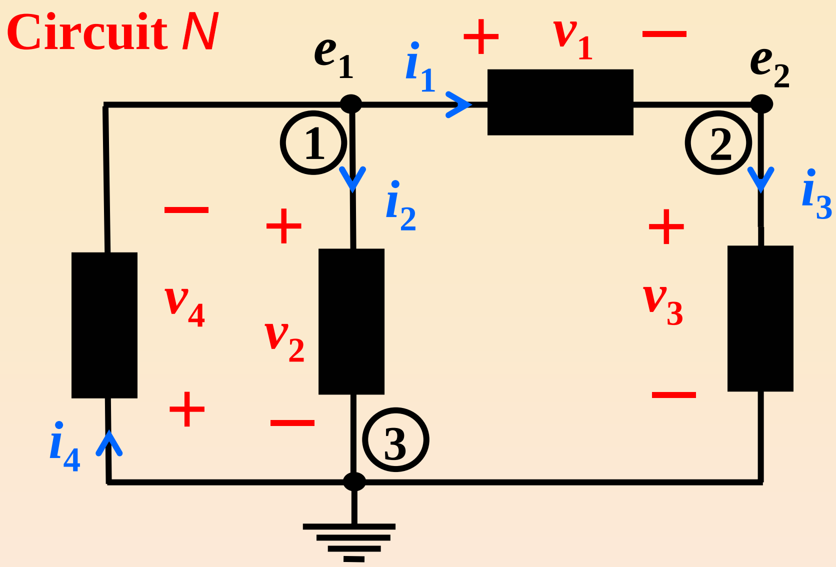
There are infinitely many sets of branch currents (i_1, i_2, i_3, i_4) which satisfy KCL for G.

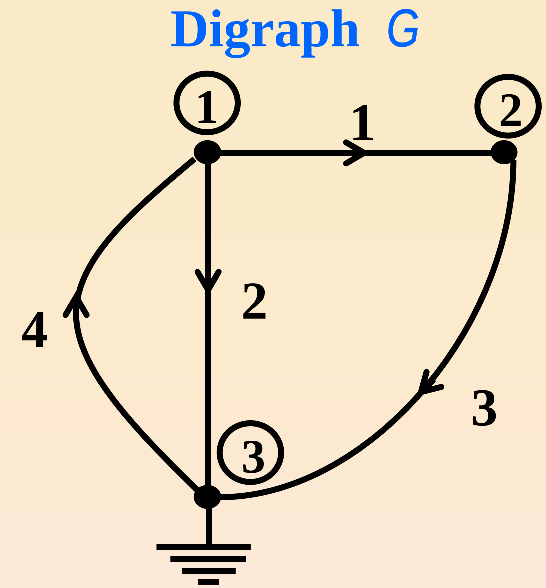
2 Examples satisfying KCL:

KCL solution 1: $i_1 = 3A, i_2 = 2A, i_3 = 3A, i_4 = 5A$

KCL solution 2: $\hat{i}_1 = 6A, \hat{i}_2 = -4A, \hat{i}_3 = 6A, \hat{i}_4 = 2A$

NOTE: So far we have not specified what circuit elements are used in this circuit. This explains why the voltage and current solutions are not unique.





Example 1

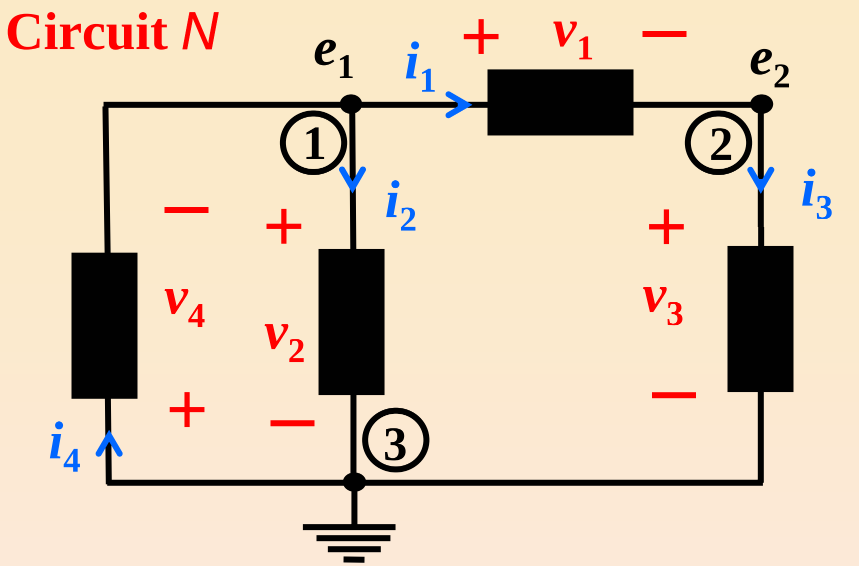
KVL solution: Choose $v_1 = -3V, v_2 = 2V, v_3 = 5V, v_4 = -2V$

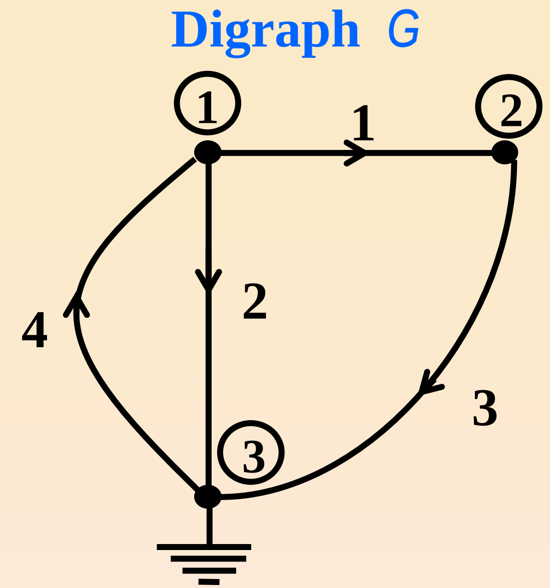
KCL solution: Choose $i_1 = 3A, i_2 = 2A, i_3 = 3A, i_4 = 5A$

$$\sum_{j=1}^{4} v_j \, i_j = (-3)(3) + (2)(2) + (5)(3) + (-2)(5)$$

$$= -9 + 4 + 15 - 10$$

$$= 0$$





Example 2

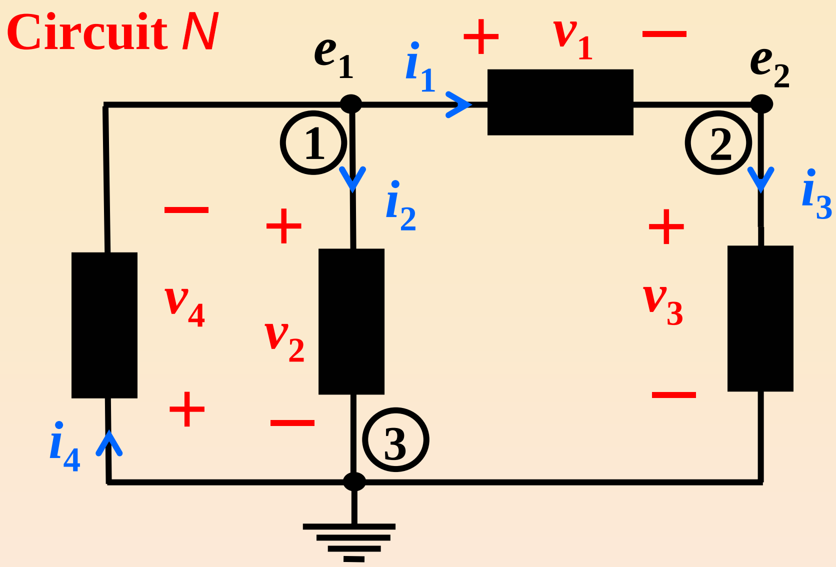
KVL solution: Choose $\hat{v}_1 = 2V, \hat{v}_2 = 4V, \hat{v}_3 = 2V, \hat{v}_4 = -4V$

