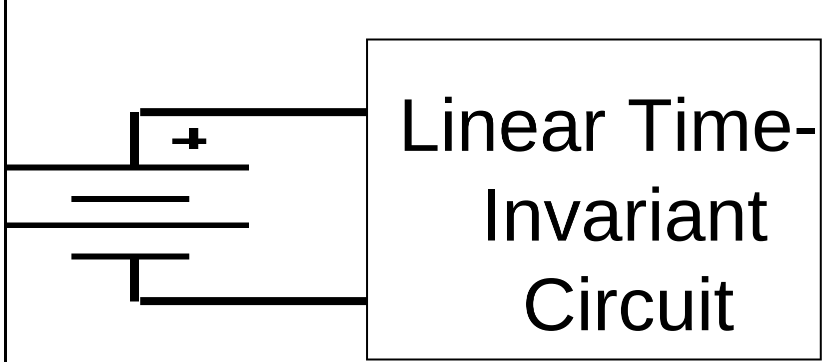
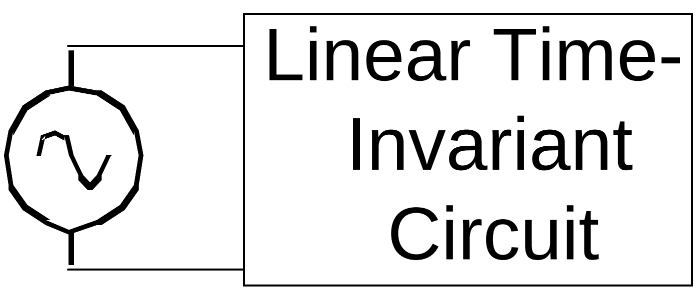
Chapter 9 and Chapter 1 from reader

- OUTLINE
  - Phasors as notation for Sinusoids
  - Arithmetic with Complex Numbers
  - Complex impedances
  - Circuit analysis using complex impdenaces
  - Dervative/Integration as multiplication/division
  - Phasor Relationship for Circuit Elements
  - Frequency Response and Bode plots
- Reading
  - Chapter 9 from your book
  - Chapter 1 from your reader

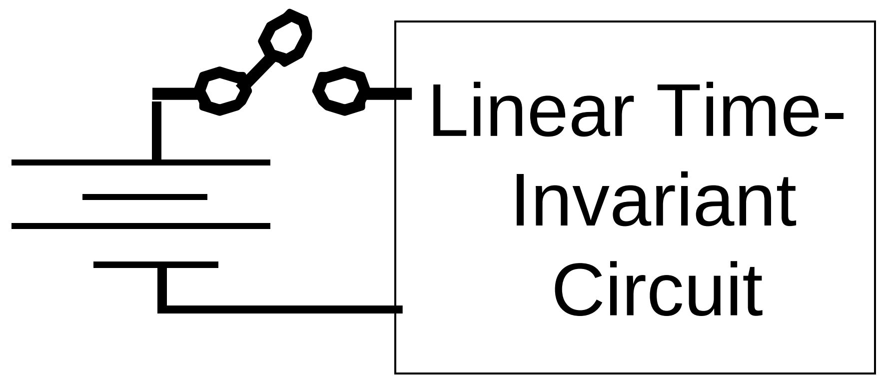
Types of Circuit Excitation



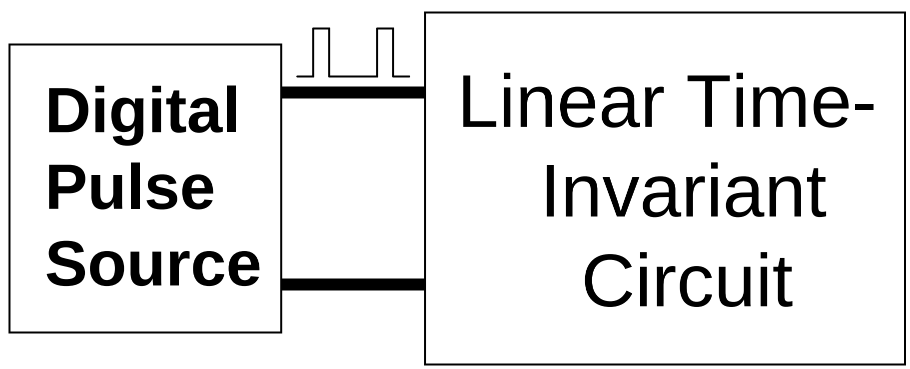
Steady-State Excitation (DC Steady-State)



Sinusoidal (Single-Frequency) Excitation →AC Steady-State



OR

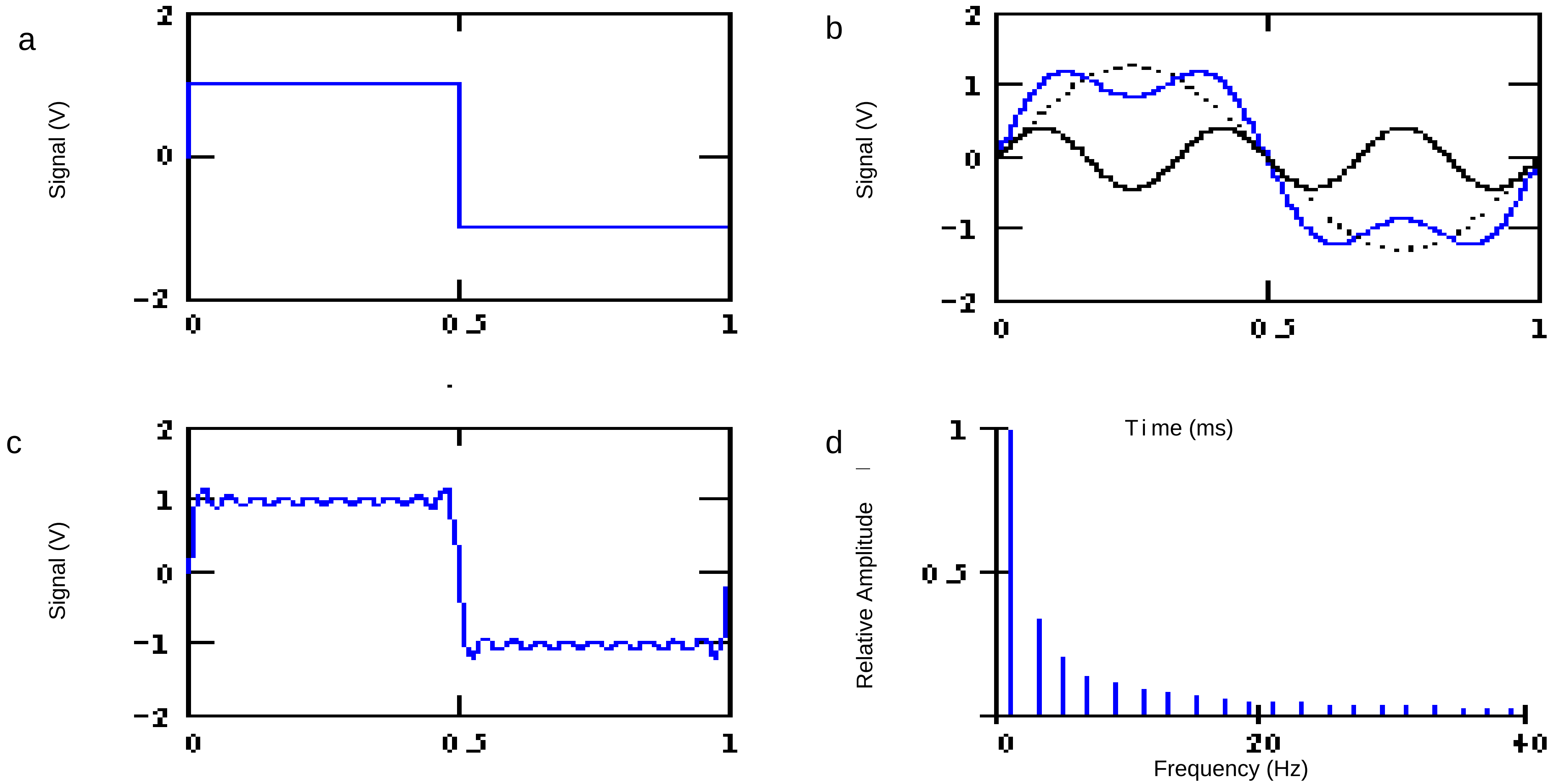


Transient Excitation

Why is Single-Frequency Excitation Important?

- Some circuits are driven by a single-frequency sinusoidal source.
- Some circuits are driven by sinusoidal sources whose frequency changes slowly over time.
- You can express any periodic electrical signal as a sum of single-frequency sinusoids – so you can analyze the response of the (linear, time-invariant) circuit to each individual frequency component and then sum the responses to get the total response.
- This is known as Fourier Transform and is tremendously important to all kinds of engineering disciplines!

Representing a Square Wave as a Sum of Sinusoids

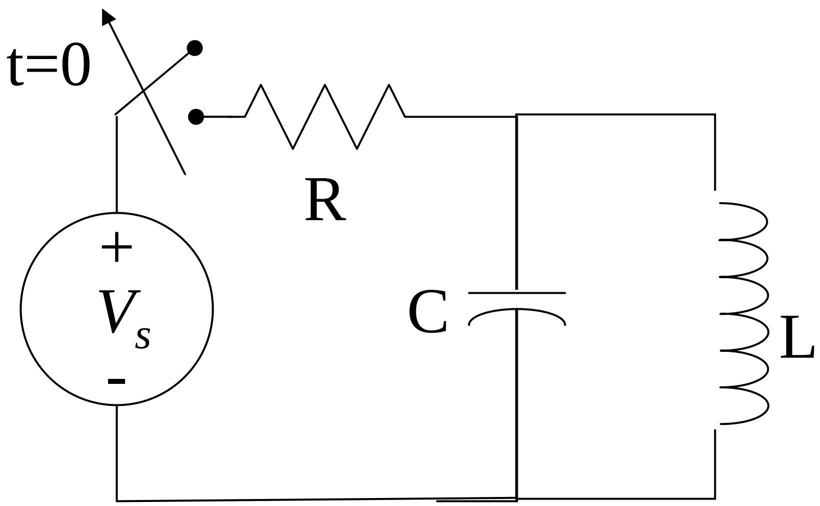


(a)Square wave with 1-second period. (b) Fundamental component (dotted) with 1-second period, third-harmonic (solid black) with1/3-second period, and their sum (blue). (c) Sum of first ten components. (d) Spectrum with 20 terms.

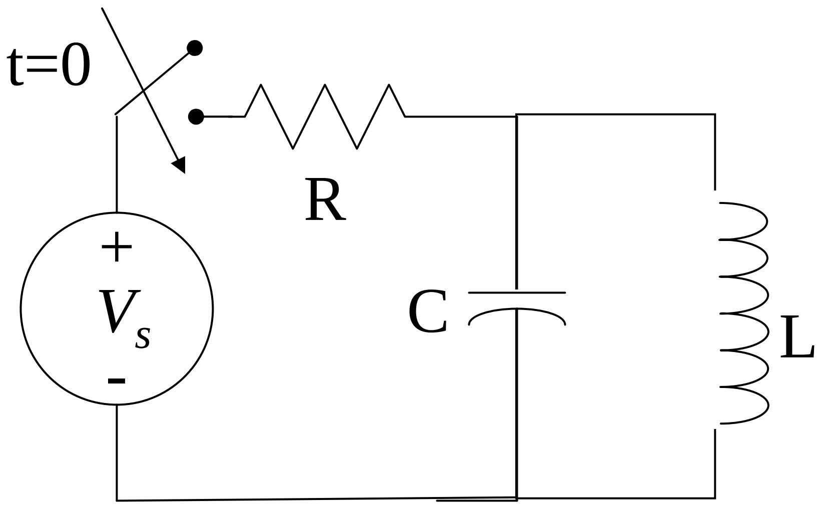
Steady-State Sinusoidal Analysis

- Also known as AC steady-state
- Any steady state voltage or current in a linear circuit with a sinusoidal source is a sinusoid.
  - This is a consequence of the nature of particular solutions for sinusoidal forcing functions.
- All AC steady state voltages and currents have the same frequency as the source.
- In order to find a steady state voltage or current, all we need to know is its magnitude and its phase relative to the source
  - We already know its frequency.
- Usually, an AC steady state voltage or current is given by the particular solution to a differential equation.

Example 1: 2nd Order RLC Circuit



Example 2: 2nd Order RLC Circuit



Sinusoidal Sources Create Too Much Algebra

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

Guess a solution

Two terms to be general

Dervatives

Addition

$$x_P(t) = A\sin(wt) + B\cos(wt)$$

$$(A\sin(wt) + B\cos(wt)) + \tau \frac{d(A\sin(wt) + B\cos(wt))}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$(A - \tau B - F_A)\sin(wt) + (B + \tau A - F_B)\cos(wt) = 0$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

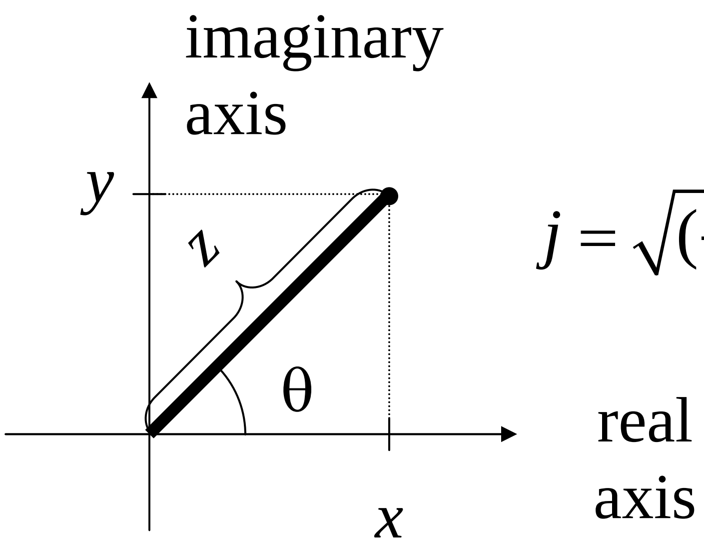
$$(A - \tau B - F_A) = 0$$

$$(B + \tau A - F_B) = 0$$

$$A = \frac{F_A + \tau F_B}{\tau^2 + 1} \qquad B = -\frac{\tau F_A - F_B}{\tau^2 + 1}$$

Phasors (vectors that rotate in the complex plane) are a clever alternative.

