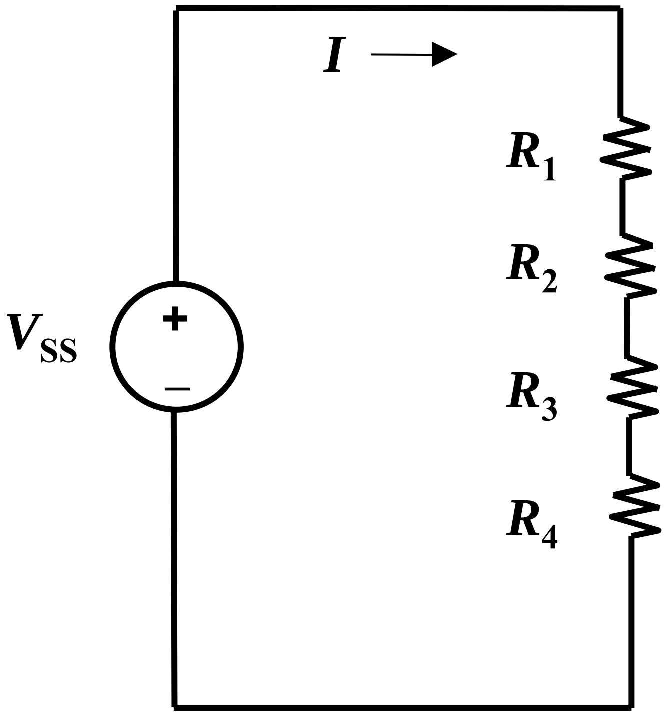
# Chapters 3 and 4

- Outline
  - Resistors in Series – Voltage Divider
  - Conductances in Parallel – Current Divider
  - Node-Voltage Analysis
  - Mesh-Current Analysis
  - Superposition
  - Thévenin equivalent circuits
  - Norton equivalent circuits
  - Maximum Power Transfer

# Resistors in Series

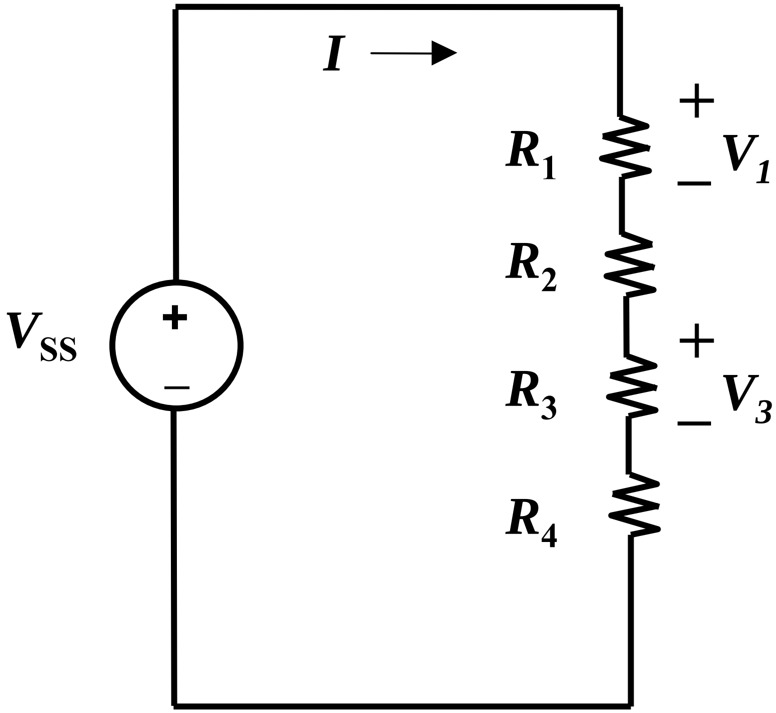
Consider a circuit with multiple resistors connected in series. Find their "equivalent resistance".