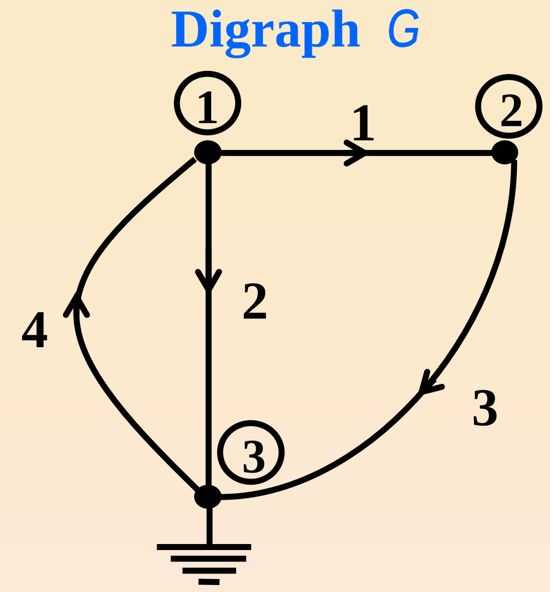
KCL solution: Choose $i_1 = 3\,A, i_2 = 2\,A, i_3 = 3A, i_4 = 5\,A$

$$\sum_{j=1}^{4} \hat{v}_j\, i_j = (2)(3) + (4)(2) + (2)(3) + (-4)(5)$$

$$= 6 + 8 + 6 - 20$$

$$= 0$$



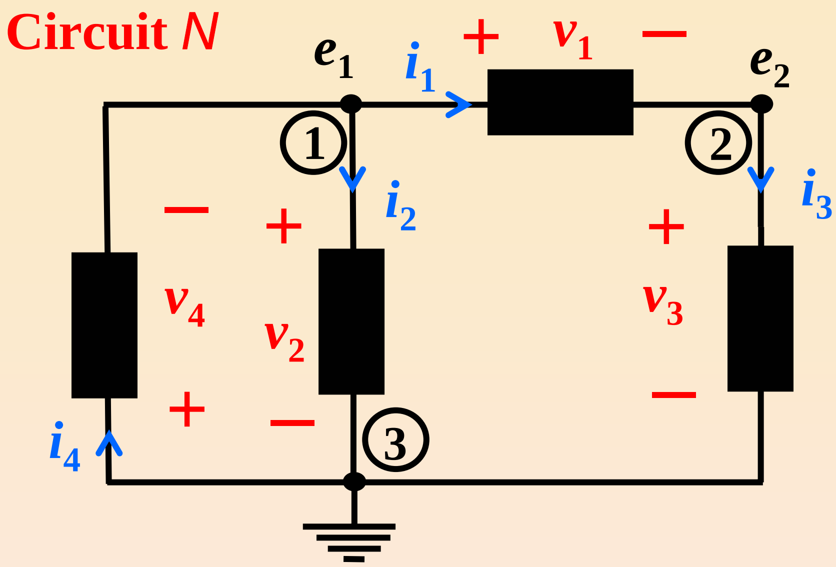


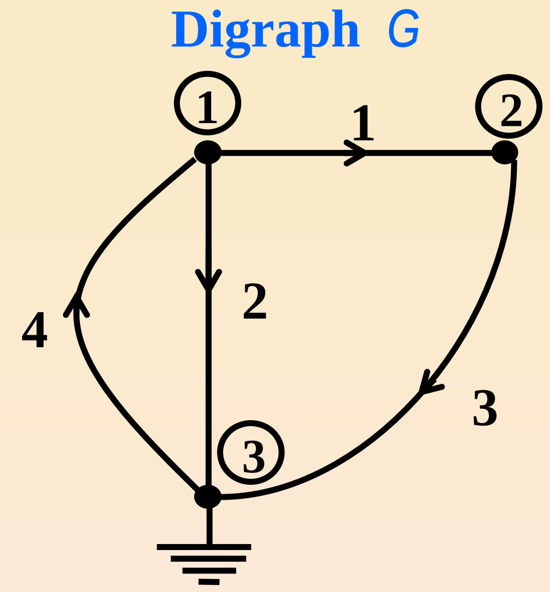
Example 3

KVL solution: Choose $v_1 = -3V, v_2 = 2V, v_3 = 5V, v_4 = -2V$

KCL solution: Choose $\hat{i}_1 = 6\,A, \hat{i}_2 = -4\,A, \hat{i}_3 = 6A, \hat{i}_4 = 2\,A$

$$\sum_{j=1}^{4} v_j\,\hat{i}_j = (-3)(6) + (2)(-4) + (5)(6) + (-2)(2)$$
$$= -18 - 8 + 30 - 4$$
$$= 0$$





Example 4

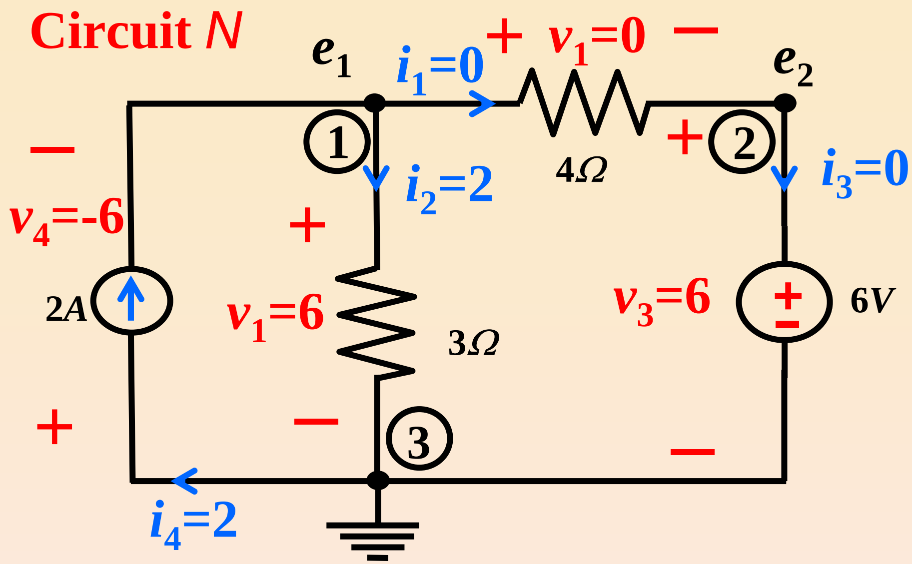
KVL solution: Choose $\hat{v}_1 = 2V, \hat{v}_2 = 4V, \hat{v}_3 = 2V, \hat{v}_4 = -4V$