Complex Numbers (1)



$$j = \sqrt{(-1)}$$

- x is the real part
- y is the imaginary part
- z is the magnitude
- θ is the phase

$$x = z\cos\theta \qquad y = z\sin\theta$$

$$z = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$\mathbf{Z} = z(\cos\theta + j\sin\theta)$$

- Rectangular Coordinates

  $\mathbf{Z} = x + jy$

- Polar Coordinates:

  $\mathbf{Z} = z \angle \theta$

- Exponential Form:

  $$\mathbf{Z} = |\mathbf{Z}|e^{j\theta} = ze^{j\theta}$$

$$1 = 1e^{j0} = 1\angle 0°$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90°$$

Complex Numbers (2)

Euler's Identities

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Exponential Form of a complex number

$$\mathbf{Z} = \left|\mathbf{Z}\right| e^{j\theta} = z e^{j\theta} = z\angle\theta$$

Arithmetic With Complex Numbers

- To compute phasor voltages and currents, we need to be able to perform computation with complex numbers.
  - Addition
  - Subtraction
  - Multiplication
  - Division
- (And later use multiplication by $j\omega$ to replace
  - Diffrentiation
  - Integration

Addition

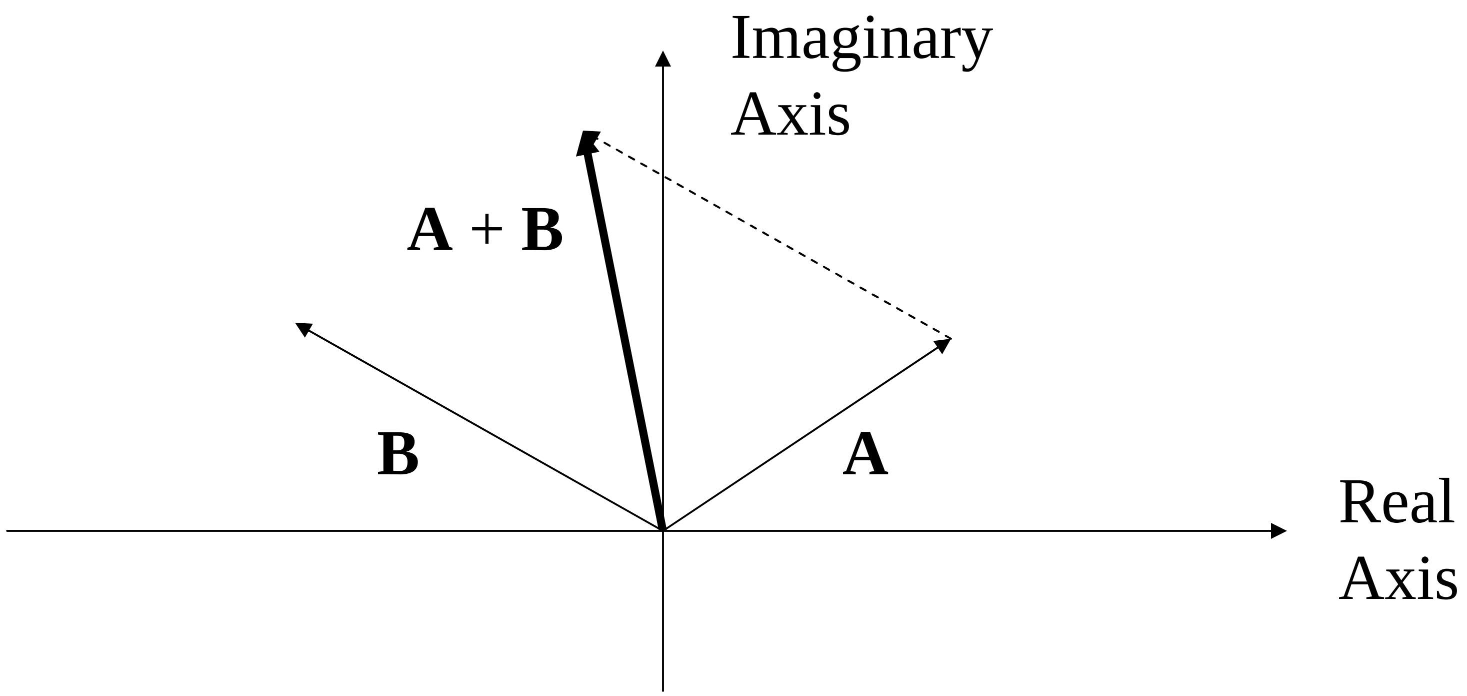
- Addition is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} + \mathbf{B} = (x + z) + j(y + w)$$

Addition



Subtraction

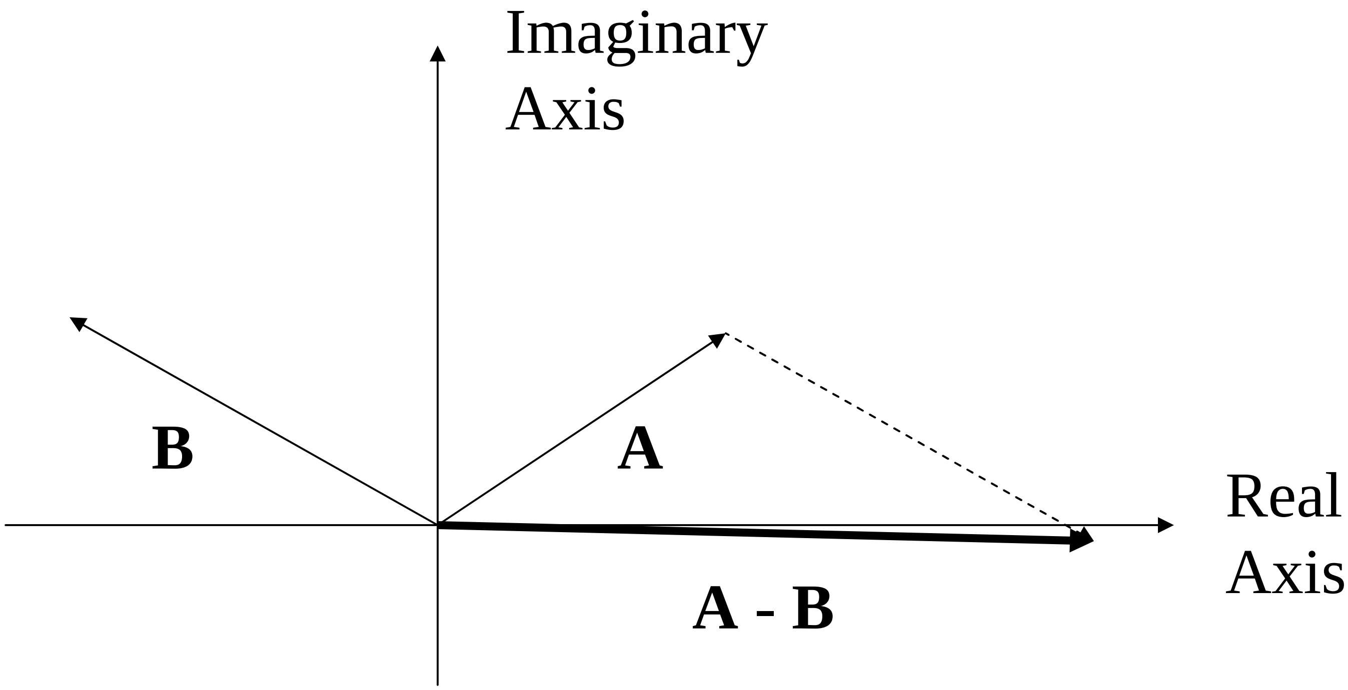
- Subtraction is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} - \mathbf{B} = (x - z) + j(y - w)$$

Subtraction



Multiplication

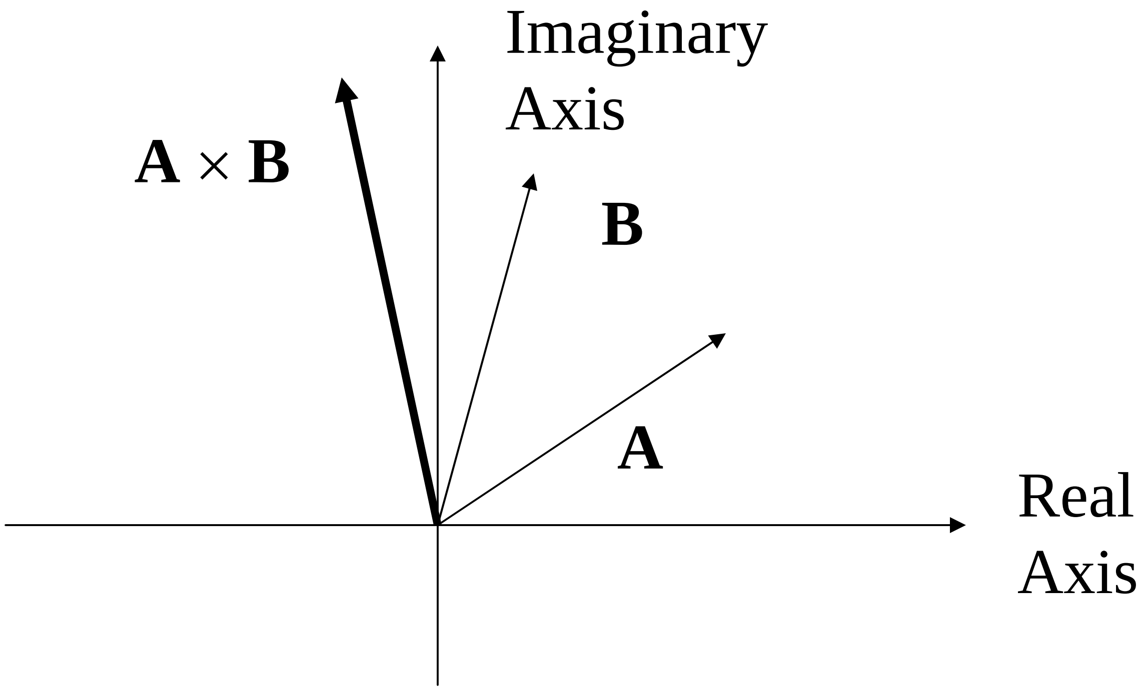
- Multiplication is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} \times \mathbf{B} = (A_M \times B_M) \angle (\theta + \phi)$$

Multiplication



Division

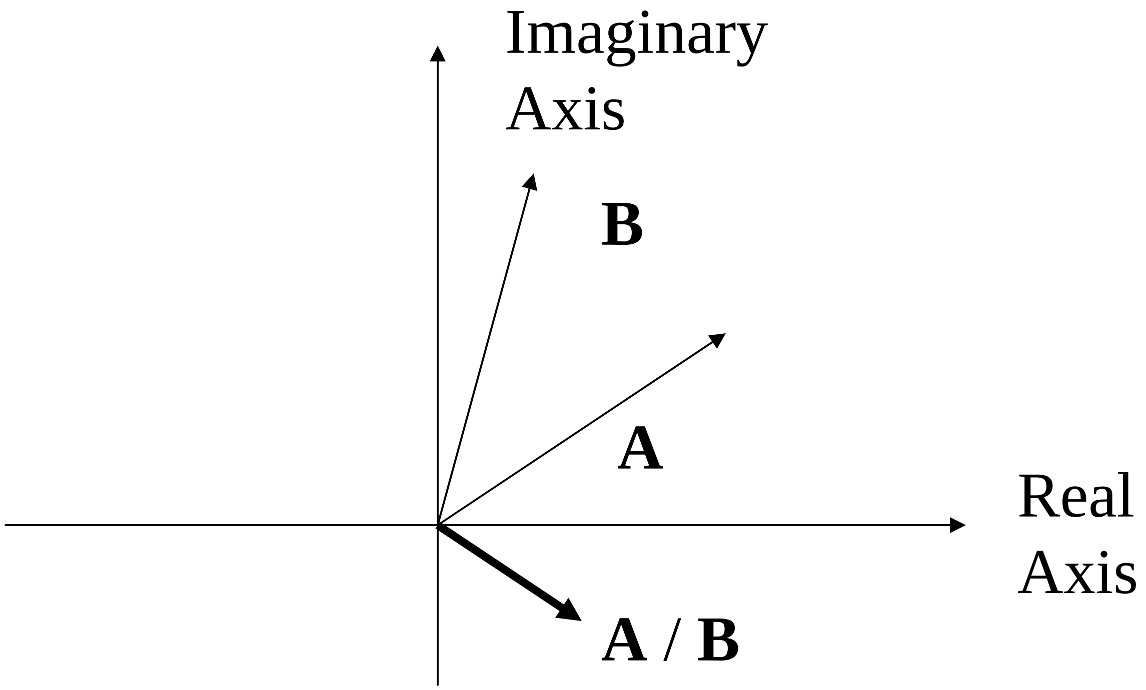
- Division is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} / \mathbf{B} = (A_M / B_M) \angle (\theta - \phi)$$

Division



Arithmetic Operations of Complex Numbers

- Add and Subtract: it is easiest to do this in rectangular format
  - Add/subtract the real and imaginary parts separately
- Multiply and Divide: it is easiest to do this in exponential/polar format
  - Multiply (divide) the magnitudes
  - Add (subtract) the phases

$$\mathbf{Z_1} = z_1 e^{j\theta_1} = z_1 \angle \theta_1 = z_1 \cos\theta_1 + j z_1 \sin\theta_1$$

$$\mathbf{Z}_2 = z_2 e^{j\theta_2} = z_2 \angle \theta_2 = z_2 \cos\theta_2 + j z_2 \sin\theta_2$$

$$\mathbf{Z_1} + \mathbf{Z}_2 = (z_1 \cos\theta_1 + z_2 \cos\theta_2) + j(z_1 \sin\theta_1 + z_2 \sin\theta_2)$$

$$\mathbf{Z_1} - \mathbf{Z}_2 = (z_1 \cos\theta_1 - z_2 \cos\theta_2) + j(z_1 \sin\theta_1 - z_2 \sin\theta_2)$$

$$\mathbf{Z_1} \times \mathbf{Z}_2 = (z_1 \times z_2) e^{j(\theta_1 + \theta_2)} = (z_1 \times z_2) \angle (\theta_1 + \theta_2)$$

$$\mathbf{Z_1} / \mathbf{Z}_2 = (z_1 / z_2) e^{j(\theta_1 - \theta_2)} = (z_1 / z_2) \angle (\theta_1 - \theta_2)$$