- KCL tells us that the same current (***I***) flows through every resistor
- KVL tells us

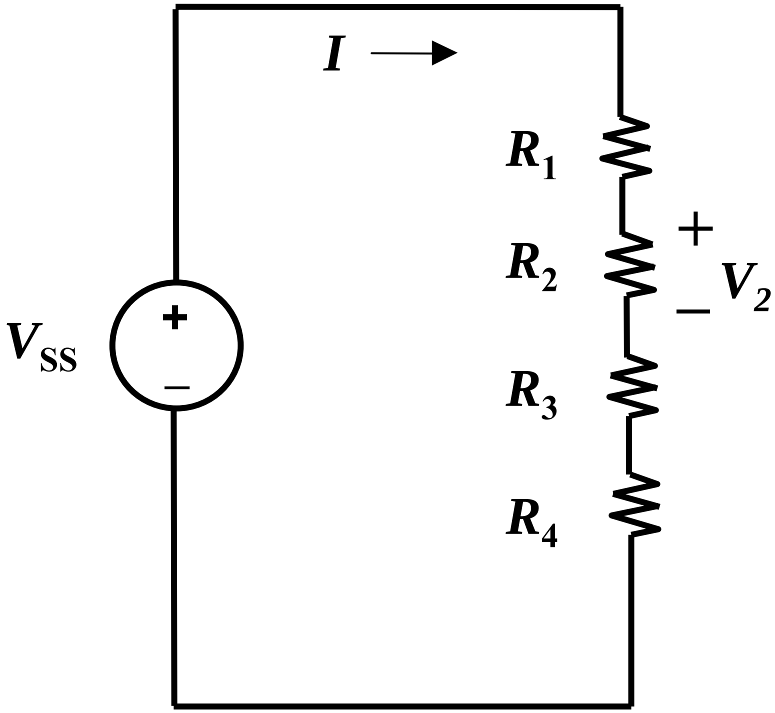
**Equivalent resistance of resistors in series is the sum**