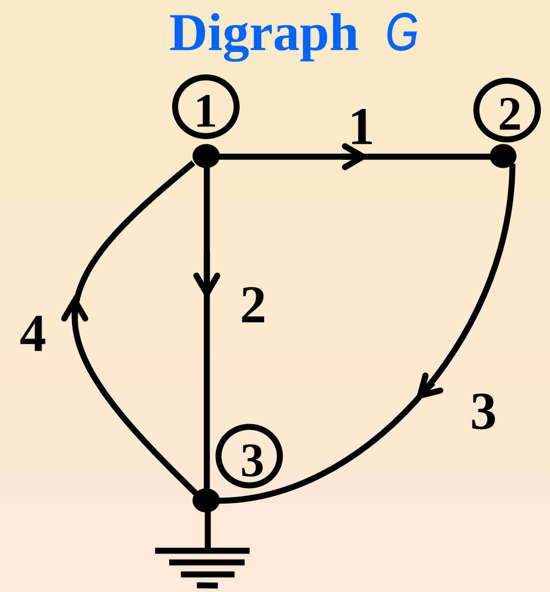
KCL solution: Choose $\hat{i}_1 = 6\,A, \hat{i}_2 = -4\,A, \hat{i}_3 = 6A, \hat{i}_4 = 2\,A$

$$\sum_{j=1}^{4} \hat{v}_j\, \hat{i}_j = (2)(6) + (4)(-4) + (2)(6) + (-4)(2)$$

$$= 12 - 16 + 12 - 8$$

$$= 0$$





Solution :

$$e_1 = 6V, e_2 = 6V$$

$$v_1 = 0V, v_2 = 6V, v_3 = 6V, v_4 = -6V$$

$$i_1 = 0\,A, i_2 = 2\,A, i_3 = 0\,A, i_4 = 2\,A$$

Verifying the solution satisfying Tellegen's Theorem:

$$\begin{aligned}\sum_{j=1}^{4} v_j\, i_j &= (v_1\, i_1) + (v_2\, i_2) + (v_3\, i_3) + (v_4\, i_4) \\ &= (0)(0) + (6)(2) + (6)(0) + (-6)(2) \\ &= 0 + 12 + 0 - 12 \\ &= 0\end{aligned}$$

How to write An Independent System of KCL and KVL Equations

Let N be any connected circuit and let the **digraph** G associated with N contain "*n*" nodes and "*b*" branches. Choose an arbitrary datum node and define the associated **node-to-datum voltage** vector $\mathbf{e}$, the **branch voltage vector** $\mathbf{v}$, and the **branch current vector** $\mathbf{i}$. Then we have the following system of **independent** KCL and KVL equations.

(n-1) Independent KCL Equations :

$$\mathbf{A}\,\mathbf{i} = \mathbf{0}$$

b Independent KVL Equations :

$$\mathbf{v} = \mathbf{A}^T\mathbf{e}$$

Element Constitutive Relations

Element 1: Resistor

Described by Ohm's Law : $v_1 = 4\, i_1$

Element 2: Resistor

Described by Ohm's Law : $v_2 = 3\, i_2$

Element 3: Voltage source

Described by : $v_3 = 6$

Element 4: Current source

Described by : $i_4 = 2$

Rearranging these equations so that circuit variables appear on the left-hand side, we obtain

Observe we have obtained 4 additional independent equations.

Element Equations

$$\boxed{\begin{aligned} v_1 - 4\, i_1 &= 0 \\ v_2 - 3\, i_2 &= 0 \\ v_3 &= 6 \\ i_4 &= 2 \end{aligned}}$$

Equations obtained from the element constitutive relations are guaranteed to be **independent** because different elements involved different circuit variables.

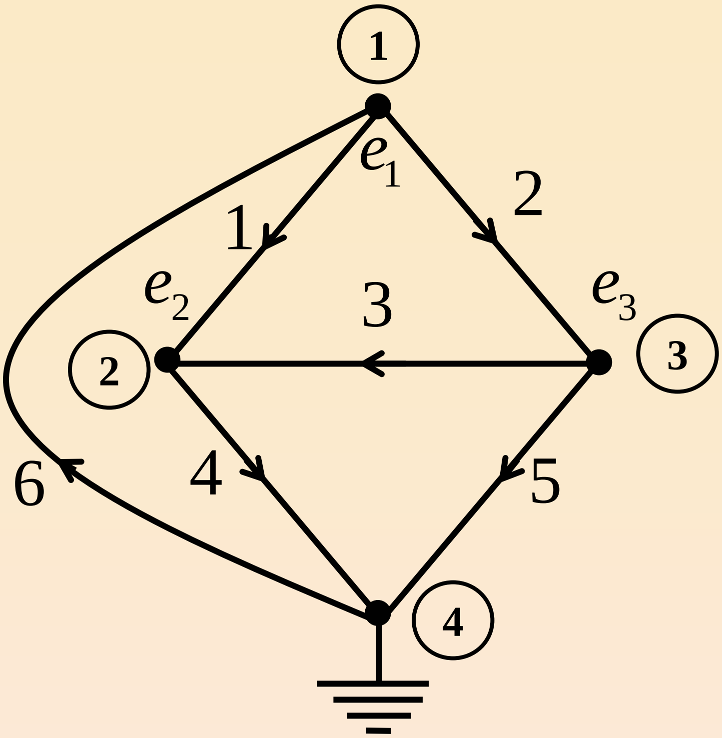
We can always recast **any** system of **linear** constitutive equations into the following standard matrix form

$$\underbrace{\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\mathbf{H}_v \qquad\qquad\qquad \mathbf{H}_i} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hline i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6 \\ 2 \end{bmatrix}}_{\mathbf{u}}$$

independent source vector

$$\Downarrow$$

$$\boxed{\mathbf{H}_v \, \mathbf{v} + \mathbf{H}_i \, \mathbf{i} = \mathbf{u}}$$



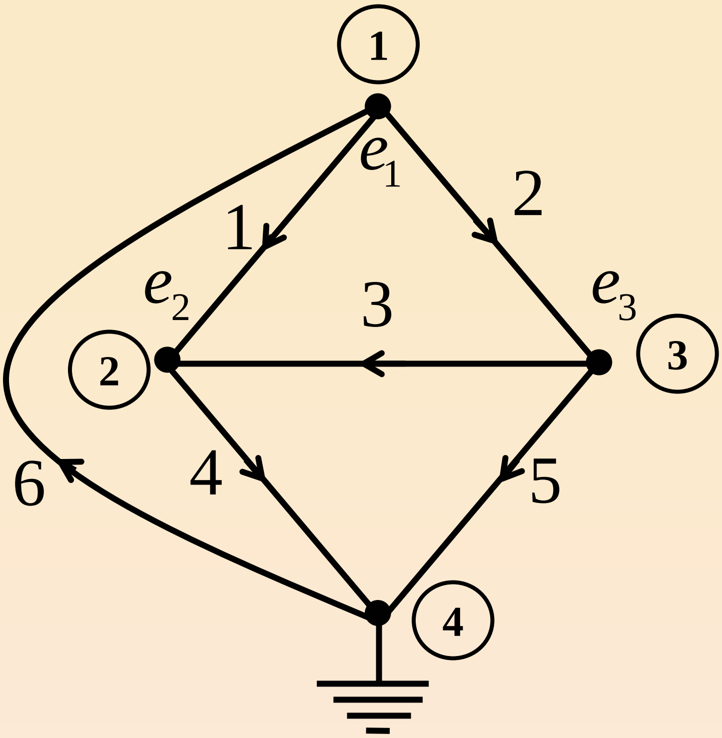
KCL Equations:

$$\text{①}\quad i_1 + i_2 - i_6 = 0$$

$$\text{②}\quad -i_1 - i_3 + i_4 = 0$$

$$\text{③}\quad -i_2 + i_3 + i_5 = 0$$

$$\mathbf{A}\,\mathbf{i} = \mathbf{0} \quad \Rightarrow \quad \begin{array}{c|c} & \text{Branch no.} \\ \text{node no.} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} ① \\ ② \\ ③ \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