Phasors

- Assuming a source voltage is a sinusoid time-varying function

$$v(t) = V \cos(\omega t + \theta)$$

- We can write:

$$v(t) = V\cos(\omega t + \theta) = V\,\mathrm{Re}\left[e^{j(\omega t + \theta)}\right] = \mathrm{Re}\left[Ve^{j(\omega t + \theta)}\right]$$

$$Define\ \ Phasor\ \ as\ \ Ve^{j\theta} = V\angle\theta$$

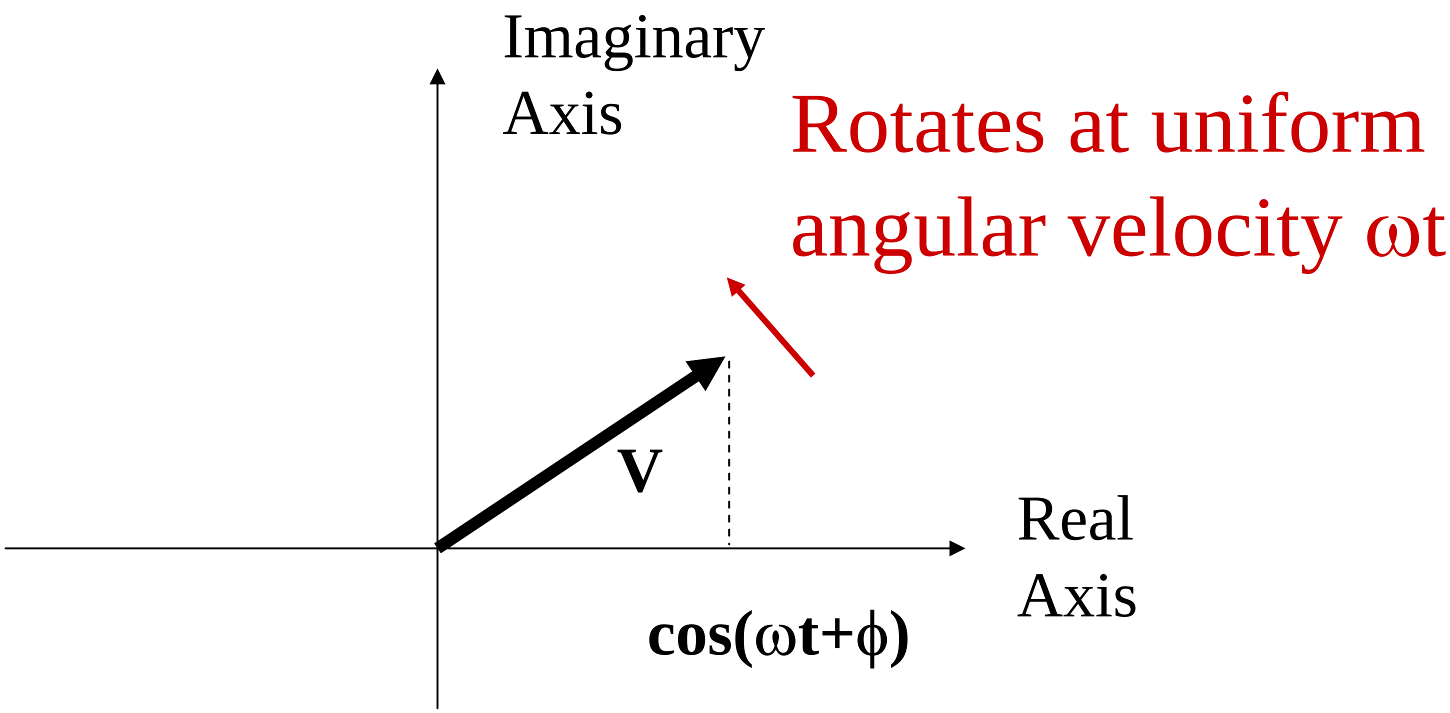
- Similarly, if the function is $v(t) = V \sin(\omega t + \theta)$

$$v(t) = V\sin(\omega t + \theta) = V\cos(\omega t + \theta - \frac{\pi}{2}) = \mathrm{Re}\left[Ve^{j(\omega t + \theta - \frac{\pi}{2})}\right]$$

$$Phasor\ = V\angle\left(\theta - \frac{\pi}{2}\right)$$

Phasor: Rotating Complex Vector

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{Ve^{j\phi}e^{jwt}\right\} = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right)$$

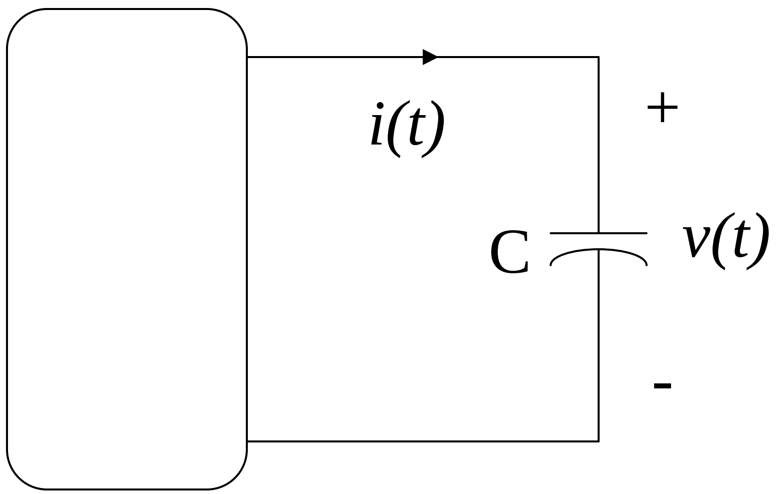


The head start angle is φ.

Complex Exponentials

- We represent a real-valued sinusoid as the real part of a complex exponential after multiplying by $e^{j\omega t}$.
- Complex exponentials
  - provide the link between time functions and phasors.
  - Allow dervatives and integrals to be replaced by multiplying or dividing by $j\omega$
  - make solving for AC steady state simple algebra with complex numbers.
- Phasors allow us to express current-voltage relationships for inductors and capacitors much like we express the current-voltage relationship for a resistor.

I-V **Relationship for a Capacitor**



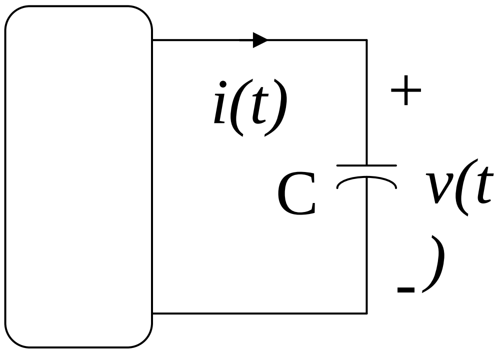
$$i(t) = C\frac{dv(t)}{dt}$$

Suppose that *v(t)* is a sinusoid:

$$v(t) = \mathrm{Re}\{V_M\, e^{j(\omega t+\theta)}\}$$

Find *i(t)*.

Capacitor Impedance (1)



$$i(t) = C\frac{dv(t)}{dt}$$

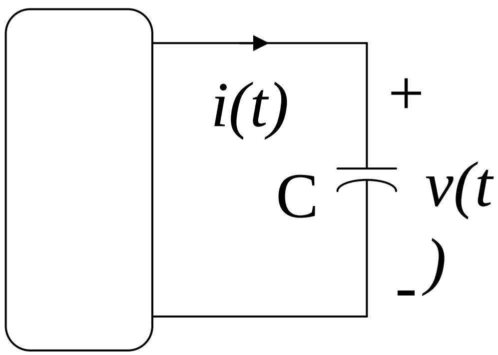
$$v(t) = V\cos(\omega t + \theta) = \frac{V}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{CV}{2}\frac{d}{dt}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] = \frac{CV}{2}j\omega\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right]$$

$$= \frac{-\omega CV}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right] = -\omega CV\sin(\omega t + \theta) = \omega CV\cos(\omega t + \theta + \frac{\pi}{2})$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\left(\theta + \frac{\pi}{2}\right)} = \frac{V}{\omega CV}\angle(\theta - \theta - \frac{\pi}{2}) = \frac{1}{\omega C}\angle(-\frac{\pi}{2}) = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

Capacitor Impedance (2)



$$i(t) = C\frac{dv(t)}{dt}$$

Phasor definition

$$v(t) = V\cos(\omega t + \theta) = \mathrm{Re}\left[Ve^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{V} = V\angle\theta$$

$$i(t) = C\frac{dv(t)}{dt} = \mathrm{Re}\left[CV\frac{de^{j(\omega t+\theta)}}{dt}\right] = \mathrm{Re}\left[j\omega CVe^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{I} = I\angle\theta$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\theta} = \frac{V}{j\omega CV}\angle(\theta-\theta) = \frac{1}{j\omega C}$$

Example

$$v(t) = 120\text{V} \cos(377t + 30°)$$

$$C = 2\mu\text{F}$$

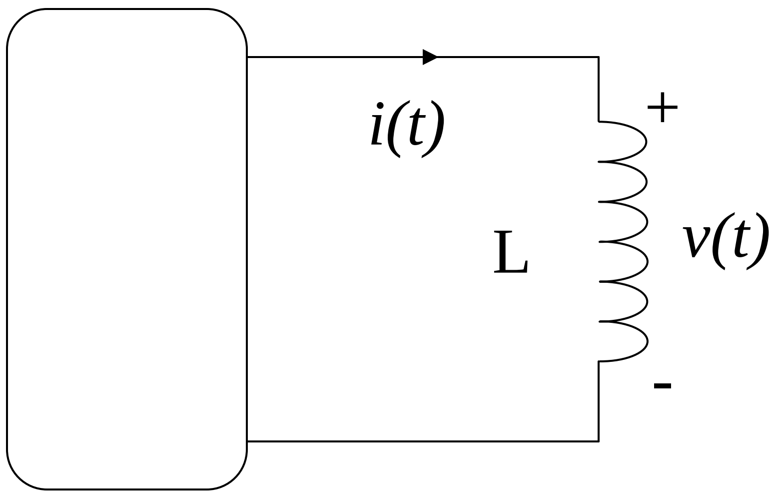
- What is **V**?
- What is **I**?
- What is *i(t)*?

Computing the Current

Note: The differentiation and integration operations become algebraic operations

$$\frac{d}{dt} \Rightarrow j\omega \qquad\qquad \int dt \Rightarrow \frac{1}{j\omega}$$

Inductor Impedance



$$v(t) = L\frac{di(t)}{dt}$$

$$\mathbf{V} = j\omega L\ \mathbf{I}$$

Example

$$i(t) = 1\mu\text{A} \cos(2\pi\ 9.15\ 10^7 t + 30°)$$

$$L = 1\mu\text{H}$$

- What is **I**?
- What is **V**?
- What is *v(t)*?

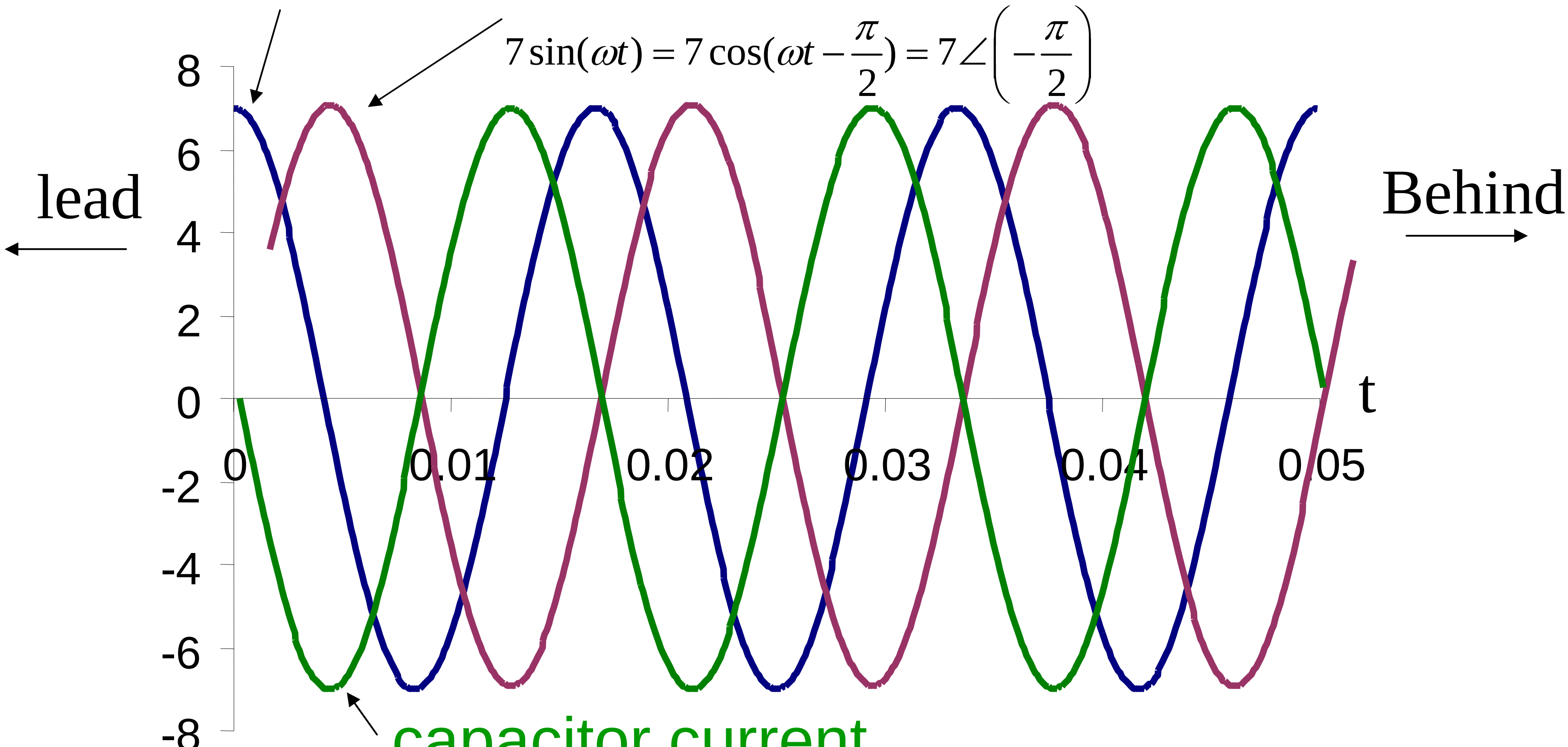
Phase

Voltage

$7\cos(\omega t) = 7\angle 0°$

inductor current

$7\sin(\omega t) = 7\cos(\omega t - \frac{\pi}{2}) = 7\angle\left(-\frac{\pi}{2}\right)$



capacitor current

$-7\sin(\omega t) = 7\cos(\omega t + \frac{\pi}{2}) = 7\angle\left(+\frac{\pi}{2}\right)$