KCL Equations:

$$\text{①}\quad i_1 + i_2 - i_6 = 0$$
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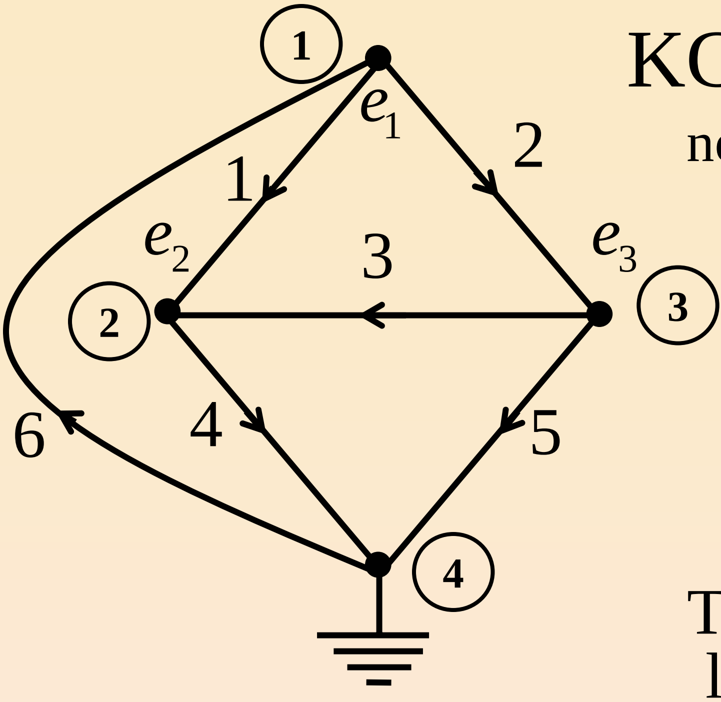
$$\text{③}\quad -i_2 + i_3 + i_5 = 0$$

$$\mathbf{A}\,\mathbf{i} = \mathbf{0} \quad \Rightarrow \quad \begin{array}{c} \text{node no.} \\ 1 \\ 2 \\ 3 \end{array} \underbrace{\overset{\text{Branch no.} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix}}}_{\mathbf{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

A is called the **reduced Incidence Matrix** of the diagraph G relative to datum node 4.



KCL Node Equations:

node No.

(1) $i_1 + i_2 - i_6 = 0$

(2) $-i_1 - i_3 + i_4 = 0$

(3) $-i_2 + i_3 + i_5 = 0$

(4) $-i_4 - i_5 + i_6 = 0$

These 4 equations are linearly-dependent.

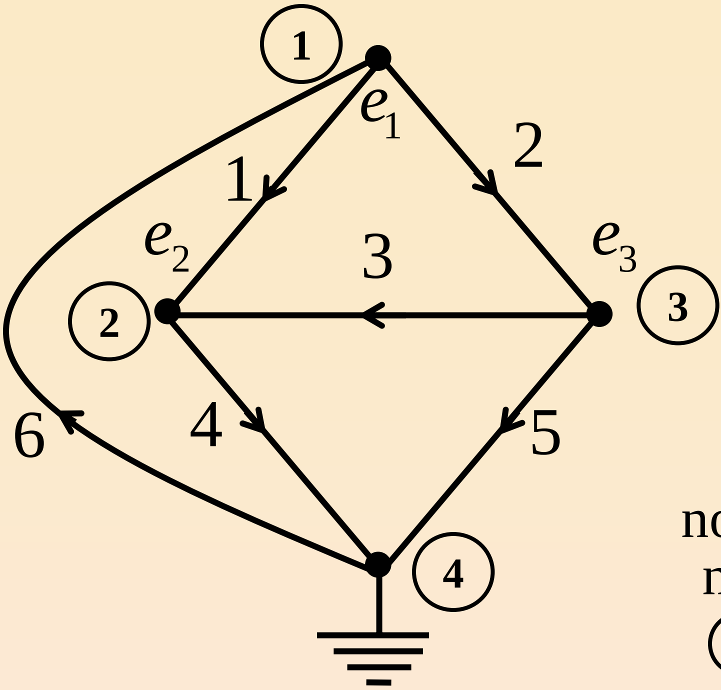
Matrix Formulation:

$$\begin{array}{cc} \text{node no.} & \text{Branch no.} \\ & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}}_{\mathbf{A}_a} \end{array} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

INCIDENCE MATRIX $\longrightarrow \mathbf{A}_a$

$$\mathbf{A}_a \mathbf{i} = \mathbf{0}$$

$$a_{jk} = \begin{cases} 1 & \text{if branch } k \text{ leaves node } (j) \\ -1 & \text{if branch } k \text{ enters node } (j) \\ 0 & \text{if branch } k \text{ is not connected to node } (j) \end{cases}$$



KCL Equations:

$$\text{①}\quad i_1 + i_2 - i_6 = 0$$
$$\text{②}\quad -i_1 - i_3 + i_4 = 0$$
$$\text{③}\quad -i_2 + i_3 + i_5 = 0$$

$$\mathbf{A}\,\mathbf{i} = \mathbf{0} \quad \Rightarrow \quad \begin{array}{c|c} \text{node no.} & \text{Branch no. } \;1\quad 2\quad 3\quad 4\quad 5\quad 6 \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

KVL Equations:

$$\begin{cases} v_1 = e_1 - e_2 \\ v_2 = e_1 - e_3 \\ v_3 = e_3 - e_2 \\ v_4 = e_2 \\ v_5 = e_3 \\ v_6 = -e_1 \end{cases}$$

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}^T} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{\mathbf{e}} \Leftarrow$$

KVL: $\boxed{\mathbf{v} = \mathbf{A}^T\,\mathbf{e}}$

Since v_j is present only in the jth equation, these k equations are **linearly - independent**.

Theorem

$$\boxed{\mathbf{A}\,\mathbf{i} = \mathbf{0}}$$

gives the **maximum possible** number of **linearly-independent KCL equations** for a connected circuit.

Reduced Incidence Matrix

Let G be a connected **digraph** with "n" nodes and "b" branches. Let $\mathbf{A}_a$ be the **Incidence Matrix** of G. The $(n\text{-}1) \times b$ matrix $\mathbf{A}$ obtained by deleting any one row of $\mathbf{A}_a$ is called a **Reduced-Incidence Matrix** of G.

Observation : The 4 KCL node equations are ***not*** linearly independent.

Adding the left side of the 4 KCL node equations, we obtain:

$$\underbrace{(i_1+i_2-i_6)}_{1}+\underbrace{(-i_1-i_3+i_4)}_{2}+\underbrace{(-i_2+i_3+i_5)}_{3}$$

$$+\underbrace{(-i_4-i_5+i_6)}_{4}\equiv 0$$

This means we can derive any one of these 4 equations from the other 3.

Example: Derive KCL equations at node 4:

Adding the first 3 node equations gives:

$$\underbrace{(i_1+i_2-i_6)}_{1}+\underbrace{(-i_1-i_3+i_4)}_{2}+\underbrace{(-i_2+i_3+i_5)}_{3}$$

$$=\underbrace{i_4+i_5-i_6}_{4}$$

Reduced Incidence Matrix **A**

Let G be a connected **digraph** with "n" nodes and "b" branches, the **reduced incidence matrix** **A** relative to **datum node** $\textcircled{n}$ is an $(n\text{-}1) \times b$ matrix whose coefficients a_{jk} are obtained from the $(n\text{-}1)$ KCL equations written at the n-1 non-datum nodes:

$$a_{jk} = \begin{cases} 1 & \text{if branch } k \text{ leaves node } \textcircled{j} \\ -1 & \text{if branch } k \text{ enters node } \textcircled{j} \\ 0 & \text{if branch } k \text{ is not connected to node } \textcircled{j} \end{cases}$$

By applying the various versions of KCL, we can write many different KCL equations for each circuit. However, these equations are usually **not** linearly independent in the sense that each equation can be derived by a linear combination of the others.

How can we write a maximum set of **linearly-independent** KCL equations?

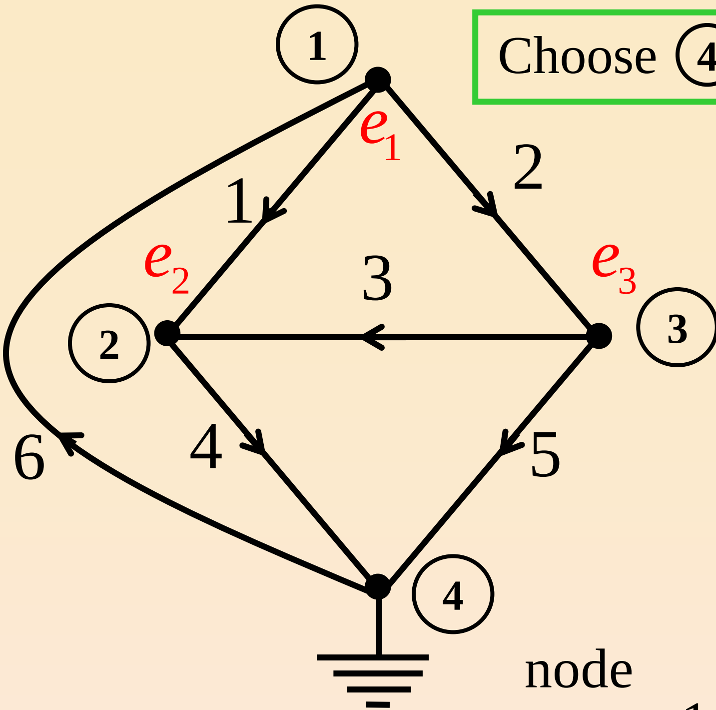
Simplest Method to write linearly-Independent KCL Equations.

Given a connected circuit with "*n*" nodes, choose an arbitrary node as **datum**. Write a KCL equation at each of the remaining (n-1) nodes.

Relationship between **A** and $\mathbf{A}_a$

Let $\mathbf{A}_a$ be the $n \times b$ **Incidence matrix** of a connected digraph G with "n" nodes and "b" branches.

By deleting any row corresponding to node m from $\mathbf{A}_a$, we obtain the **reduced incidence matrix** **A** of G relative to the datum node m .



Choose 4 as datum node for digraph G

KCL Equations:

$$\text{①}\quad i_1 + i_2 - i_6 = 0$$
$$\text{②}\quad -i_1 - i_3 + i_4 = 0$$
$$\text{③}\quad -i_2 + i_3 + i_5 = 0$$

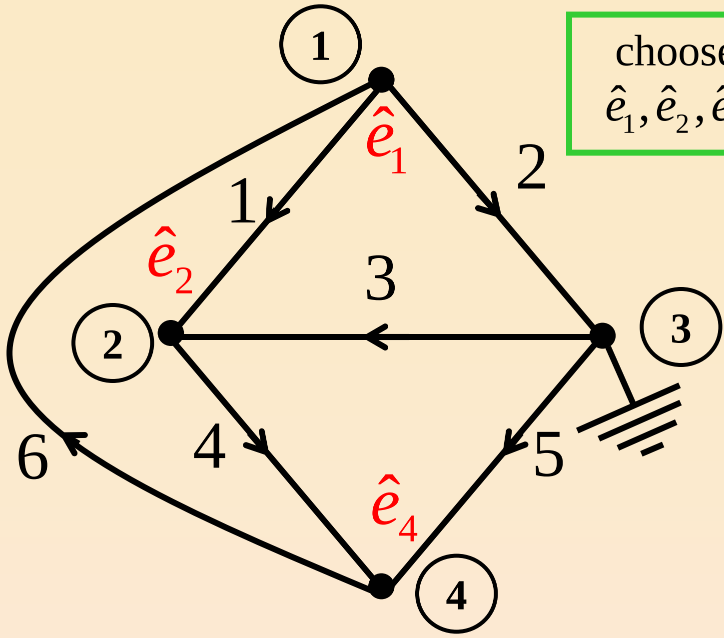
Independent KCL Equations

$$\mathbf{A}\,\mathbf{i} = \mathbf{0} \quad \Rightarrow \quad \underbrace{\begin{array}{c|cccccc} \text{node no.} \backslash \text{Branch no.} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & -1 \\ 2 & -1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 0 & -1 & 1 & 0 & 1 & 0 \end{array}}_{\mathbf{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

Independent KVL Equations

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}^T} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{\mathbf{e}} \Rightarrow$$

$$\boxed{\begin{aligned} v_1 &= e_1 - e_2 \\ v_2 &= e_1 - e_3 \\ v_3 &= -e_2 + e_3 \\ v_4 &= e_2 \\ v_5 &= e_3 \\ v_6 &= -e_1 \end{aligned}}$$



choose 3 as datum and let $\hat{e}_1, \hat{e}_2, \hat{e}_4$ be new node-to-datum voltages.

KCL Equations:

1 $\quad i_1 + i_2 - i_6 = 0$

2 $\quad -i_1 - i_3 + i_4 = 0$

4 $\quad -i_4 - i_5 + i_6 = 0$

Independent KCL Equations

$$\hat{\mathbf{A}}\,\mathbf{i} = \mathbf{0} \quad \Rightarrow \quad \begin{array}{c} \text{Branch no.} \\ \begin{array}{c|cccccc} \text{node no.} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & & & \\ 2 & & & & & & \\ 4 & & & & & & \end{array} \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}}_{\hat{\mathbf{A}}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

Independent KVL Equations

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\hat{\mathbf{A}}^T} \underbrace{\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_4 \end{bmatrix}}_{\hat{\mathbf{e}}} \quad \Rightarrow$$

$$\boxed{\begin{aligned} v_1 &= \hat{e}_1 - \hat{e}_2 \\ v_2 &= \hat{e}_1 \\ v_3 &= -\hat{e}_2 \\ v_4 &= \hat{e}_2 - \hat{e}_4 \\ v_5 &= -\hat{e}_4 \\ v_6 &= -\hat{e}_1 + \hat{e}_4 \end{aligned}}$$