Phasor Diagrams

- A phasor diagram is just a graph of several phasors on the complex plane (using real and imaginary axes).
- A phasor diagram helps to visualize the relationships between currents and voltages.
- Capacitor: I leads V by 90^o
- Inductor: V leads I by 90^o

Impedance

- AC steady-state analysis using phasors allows us to express the relationship between current and voltage using a formula that looks likes Ohm's law:

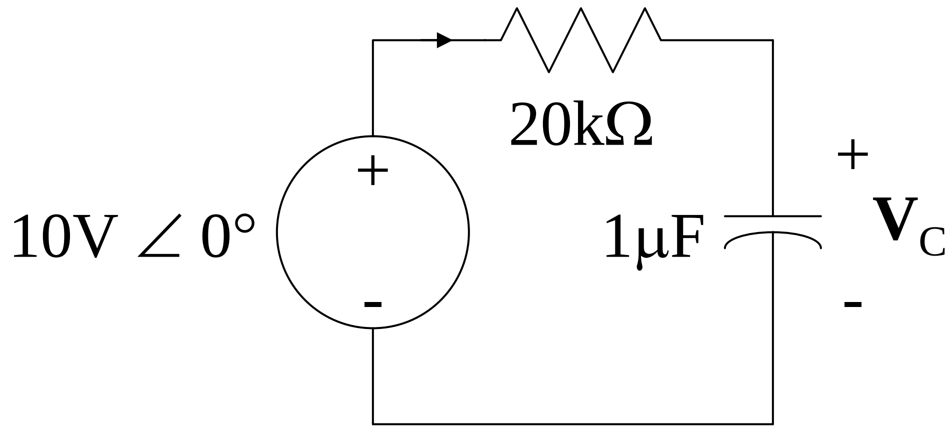
$$\mathbf{V} = \mathbf{I}\,\mathbf{Z}$$

- **Z** is called impedance.

Some Thoughts on Impedance

- Impedance depends on the frequency ω.
- Impedance is (often) a complex number.
- Impedance allows us to use the same solution techniques for AC steady state as we use for DC steady state.

Example: Single Loop Circuit



f=60 Hz, $\mathbf{V}_C$=?

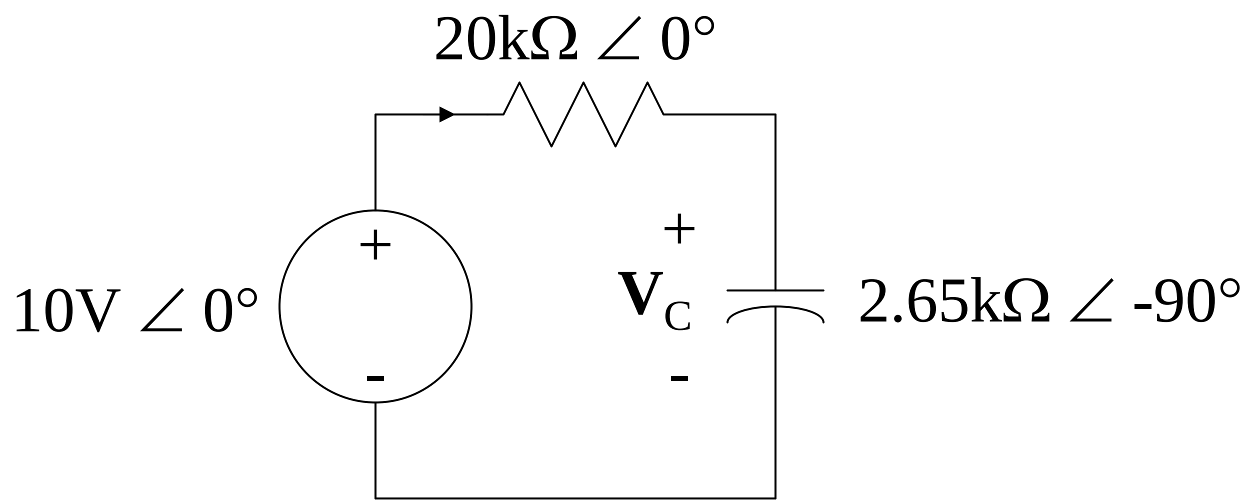
How do we find $\mathbf{V}_C$?

First compute impedances for resistor and capacitor:

$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ ∠ 0°

$\mathbf{Z}_C$ = 1/j (2πf x 1μF) = 2.65kΩ ∠ -90°

Impedance Example

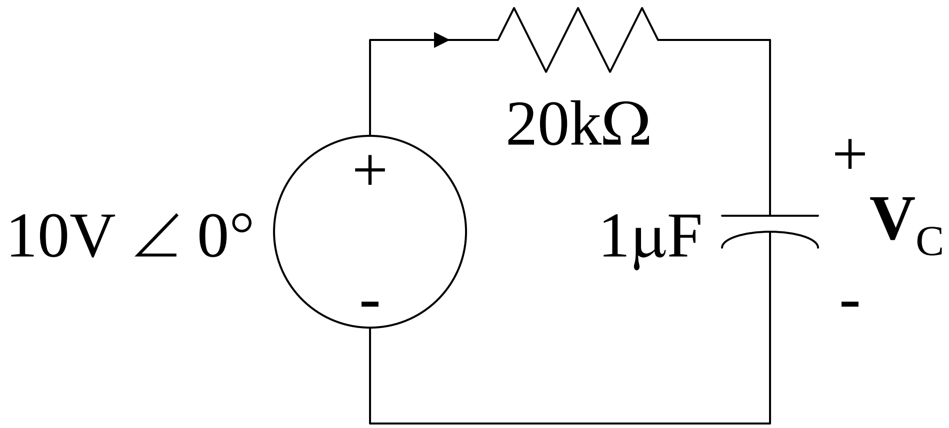


Now use the voltage divider to find $\mathbf{V}_C$:

$$\mathbf{V}_C = 10\text{V} \angle 0° \left(\frac{2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right)$$

$$\mathbf{V}_C = 1.31\text{V} \angle -82.4°$$

What happens when ω changes?



$\omega = 10$

Find $\mathbf{V}_C$

Circuit Analysis Using Complex Impedances

- Suitable for AC steady state.
- KVL

$$v_1(t) + v_2(t) + v_3(t) = 0$$

$$V_1 \cos\left(\omega t + \theta_1\right) + V_2 \cos\left(\omega t + \theta_2\right) + V_3 \cos\left(\omega t + \theta_3\right) = 0$$

$$\mathrm{Re}\left[V_1 e^{j(\omega t + \theta_1)} + V_2 e^{j(\omega t + \theta_2)} + V_3 e^{j(\omega t + \theta_3)}\right] = 0$$

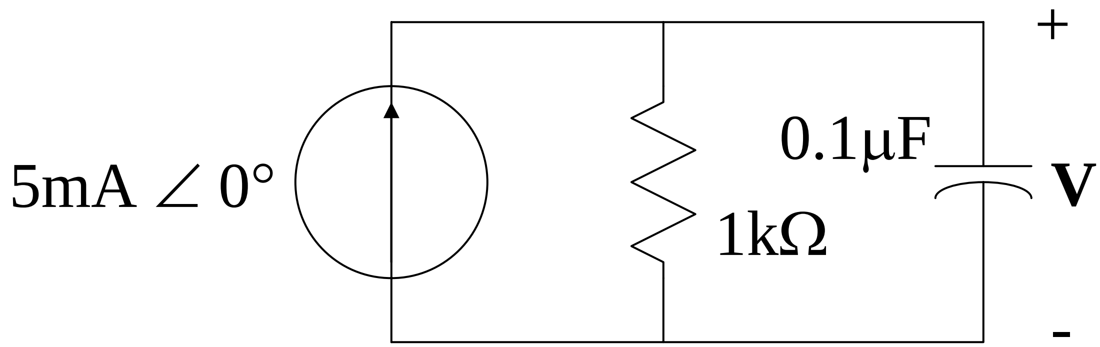
Phasor Form KVL

$$V_1 e^{j(\theta_1)} + V_2 e^{j(\theta_2)} + V_3 e^{j(\theta_3)} = 0$$

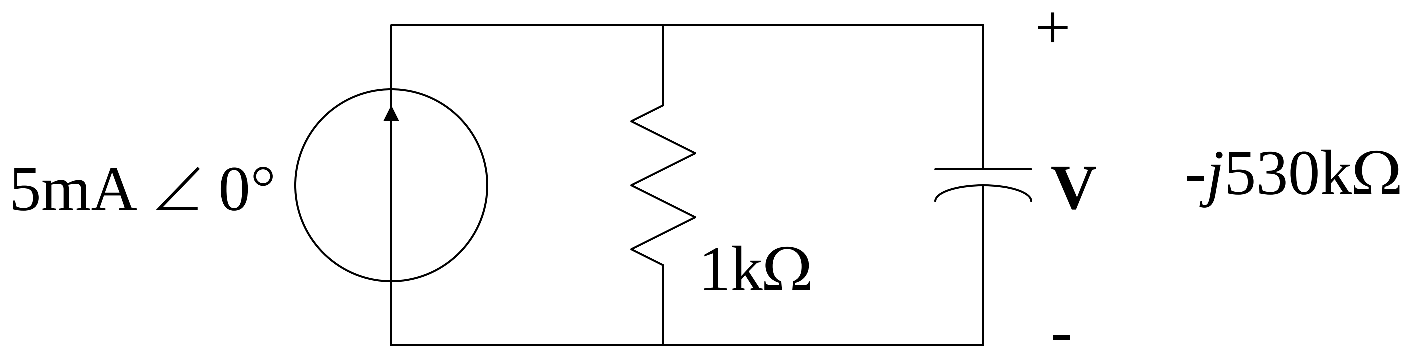
$$\mathbf{V_1} + \mathbf{V_2} + \mathbf{V_3} = 0$$

- Phasor Form KCL $\quad \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} = 0$
- Use complex impedances for inductors and capacitors and follow same analysis as in chap 2.

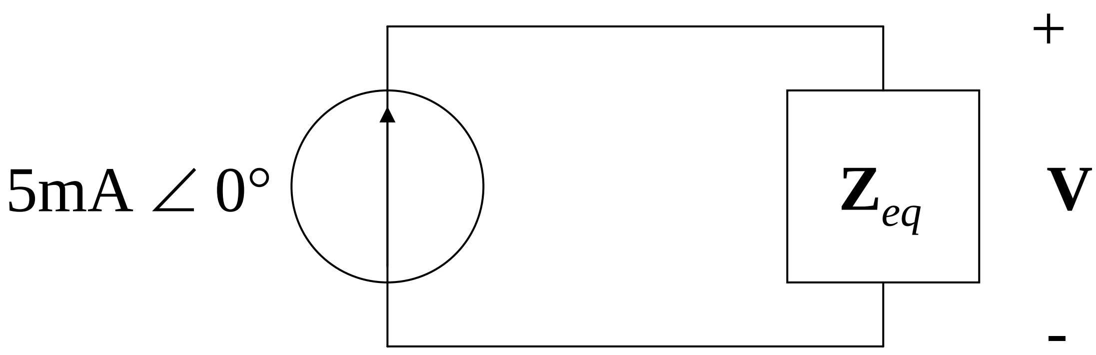
Steady-State AC Analysis



Find *v*(*t*) for ω=2π 3000



Find the Equivalent Impedance



$$\mathbf{Z}_{eq} = \frac{1000(-j530)}{1000 - j530} = \frac{10^3\angle 0° \times 530\angle -90°}{1132\angle -27.9°}$$

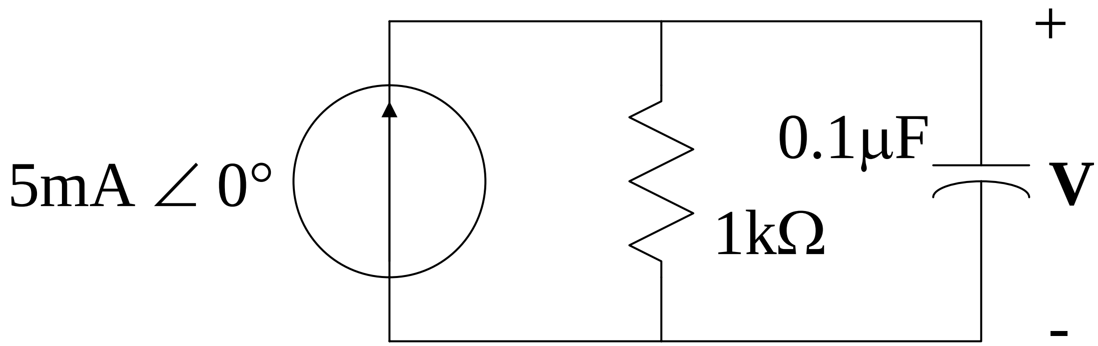
$$\mathbf{Z}_{eq} = 468.2\Omega\angle -62.1°$$

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = 5\text{mA}\angle 0° \times 468.2\Omega\angle -62.1°$$