We can always recast **any** system of **linear** constitutive equations into the following standard matrix form

$$\underbrace{\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\mathbf{H}_v \qquad\qquad \mathbf{H}_i} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hdashline i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\begin{bmatrix} \mathbf{v} \\ \mathbf{i} \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 6 \\ 2 \end{bmatrix}}_{\mathbf{u}}$$

independent source vector

$$\Downarrow$$

$$\boxed{\mathbf{H}_v \, \mathbf{v} + \mathbf{H}_i \, \mathbf{i} = \mathbf{u}}$$

$$\text{KCL}\begin{cases} i_1 + i_2 - i_4 = 0 & (1) \\ -i_1 + i_3 = 0 & (2) \end{cases}$$

$$\text{KVL}\begin{cases} v_1 = e_1 - e_2 & (3) \\ v_2 = e_1 & (4) \\ v_3 = e_2 & (5) \\ v_4 = -e_1 & (6) \end{cases}$$

$$\begin{array}{l}\text{Element} \\ \text{Constitive} \\ \text{Relation}\end{array}\begin{cases} v_1 = 4\,i_1 & (7) \\ v_2 = 3\,i_2 & (8) \\ v_3 = 6 & (9) \\ i_4 = 2 & (10) \end{cases}$$

$\left.\right\}$ **10 independent linear equations involving 10 variables**

We can always find the solution using **Cramer's rule**.

For simple circuits, we can often find the solution by ***as hoc*** elimination and substitution of variables:

EXAMPLE : (5) and (9) $\Rightarrow$ $e_2 = 6$ (11)

(1) and (10) $\Rightarrow$ $i_1 + i_2 = 2$ (12)

(3) , (7) and (11) $\Rightarrow$ $i_1 = \frac{1}{4}(e_1 - 6)$ (13)

(4) and (8) $\Rightarrow$ $i_2 = \frac{1}{3}e_1$ (14)

Substituting (10), (11), (12), and (13) into (1), we obtain

$$\frac{1}{4}(e_1 - 6) + \frac{1}{3}e_1 - 2 = 0$$

$$\Rightarrow \boxed{e_1 = 6} \qquad (15)$$

Complete Solution:

$$\boxed{\begin{array}{l} e_1 = 6V, e_2 = 6V \\ v_1 = 0V, v_2 = 6V, v_3 = 6V, v_4 = -6V \\ i_1 = 0\,A, i_2 = 2\,A, i_3 = 0\,A, i_4 = 2\,A \end{array}}$$

Verification of solution via Tellegen's Theorem

$$\sum_{j=1}^{4} v_j\, i_j = (v_1\, i_1) + (v_2\, i_2) + (v_3\, i_3) + (v_4\, i_4)$$

$$= (0)(0) + (6)(2) + (6)(0) + (-6)(2) \stackrel{?}{=} 0$$

Tellegen's Theorem

Let G be a diagraph with "b" branches.

Let $(v_1, v_2, \cdots, v_b)$ be any set of b voltages of G which satisfy KVL.

Let $(i_1, i_2, \cdots, i_b)$ be any set of b currents of G which satisfy KCL.

Then
$$\boxed{\sum_{\text{j}=1}^{\text{b}} v_j \, i_j = 0}$$

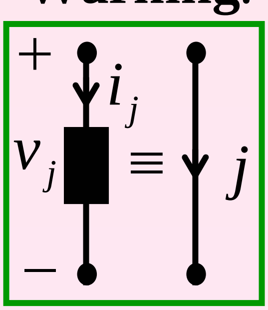
Proof: suppose

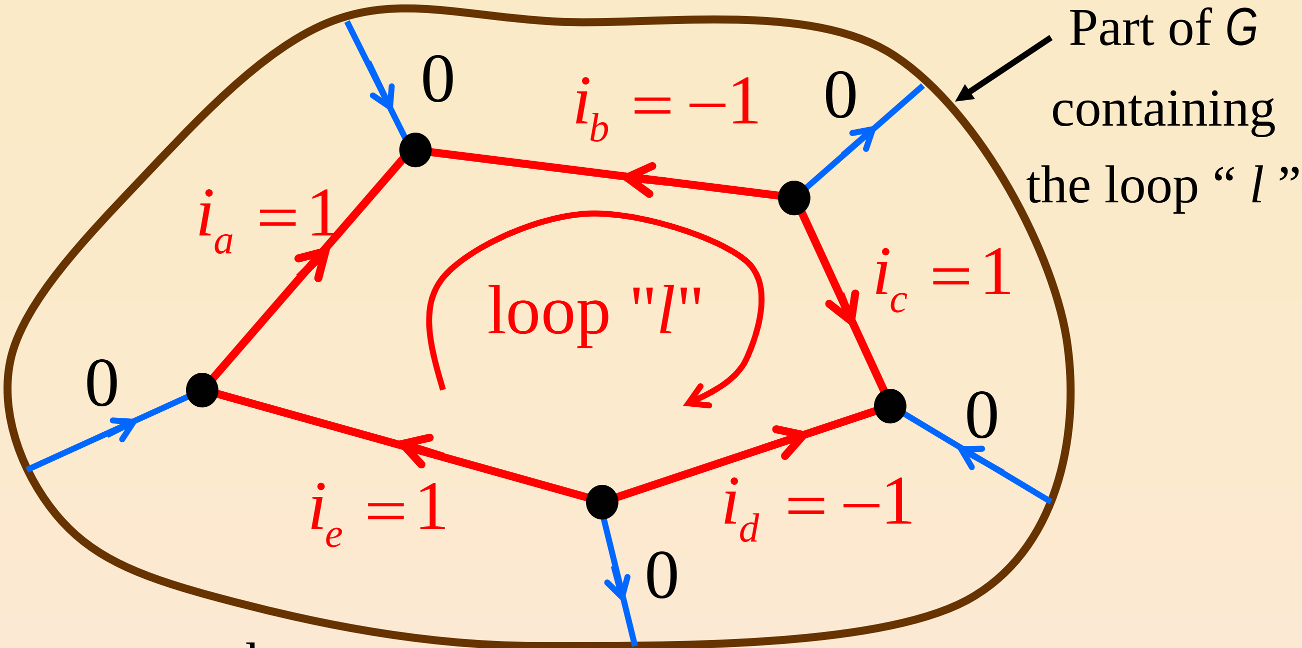
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_b \end{bmatrix} \text{ satisfies KVL for } G \quad , \quad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_b \end{bmatrix} \text{ satisfies KCL for } G$$

Then $\displaystyle\sum_{j=1}^{b} v_j i_j = \mathbf{v}^T \mathbf{i} = \underbrace{\left(\mathbf{A}^T \mathbf{e}\right)^T}_{\textbf{KVL}} \mathbf{i} = \mathbf{e}^T \underbrace{\left(\mathbf{A}\,\mathbf{i}\right)}_{\textbf{KCL}} = 0$

where $\mathbf{e} = [e_1, e_2, \cdots, e_{n-1}]^T$ is any node-to-datum voltage. ■

Warning: By definition of a diagraph, each branch voltage v_j and branch current i_j associated with branch j must follow the Associated reference convention: i_j flows from the positive terminal to the negative terminal.