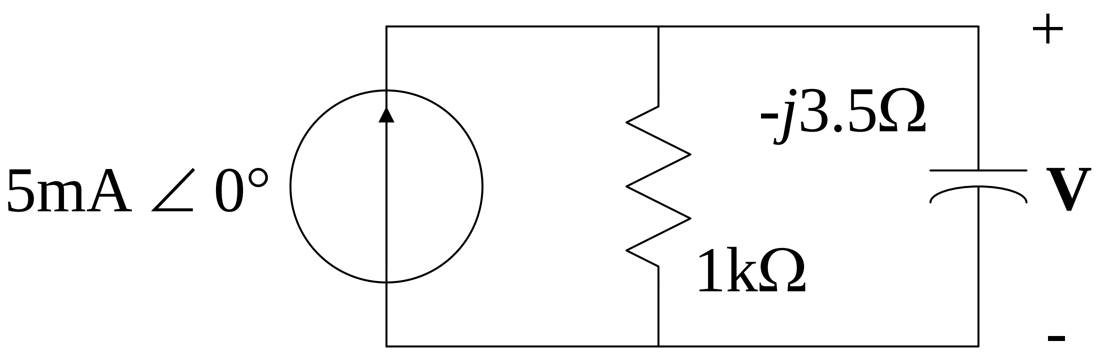
$$\mathbf{V} = 2.34\text{V}\angle -62.1°$$

$$v(t) = 2.34\text{V}\cos(2\pi 3000\text{t} - 62.1°)$$

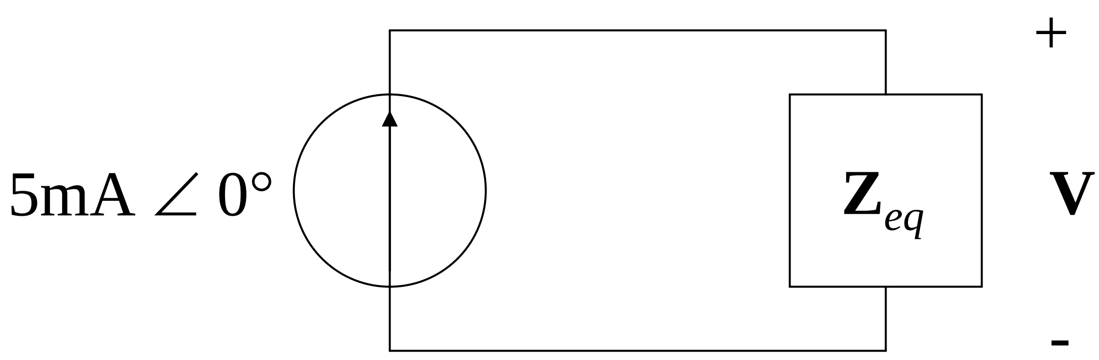
Change the Frequency



Find $v(t)$ for $\omega = 2\pi\ 455000$



Find an Equivalent Impedance



$$\mathbf{Z}_{eq} = \frac{1000(-j3.5)}{1000 - j3.5} = \frac{10^3 \angle 0° \times 3.5 \angle -90°}{1000 \angle -0.2°}$$

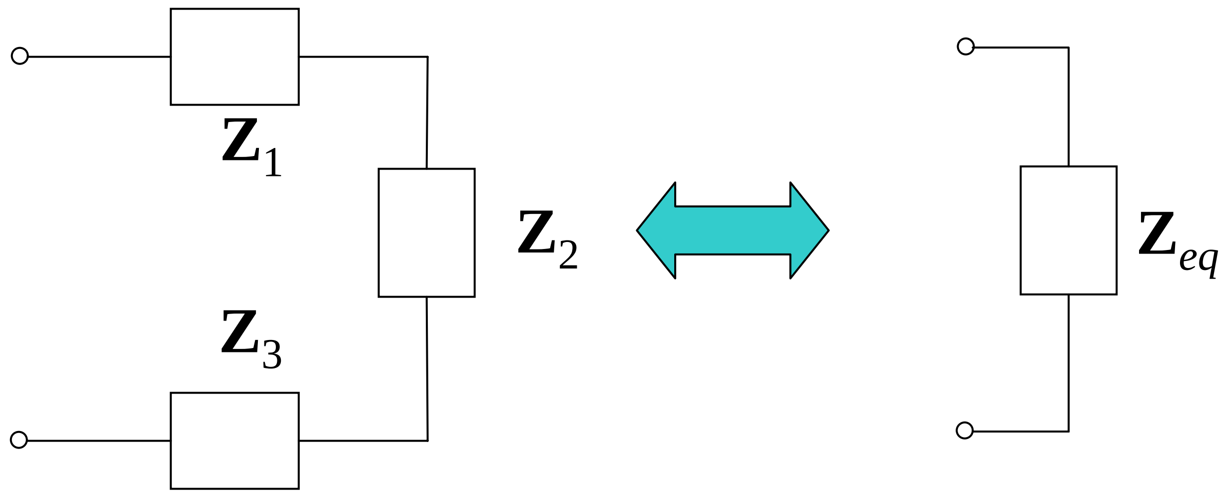
$$\mathbf{Z}_{eq} = 3.5\Omega \angle -89.8°$$

$$\mathbf{V} = \mathbf{IZ}_{eq} = 5\text{mA} \angle 0° \times 3.5\Omega \angle -89.8°$$

$$\mathbf{V} = 17.5\text{mV} \angle -89.8°$$

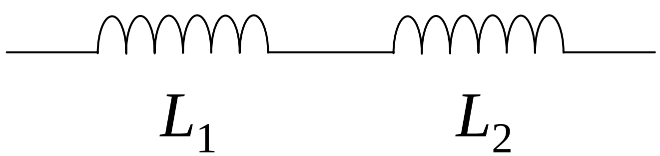
$$v(t) = 17.5\text{mV} \cos(2\pi 455000\text{t} - 89.8°)$$

Series Impedance

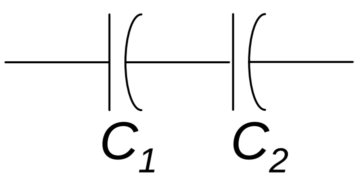


$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$

For example:

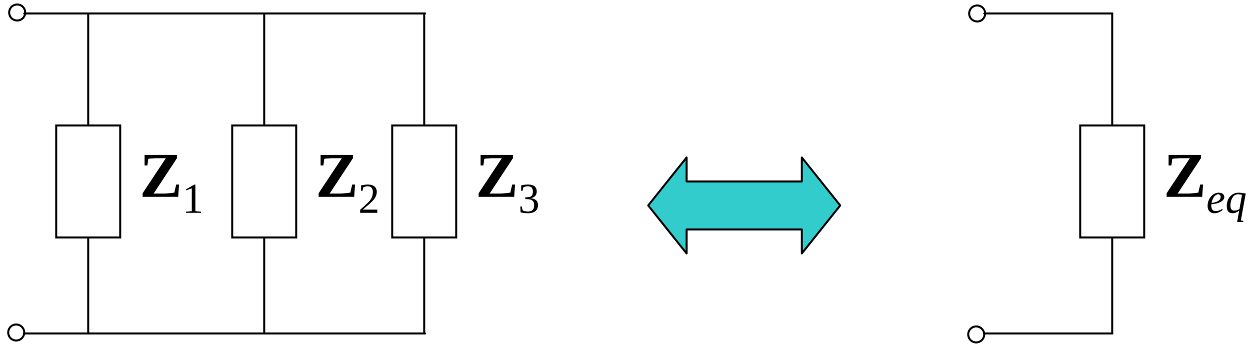


$$\mathbf{Z}_{eq} = j\omega(L_1 + L_2)$$



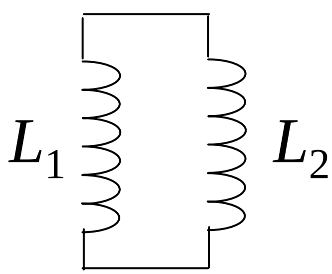
$$\mathbf{Z}_{eq} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}$$

Parallel Impedance

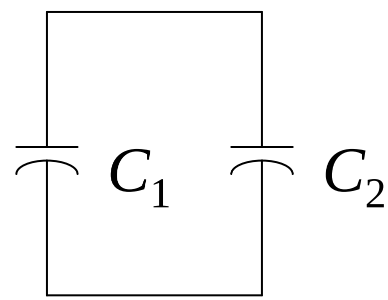


$$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3$$

For example:

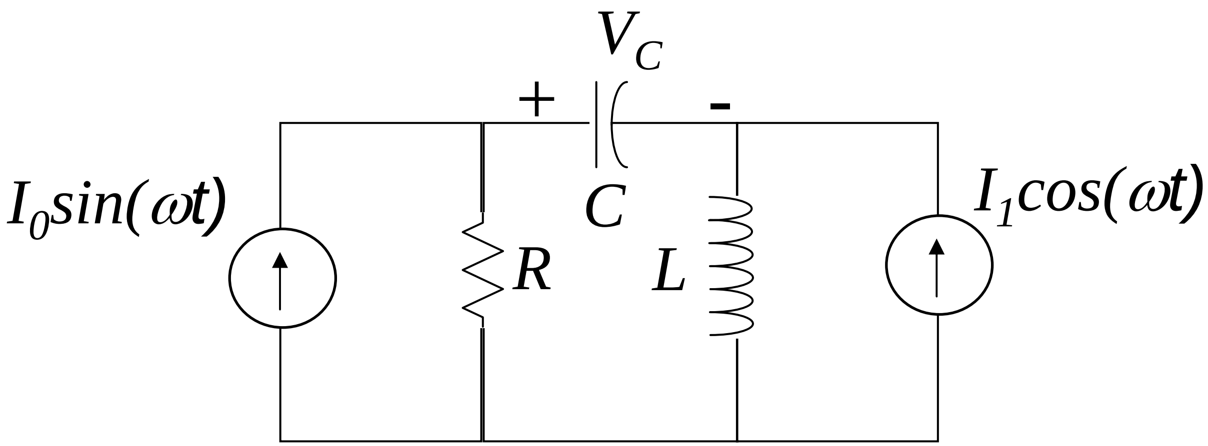


$$\mathbf{Z}_{eq} = j\omega \frac{L_1 L_2}{(L_1 + L_2)}$$



$$\mathbf{Z}_{eq} = \frac{1}{j\omega(C_1 + C_2)}$$

Steady-State AC Node-Voltage Analysis



- Try using Thevinin equivalent circuit.
- What happens if the sources are at different frequencies?

Resistor I-V relationship

$v_R = i_R R$ …………$\mathbf{V}_R = \mathbf{I}_R R$ where R is the resistance in ohms,

$\mathbf{V}_R$ = phasor voltage, $\mathbf{I}_R$ = phasor current

(boldface indicates complex quantity)

Capacitor I-V relationship

$i_C = C dv_C/dt$ ....Phasor current $\mathbf{I}_C$ = phasor voltage $\mathbf{V}_C$ / capacitive impedance $\mathbf{Z}_C$: → $\mathbf{I}_C = \mathbf{V}_C/\mathbf{Z}_C$

where $\mathbf{Z}_C = 1/j\omega C$, $j = (-1)^{1/2}$ and boldface indicates complex quantity

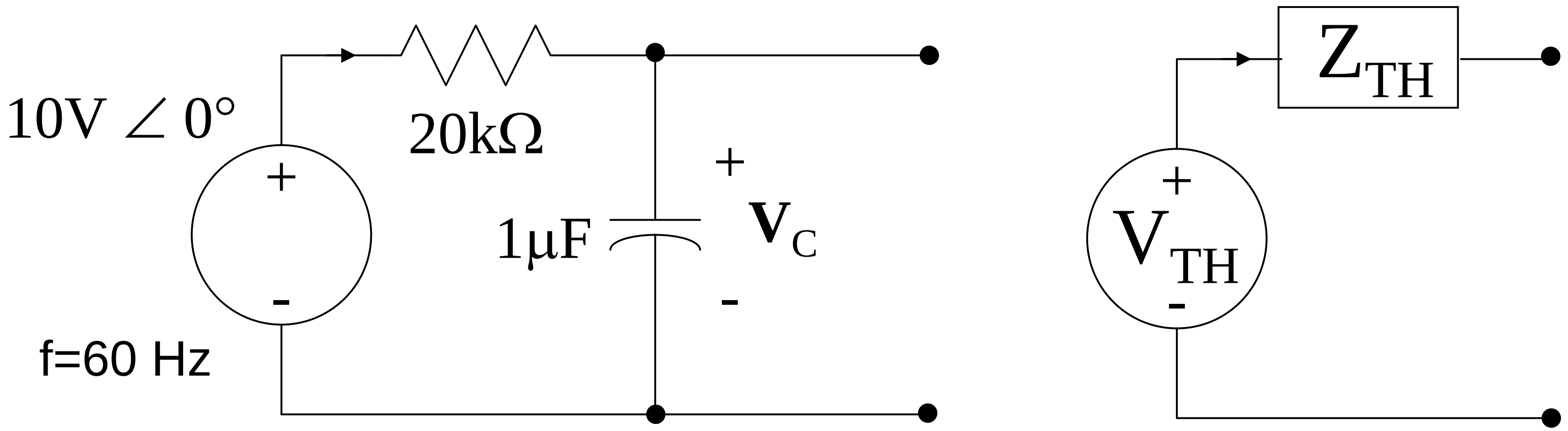
Inductor I-V relationship

$v_L = L di_L/dt$ ....Phasor voltage $\mathbf{V}_L$ = phasor current $\mathbf{I}_L$/ inductive impedance $\mathbf{Z}_L$ → $\mathbf{V}_L = \mathbf{I}_L \mathbf{Z}_L$

where $\mathbf{Z}_L = j\omega L$, $j = (-1)^{1/2}$ and boldface indicates complex quantity

R	C	L
$v_0(t) = V_0 \cos(\omega t)$	$v_0(t) = V_0 \cos(\omega t)$	$v_0(t) = V_0 \cos(\omega t)$
$\vec{V}_0 = V_0 \angle 0^\circ$	$\vec{V}_0 = V_0 \angle 0^\circ$	$\vec{V}_0 = V_0 \angle 0^\circ$
$i_0(t) = \frac{V_0}{R} \cos(\omega t)$	$i_0(t) = -\omega C V_0 \sin(\omega t)$	$i_0(t) = \frac{V_0}{\omega L} \sin(\omega t)$
$\vec{I}_0 = \frac{V_0}{R} \angle 0^\circ$	$\vec{I}_0 = \omega C V_0 \angle 90^\circ$	$\vec{I}_0 = \frac{V_0}{\omega L} \angle -90^\circ$

Thevenin Equivalent



$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ ∠ 0°

$\mathbf{Z}_C$ = 1/j (2πf x 1μF) = 2.65kΩ ∠ -90°

$$\mathbf{V}_{TH} = \mathbf{V}_{OC} = 10\text{V} \angle 0° \left(\frac{2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right) = 1.31 \angle -82.4$$

$$\mathbf{Z}_{TH} = \mathbf{Z}_R \,||\, \mathbf{Z}_C = ° \left(\frac{20\text{k}\Omega \angle 0° \cdot 2.65\text{k}\Omega \angle -90°}{2.65\text{k}\Omega \angle -90° + 20\text{k}\Omega \angle 0°} \right) = 2.62 \angle -82.4$$

Chapter 6

- OUTLINE
  - Frequency Response for Characterization
  - Asymptotic Frequency Behavior
  - Log magnitude vs log frequency plot
  - Phase vs log frequency plot
  - dB scale
  - Transfer function example

Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunications pioneer.
  - The bel is a logarithmic measure. The number of bels for a given ratio of power levels is calculated by taking the logarithm, to the base 10, of the ratio.
  - one bel corresponds to a ratio of 10:1.
  - $B = \log_{10}(P_1/P_2)$ where P_1 and P_2 are power levels.
- The bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.
  - $1\text{dB} = 10 \log_{10}(P_1/P_2)$
- dB are used to measure
  - Electric power, Gain or loss of amplifiers, Insertion loss of filters.

Logarithmic Measure for Power

- To express a power in terms of decibels, one starts by choosing a reference power, $P_{reference}$, and writing

  Power P in decibels = $10 \log_{10}(P/P_{reference})$
- Exercise:
  - Express a power of 50 mW in decibels relative to 1 watt.
  - P (dB) $=10 \log_{10} (50 \times 10^{-3}) = -13$ dB
- Exercise:
  - Express a power of 50 mW in decibels relative to 1 mW.
  - P (dB) $=10 \log_{10} (50) = 17$ dB.
- dBm to express absolute values of power relative to a milliwatt.
  - dBm = $10 \log_{10}$ (power in milliwatts / 1 milliwatt)
  - 100 mW = 20 dBm
  - 10 mW = 10 dBm

Logarithmic Measures for Voltage or Current

From the expression for power ratios in decibels, we can readily derive the corresponding expressions for voltage or current ratios.

Suppose that the voltage V (or current I) appears across (or flows in) a resistor whose resistance is R. The corresponding power dissipated, P, is V^2/R (or I^2R). We can similarly relate the reference voltage or current to the reference power, as

$$P_{\text{reference}} = (V_{\text{reference}})^2/R \text{ or } P_{\text{reference}} = (I_{\text{reference}})^2 R.$$

Hence,

$$\text{Voltage, V in decibels} = 20\log_{10}(V/V_{\text{reference}})$$
$$\text{Current, I, in decibels} = 20\log_{10}(I/I_{\text{reference}})$$

Logarithmic Measures for Voltage or Current

Note that the voltage and current expressions are just like the power expression except that they have 20 as the multiplier instead of 10 because power is proportional to the square of the voltage or current.

Exercise: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery? Let $V_{reference}$ = 1.5. The ratio in decibels is

$$20 \log_{10}(9/1.5) = 20 \log_{10}(6) = 16 \text{ dB.}$$

Logarithmic Measures for Voltage or Current

The gain produced by an amplifier or the loss of a filter is often specified in decibels.

The input voltage (current, or power) is taken as the reference value of voltage (current, or power) in the decibel defining expression:

$$\text{Voltage gain in dB} = 20\log_{10}(V_{output}/V_{input})$$
$$\text{Current gain in dB} = 20\log_{10}(I_{output}/I_{input}$$
$$\text{Power gain in dB} = 10\log_{10}(P_{output}/P_{input})$$

Example: The voltage gain of an amplifier whose input is 0.2 mV and whose output is 0.5 V is

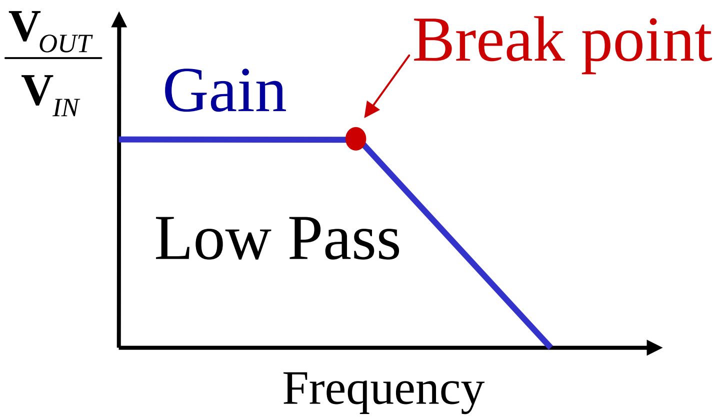
$$20\log_{10}(0.5/0.2\text{x}10^{-3}) = 68\text{ dB.}$$

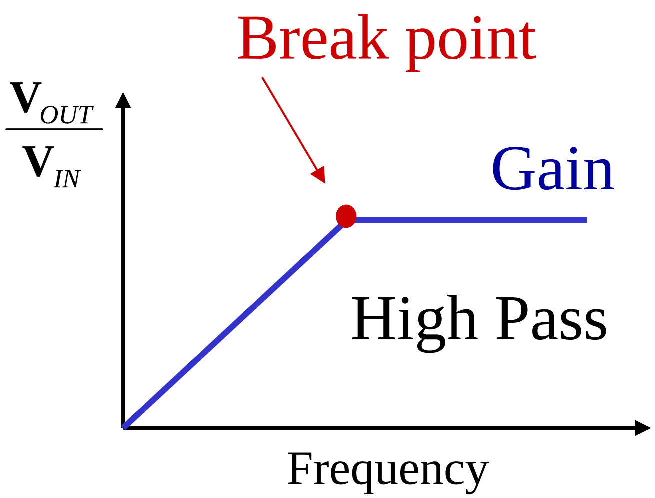
Bode Plot

- Plot of magnitude of transfer function vs. frequency
  - Both x and y scale are in log scale
  - Y scale in dB
- Log Frequency Scale
  - Decade → Ratio of higher to lower frequency = 10
  - Octave → Ratio of higher to lower frequency = 2

Frequency Response

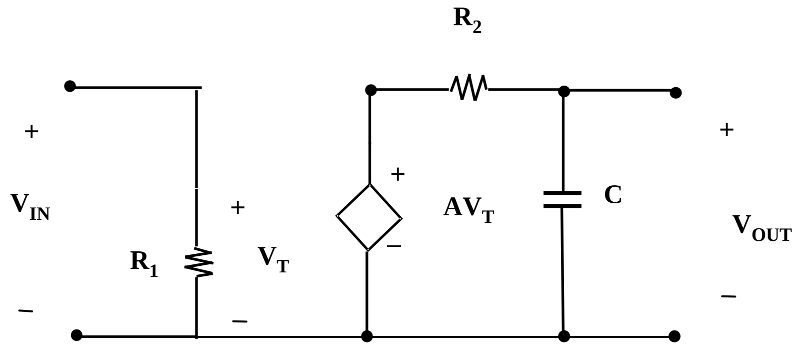
- The shape of the frequency response of the complex ratio of phasors $\mathbf{V}_{OUT}/\mathbf{V}_{IN}$ is a convenient means of classifying a circuit behavior and identifying key parameters.





FYI: These are log ratio vs log frequency plots

Example Circuit



$$TransferFunction = \frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}}$$

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{AZ_c}{Z_R + Z_c}$$

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{A(1/jwC)}{R_2 + 1/j\omega C)} = \frac{A}{(1 + j\omega R_2 C)}$$

A = 100

R_1 = 100,000 Ohms

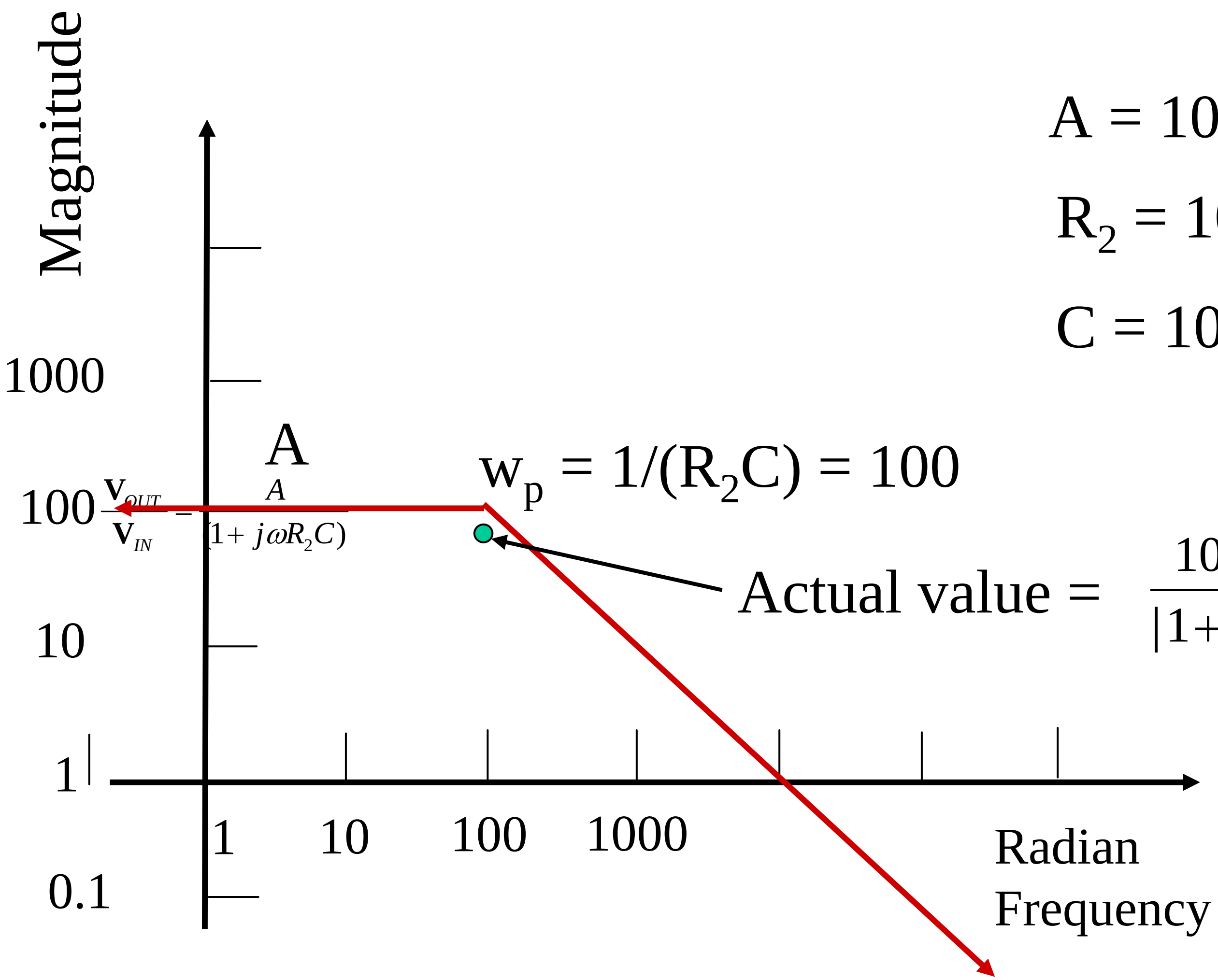
R_2 = 1000 Ohms

C = 10 uF

Break Point Values

- **When dealing with resonant circuits it is convenient to refer to the frequency difference between points at which the power from the circuit is half that at the peak of resonance.**
- **Such frequencies are known as "half-power frequencies", and the power output there referred to the peak power (at the resonant frequency) is**
- $\mathbf{10\log_{10}(P_{half\text{-}power}/P_{resonance}) = 10\log_{10}(1/2) = -3\ dB.}$

Example: Circuit in Slide #3 Magnitude



A = 100

R_2 = 1000 Ohms

C = 10 uF

Actual value = $\frac{100}{|1+j|} = \frac{100}{\sqrt{2}}$

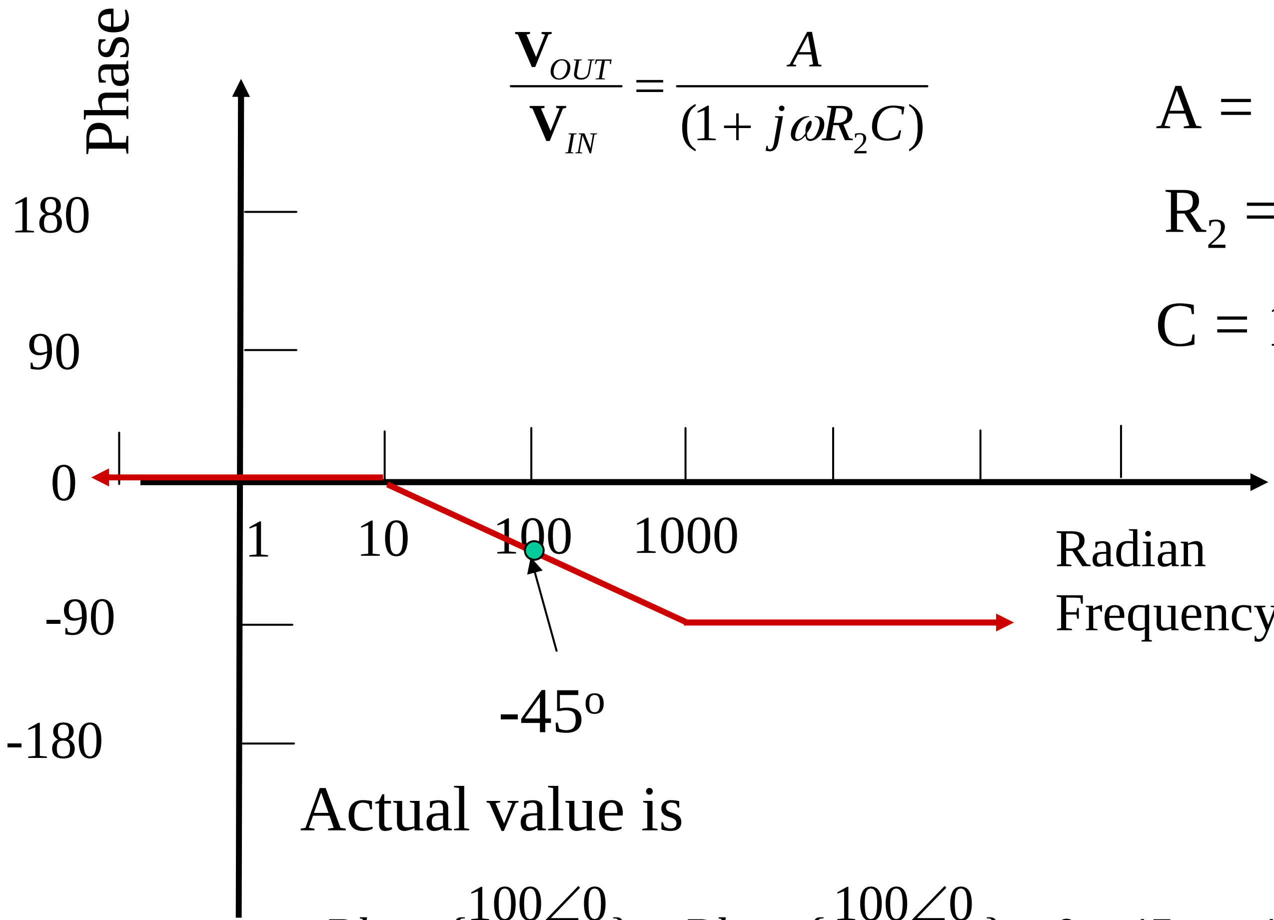
Example: Circuit in Slide #3 Phase

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{A}{(1 + j\omega R_2 C)}$$

A = 100

R_2 = 1000 Ohms

C = 10 uF



Actual value is

$$Phase\{\frac{100\angle 0}{|1+j|}\} = Phase\{\frac{100\angle 0}{\sqrt{2}\angle 45}\} = 0 - 45 = -45$$