Suppose we choose:

This choice of $\{i_1, i_2, \ldots, i_b\}$ Satisfies KCL

$i_j = 0$, if i_j is not in loop "l"

$= 1$, if i_j is in loop "l" and flows in the same direction as loop "l"

$= -1$, if i_j is in loop "l" and flows in opposite direction as loop "l"

$$\therefore \sum_{j=1}^{b} v_j\, i_j = 0$$

(because v_j chosen earlier satisfies Tellegen's theorem)

$$0 = \sum_{j=1}^{b} v_j\, i_j = \sum_{\substack{b_j \text{ belonging} \\ \text{to loop "}l\text{"}}} v_j\, i_j + \underbrace{\sum_{\substack{b_j \text{ not belonging} \\ \text{to loop "}l\text{"}}} v_j\, i_j}_{\text{equals 0 because } i_j = 0}$$

$$\Rightarrow \sum_{b_j \text{ belonging to loop "}l\text{"}} v_j\, i_j = 0 \Rightarrow \text{KVL} \quad \blacksquare$$

Relationship Between

Kirchhoff's Laws and Tellegen's Theorem

1. **KCL** and **KVL** $\longrightarrow$ **Tellegen's Theorem**

2. **KVL** and **Tellegen's Theorem** $\longrightarrow$ **KCL**

3. **Tellegen's Theorem** and **KCL** $\longrightarrow$ **KVL**

KVL and Tellegen's Theorem $\longrightarrow$ KCL

***Proof*.**

Let $\mathbf{v}$ satisfy KVL for G:

$$\mathbf{v} = \mathbf{A}^T \mathbf{e} \qquad (1)$$

Let $\mathbf{v}$ and $\mathbf{i}$ satisfy Tellegen's Theorem:

$$\mathbf{v}^T \mathbf{i} = 0 \qquad (2)$$

Substitute (1) for $\mathbf{v}$ in (2):

$$\left(\mathbf{A}^T \mathbf{e}\right)^T \mathbf{i} = 0 \qquad (3)$$

$$\mathbf{e}^T \left(\mathbf{A}\,\mathbf{i}\right) = 0 \qquad (4)$$

Since (4) is true for any node-to-datum voltages $\mathbf{e} \neq \mathbf{0}$,

(4) can be true only if

$$\boxed{\left(\mathbf{A}\,\mathbf{i}\right) = \mathbf{0}} \quad \Rightarrow \quad \textbf{KCL} \qquad \blacksquare$$

Tellegen's Theorem and **KCL** ⟶ **KVL**

Proof.

Let G be any connected digraph with b branches $\{1, 2, \ldots, b\}$.

Let $\{i_1, i_2, \ldots, i_b\}$ be any set of branch currents satisfying KCL.

Choose ***any*** subset $\{b_a, b_b, \ldots, b_n\}$ of the b branches which form a **closed loop** " l ". Let $\{v_1, v_2, \ldots, v_b\}$ be ***any*** set of branch voltages which, together with $\{i_1, i_2, \ldots, i_b\}$ satisfy Tellegen's Theorem.

Our goal is to prove that the subset of these voltages which belong to the above closed loop "l" must satisfy KVL around the loop.

Applying Tellegen's Theorem to Circuits Containing (n+1)-terminal devices

Let N be any circuit containing (n+1)-terminal devices.

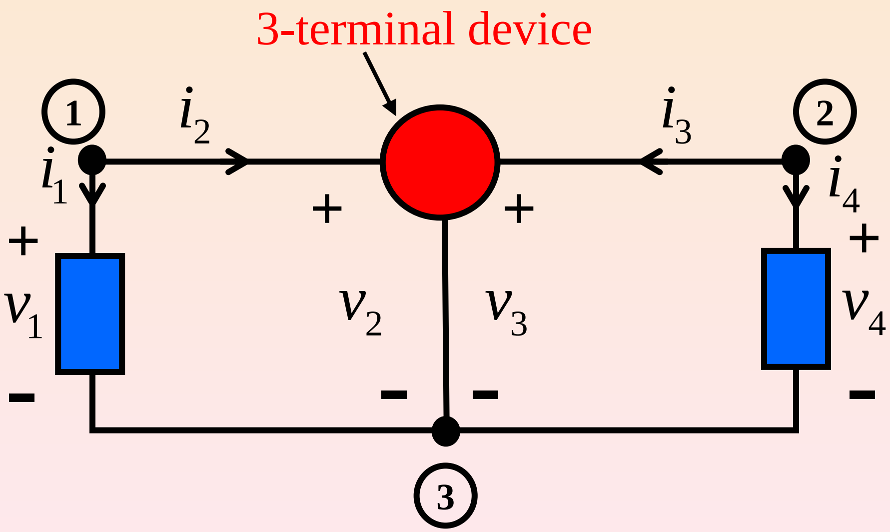
Step 1. Assign a datum to each device. Assign "n" terminal-to-datum voltages for each (n+1)-terminal device, following **associated reference convention**.

Step 2. Draw the digraph G of N.

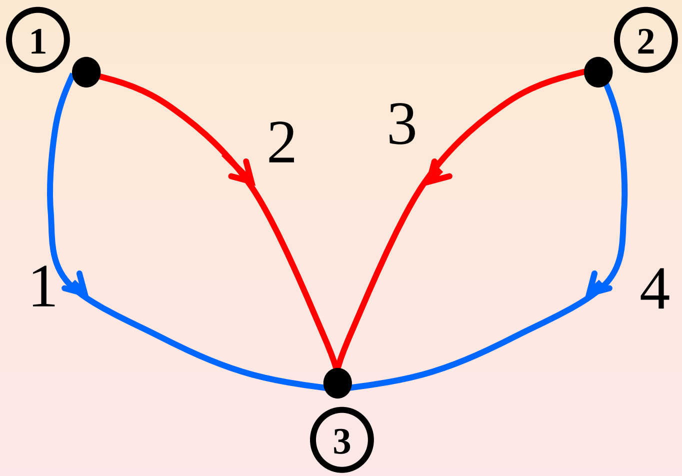
Step 3. Apply Tellegen's theorem to G.

Remarks

Tellegen's theorem can be applied directly to a circuit provided we use Associated Reference convention for all device terminal currents and voltages.



circuit N

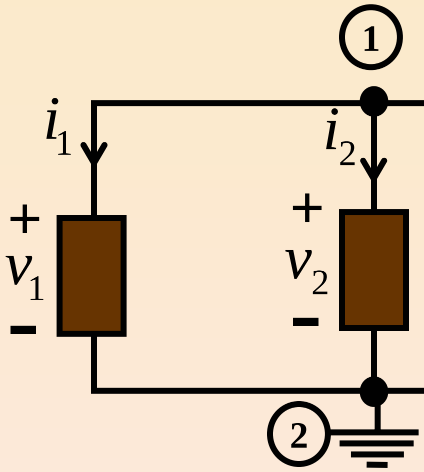


digraph G

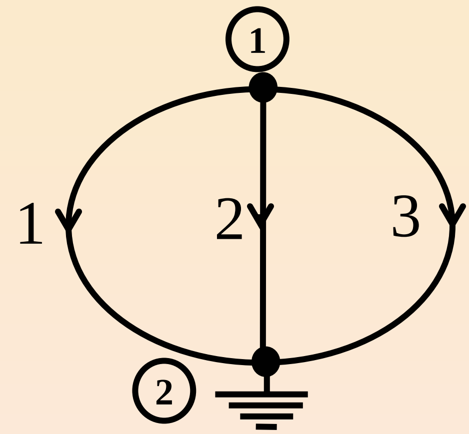
$$\sum_{j=1}^{4} v_j i_j = v_1 i_1 + v_2 i_2 + v_3 i_3 + v_4 i_4 = 0$$

(choose 3 as datum node for the 3-terminal device)

Voltage and Current Solutions are Orthogonal!



circuit N



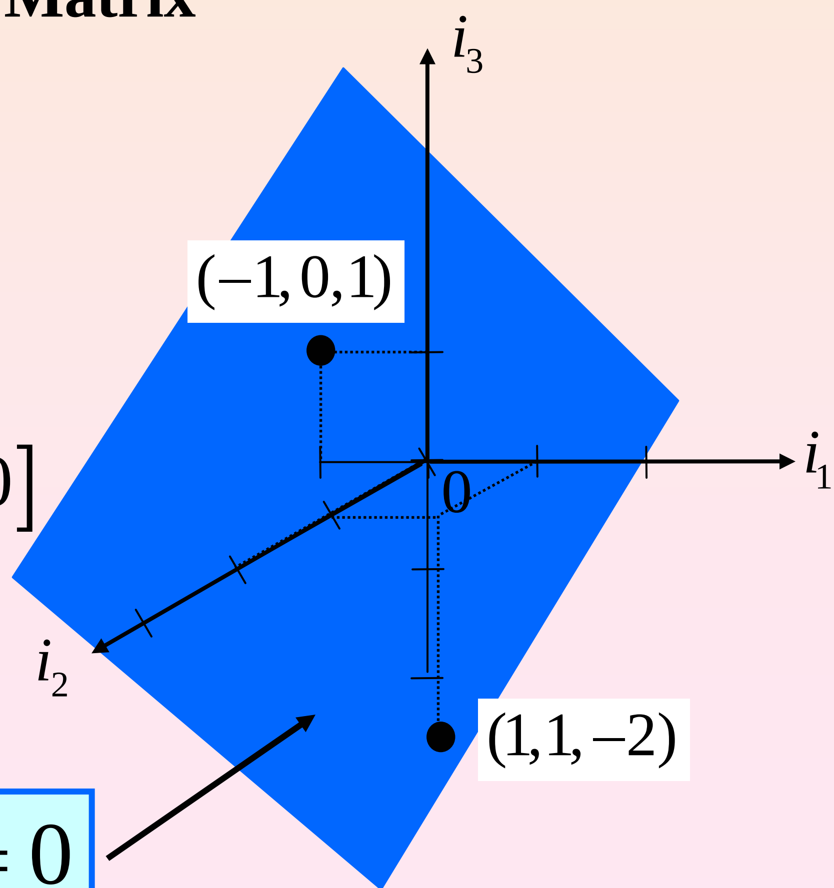
digraph G

Reduced Incidence Matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

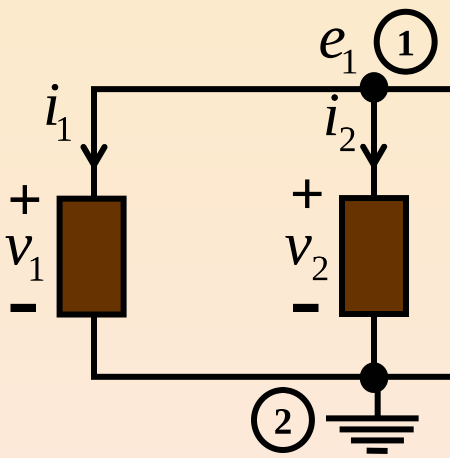
$$\text{KCL} : \mathbf{A}\,\mathbf{i} = \mathbf{0}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}}_{\mathbf{i}} = \begin{bmatrix} 0 \end{bmatrix}$$

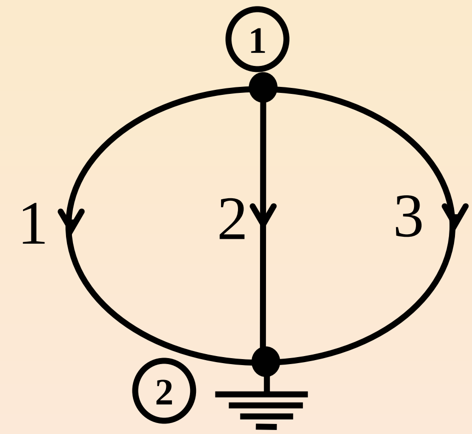


$$\boxed{i_1 + i_2 + i_3 = 0}$$

Voltage and Current Solutions are Orthogonal!



circuit N



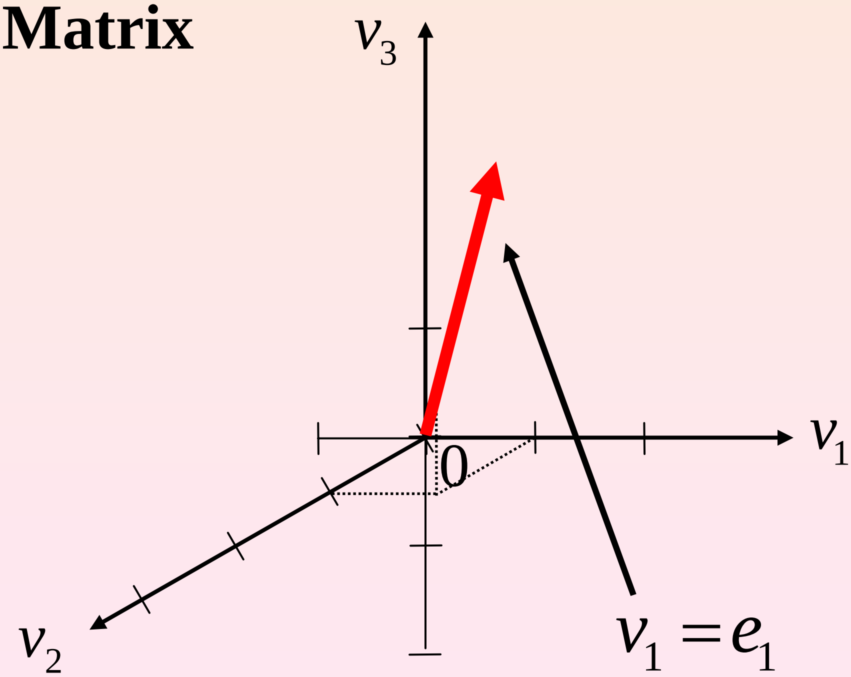
digraph G

Reduced Incidence Matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$\text{KVL}: \mathbf{v} = \mathbf{A}^T\,\mathbf{e}$$

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{A}^T} \underbrace{\begin{bmatrix} \mathbf{e}_1 \end{bmatrix}}_{\mathbf{e}}$$



$$v_1 = e_1$$
$$v_2 = e_1$$
$$v_3 = e_1$$

Geometrical Interpretation of Tellegen's Theorem

$$\text{KCL}: i_1 + i_2 + i_3 = 0$$

$$\text{KVL}: \mathrm{v}_1 = v_2 = v_3$$

$$\begin{aligned}\sum_{\mathrm{j}=1}^{3} v_j i_j &= v_1 i_1 + v_2 i_2 + v_3 i_3 \\ &= e_1 i_1 + e_1 i_2 + e_1 i_3 \\ &= e_1 \left(i_1 + i_2 + i_3 \right) = 0\end{aligned}$$