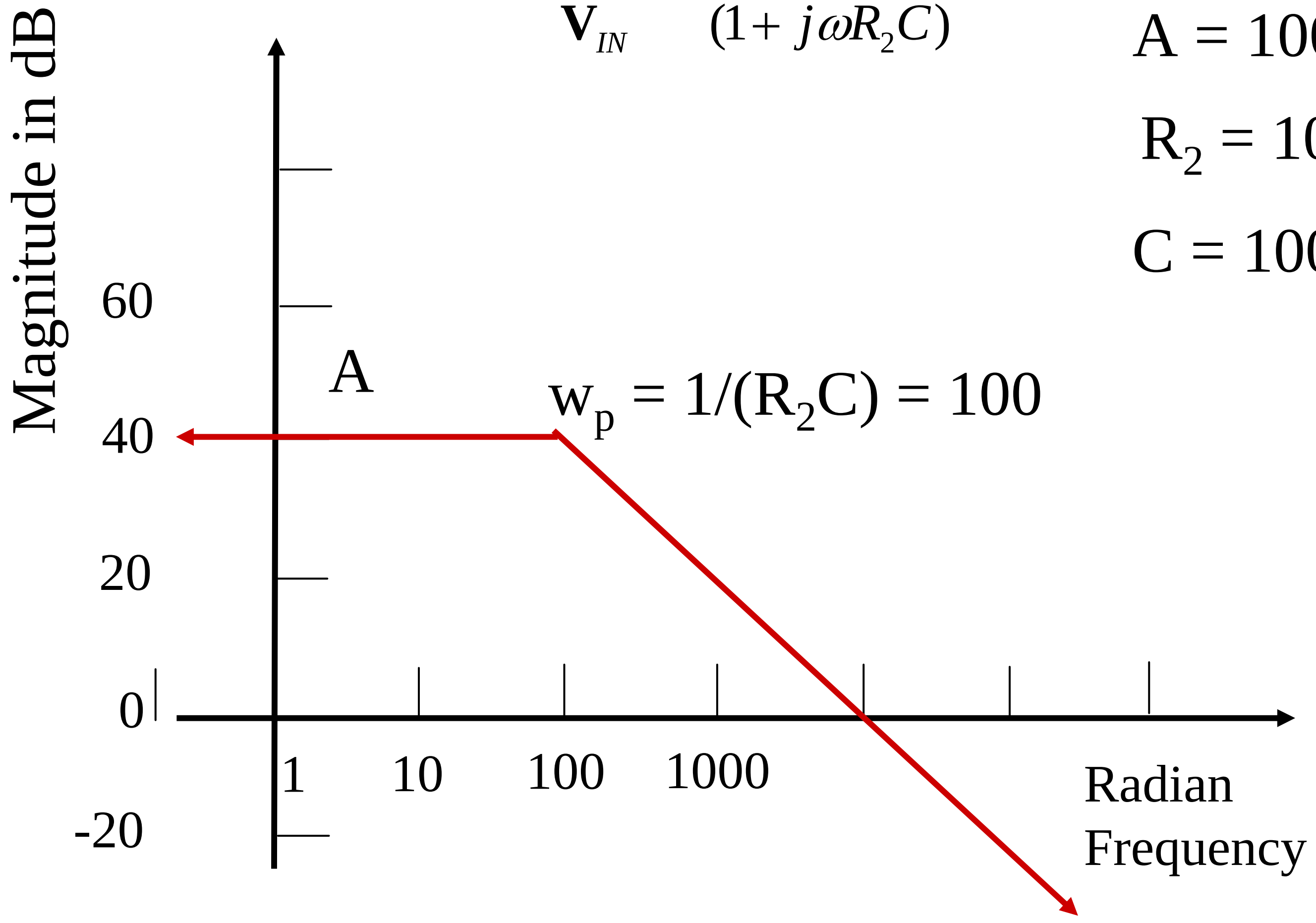
Bode Plot: Label as dB

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{A}{(1 + j\omega R_2 C)}$$

A = 100

R_2 = 1000 Ohms

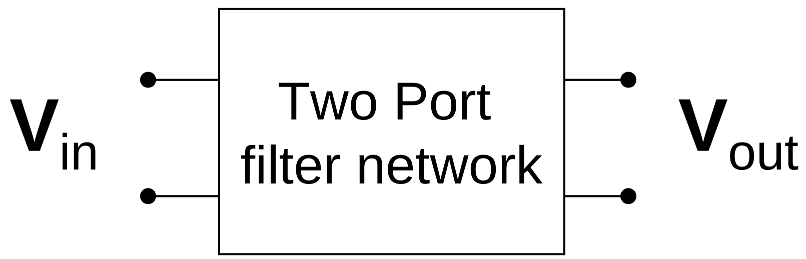
C = 100 uF



Note: Magnitude in dB = 20 $\log_{10}(V_{OUT}/V_{IN})$

Transfer Function

- Transfer function is a function of frequency
  - Complex quantity
  - Both magnitude and phase are function of frequency



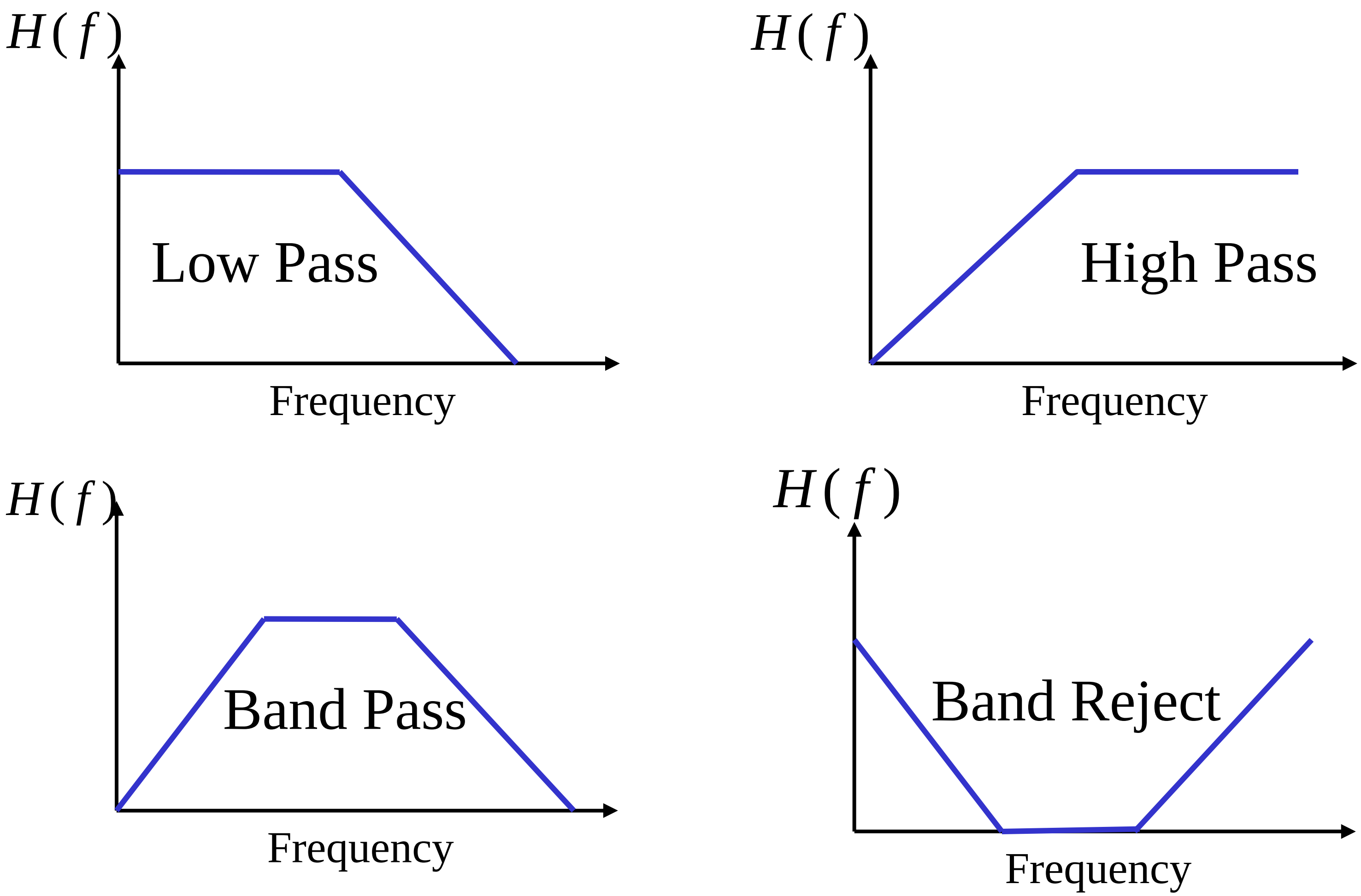
$$\mathbf{H}(f) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{out}}{V_{in}} \angle \left(\theta_{out} - \theta_{in}\right)$$

$$\mathbf{H(f)} = H(f) \angle \theta$$

Filters

- Circuit designed to retain a certain frequency range and discard others
  - *Low-pass*: pass low frequencies and reject high frequencies
  - *High-pass*: pass high frequencies and reject low frequencies
  - *Band-pass*: pass some particular range of frequencies, reject other frequencies outside that band
  - *Notch*: reject a range of frequencies and pass all other frequencies

Common Filter Transfer Function vs. Freq



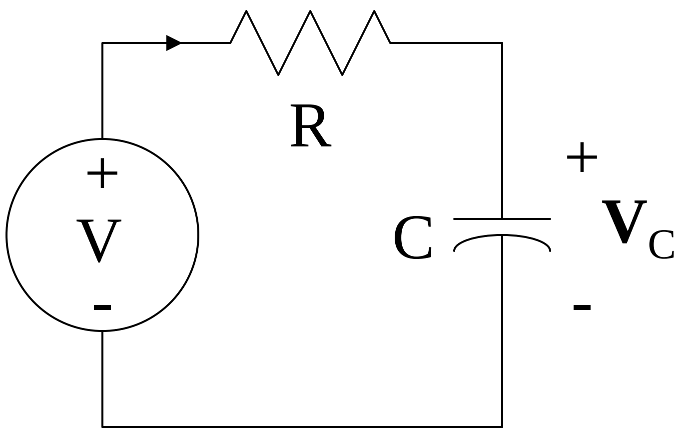
First-Order Lowpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V_C}}{\mathbf{V}} = \frac{1/(j\omega C)}{1/(j\omega C)+R} = \frac{1}{1+j\omega RC} = \frac{1}{\sqrt{1+(\omega RC)^2}} \angle -\tan^{-1}(\omega RC)$$

$$Let\ \omega_B = \frac{1}{RC} \quad and \quad f_B = \frac{1}{2\pi RC}$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

$$H(f) = \frac{1}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}}, \theta = -\tan^{-1}\left(\frac{f}{f_B}\right)$$



$$H(f_B) = \frac{1}{\sqrt{2}} = 2^{-1/2}$$

$$20\log_{10}\frac{H(f_B)}{H(0)} = 20(-\frac{1}{2})\log_{10} 2 = -3\ dB$$

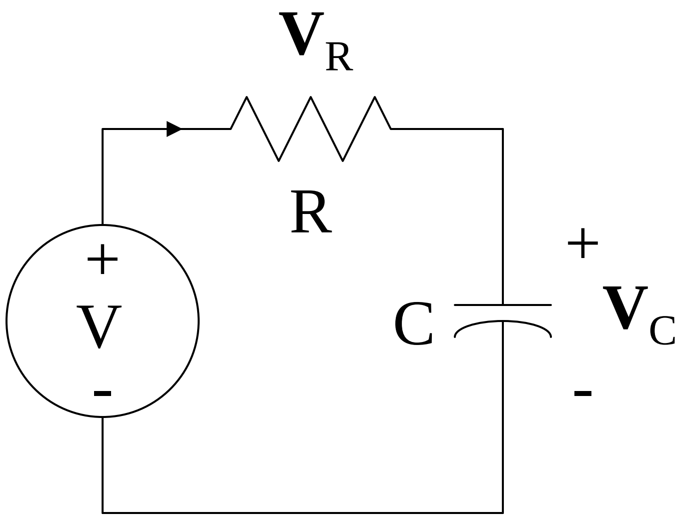
First-Order Highpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V}_R}{\mathbf{V}} = \frac{R}{1/(j\omega C)+R} = \frac{j\omega RC}{1+j\omega RC} = \frac{(\omega RC)}{\sqrt{1+(\omega RC)^2}}\angle\left[\frac{\pi}{2}-\tan^{-1}(\omega RC)\right]$$

$$H(f) = \frac{\left(\frac{f}{f_B}\right)}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}} \quad ,\theta = \frac{\pi}{2}-\tan^{-1}\left(\frac{f}{f_B}\right)$$

$$H(f_B) = \frac{1}{\sqrt{2}} = 2^{-1/2}$$

$$20\log_{10}\frac{H(f_B)}{H(0)} = 20(-\frac{1}{2})\log_{10}2 = -3\ dB$$



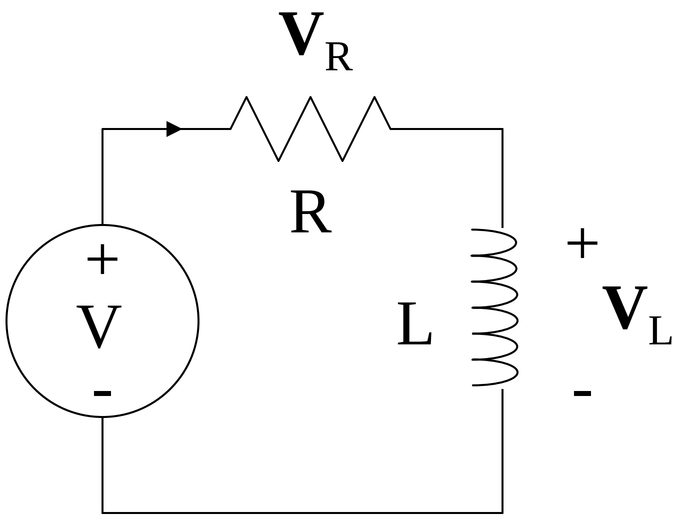
First-Order Lowpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V}_R}{\mathbf{V}} = \frac{1}{\frac{j\omega L}{R}+1} = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \angle -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$Let\ \omega_B = \frac{R}{L} \quad and \quad f_B = \frac{R}{2\pi L}$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

$$H(f) = \frac{1}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}}, \theta = -\tan^{-1}\left(\frac{f}{f_B}\right)$$



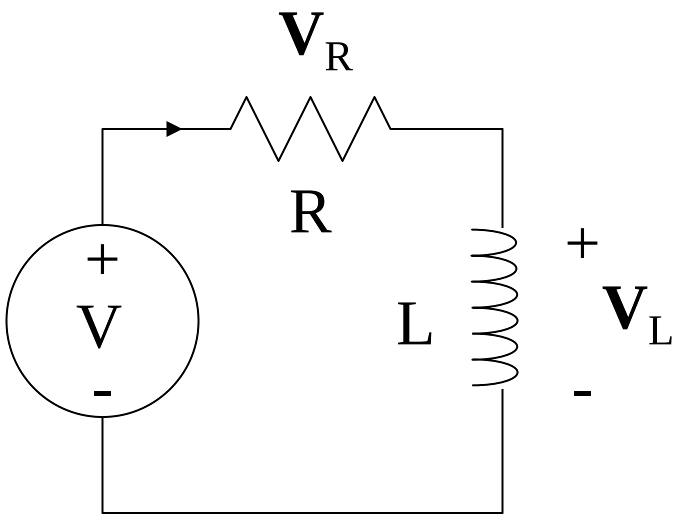
First-Order Highpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V}_L}{\mathbf{V}} = \frac{\frac{j\omega L}{R}}{\frac{j\omega L}{R}+1} = \frac{\frac{\omega L}{R}}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \angle\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R}\right)\right]$$

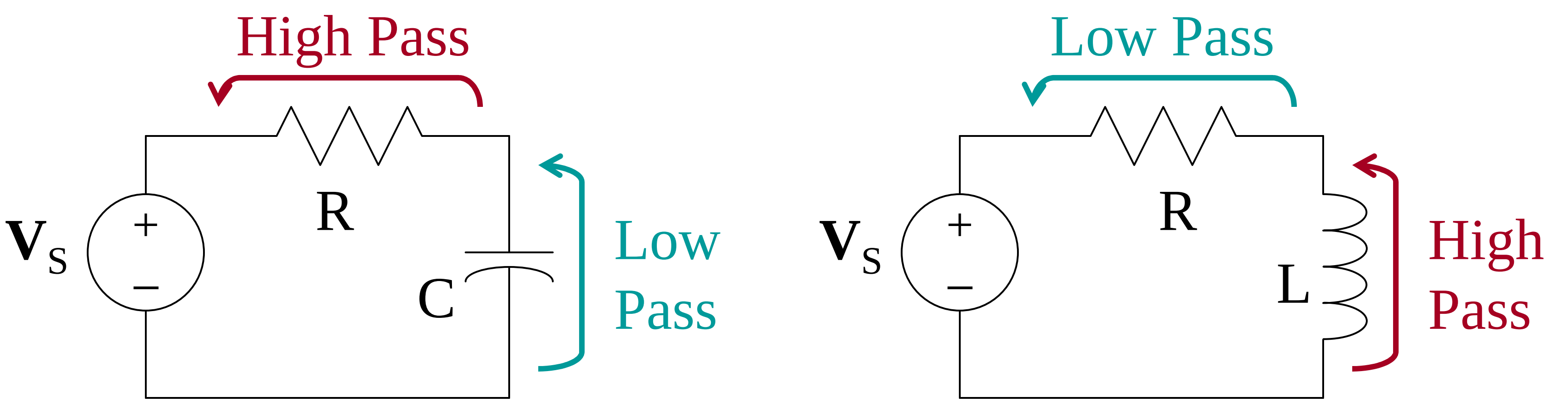
$$Let\ \omega_B = \frac{R}{L} \quad and \quad f_B = \frac{R}{2\pi L}$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

$$H(f) = \frac{\left(\frac{f}{f_B}\right)}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}}\ ,\theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{f_B}\right)$$



First-Order Filter Circuits



$H_R = R / (R + 1/j\omega C)$

$H_C = (1/j\omega C) / (R + 1/j\omega C)$

$H_R = R / (R + j\omega L)$

$H_L = j\omega L / (R + j\omega L)$

Change of Voltage or Current with A Change of Frequency

One may wish to specify the change of a quantity such as the output voltage of a filter when the frequency changes by a factor of 2 (an octave) or 10 (a decade).

For example, a single-stage RC low-pass filter has at frequencies above $\omega = 1/RC$ an output that changes at the rate -20dB per decade.

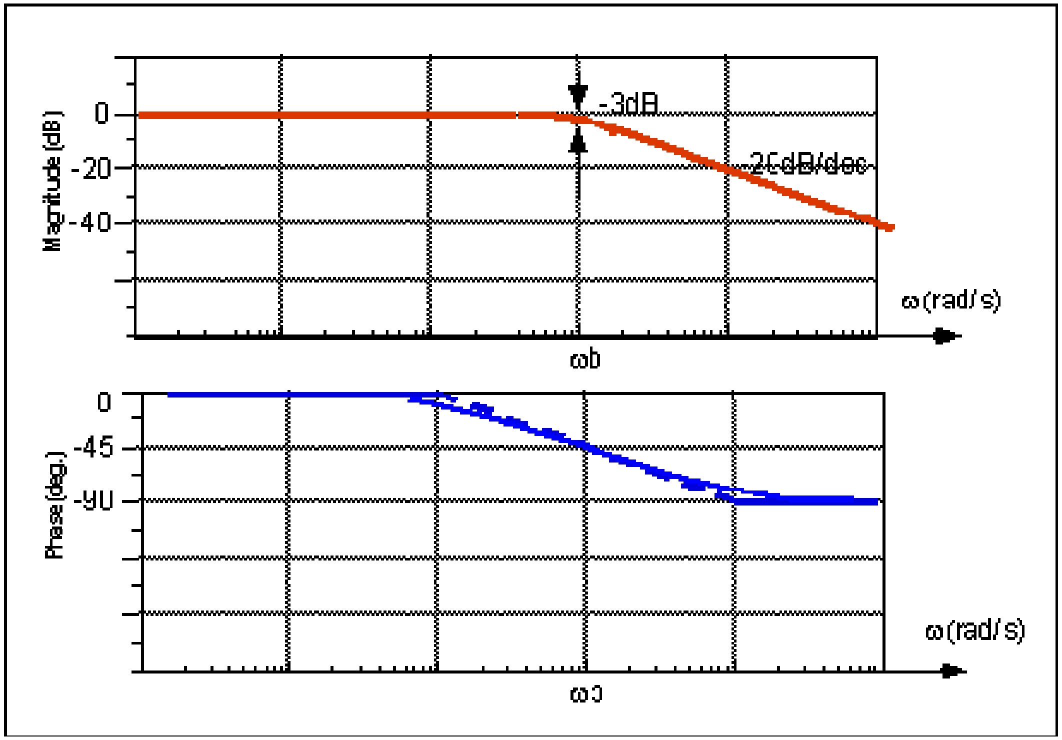
High-frequency asymptote of Lowpass filter

The high frequency asymptote of magnitude Bode plot assumes -20dB/decade slope

As $f \to \infty$

$$H(f) = \left(\frac{f}{f_B}\right)^{-1}$$

$$20\log_{10}\frac{H(10f_B)}{H(f_B)} = -20dB$$



Low-frequency asymptote of Highpass filter

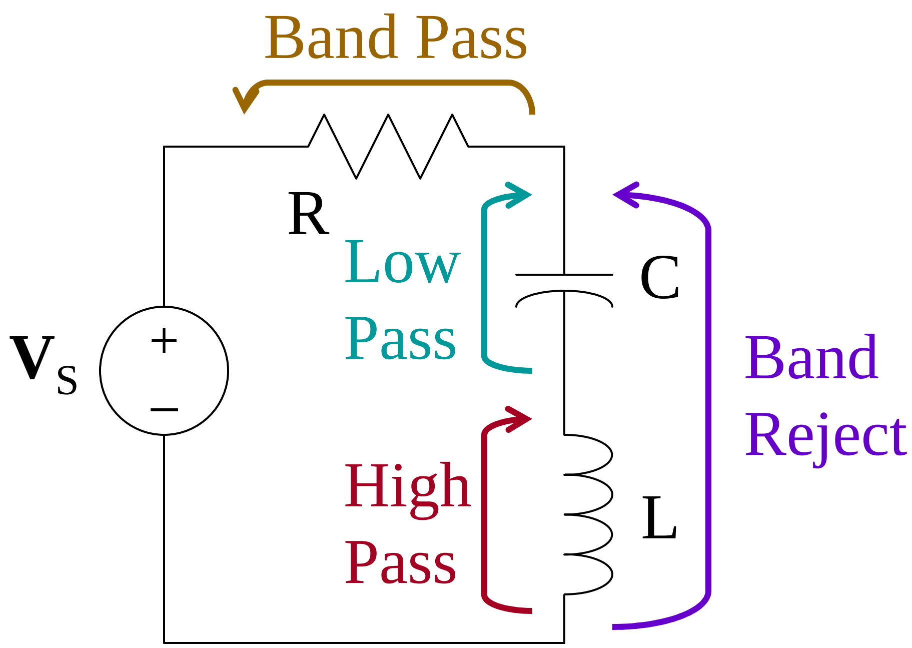
$$As \;\; f \to 0$$

$$H(f)=\frac{\left(\frac{f}{f_B}\right)}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}} \xrightarrow[f\to\infty]{} \left(\frac{f}{f_B}\right)$$

$$20\log_{10}\frac{H(f_B)}{H(0.1f_B)}=20dB$$

The low frequency asymptote of magnitude Bode plot assumes 20dB/decade slope

Second-Order Filter Circuits



$$\mathbf{Z} = R + 1/j\omega C + j\omega L$$

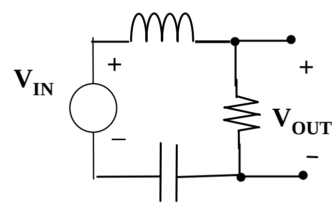
$$\mathbf{H}_{BP} = R / \mathbf{Z}$$

$$\mathbf{H}_{LP} = (1/j\omega C) / \mathbf{Z}$$

$$\mathbf{H}_{HP} = j\omega L / \mathbf{Z}$$

$$\mathbf{H}_{BR} = \mathbf{H}_{LP} + \mathbf{H}_{HP}$$

Series Resonance



Voltage divider

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{\mathbf{Z}_R}{\mathbf{Z}_L + \mathbf{Z}_R + \mathbf{Z}_C}$$

Substitute branch elements

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{R}{j\omega L + R + 1/j\omega C}$$

Arrange in resonance form

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{R}{R + j(\omega L - 1/\omega C)}$$

Maximum when $w^2 = 1/(LC)$

Resonance quality factor

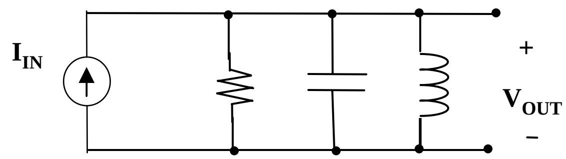
$$Q = \frac{\omega L}{R}$$

Ratio of reactance to resistance

Closely related to number of round trip cycles before 1/e decay.

Bandwidth is f_0/Q

Parallel Resonance



Admittance

$$\mathbf{V}_{OUT} = \frac{\mathbf{I}_S}{\mathbf{Y}_L + \mathbf{Y}_R + \mathbf{Y}_C}$$

Substitute branch elements

$$\mathbf{V}_{OUT} = \frac{\mathbf{I}_S}{1/j\omega L + 1/R + jwC}$$

Arrange in resonance form

$$\mathbf{V}_{OUT} = \frac{\mathbf{I}_S}{1/R + j(\omega C - 1/\omega L)}$$

Maximum = I_S/R when $w^2 = 1/(LC)$

Resonance quality factor

$$Q = \frac{\omega L}{R}$$

Ratio of reactance to resistance

Closely related to number of round trip cycles before 1/e decay.

Bandwidth is f_0/Q