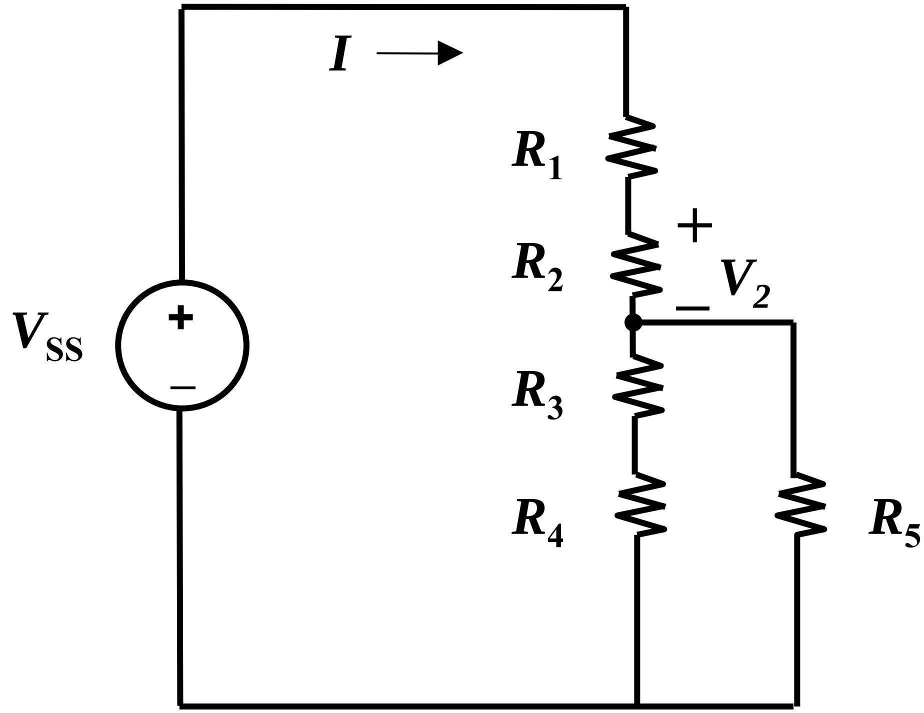
# Voltage Divider




$$I = V_{SS} / (R_1 + R_2 + R_3 + R_4)$$

# When can the Voltage Divider Formula be Used?




$$V_2 = \frac{R_2}{R_1+R_2+R_3+R_4} \cdot V_{SS}$$

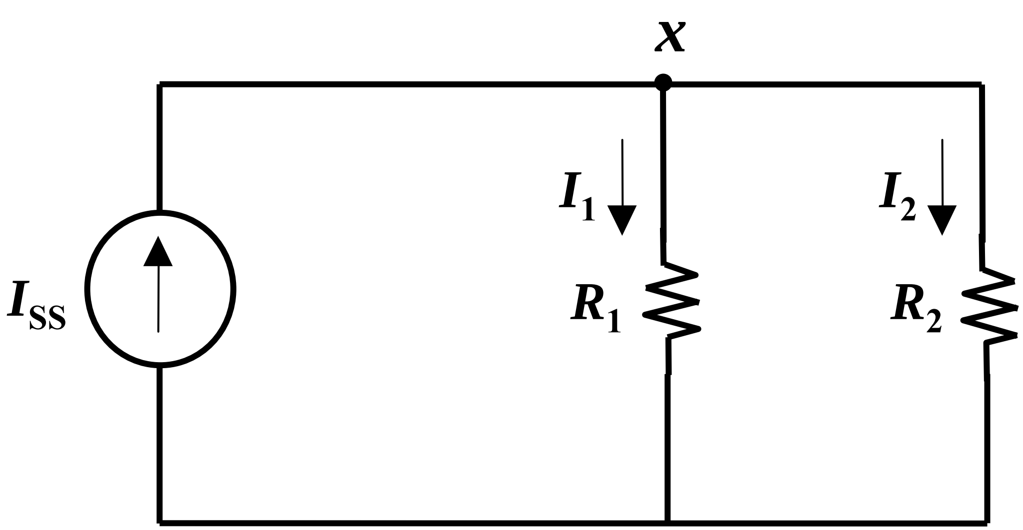
$$V_2 \neq \frac{R_2}{R_1+R_2+R_3+R_4} \cdot V_{SS}$$

Correct, if nothing else is connected to nodes

Why? What is $V_2$?

# Resistors in Parallel

Consider a circuit with two resistors connected in parallel.
Find their "equivalent resistance".



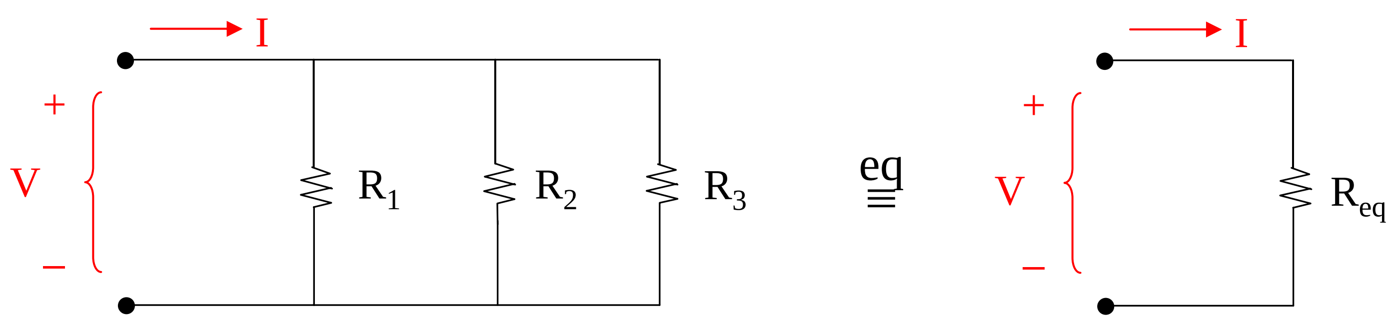

- KVL tells us that the same voltage is dropped across each resistor

$$V_x = I_1 R_1 = I_2 R_2$$

- KCL tells us

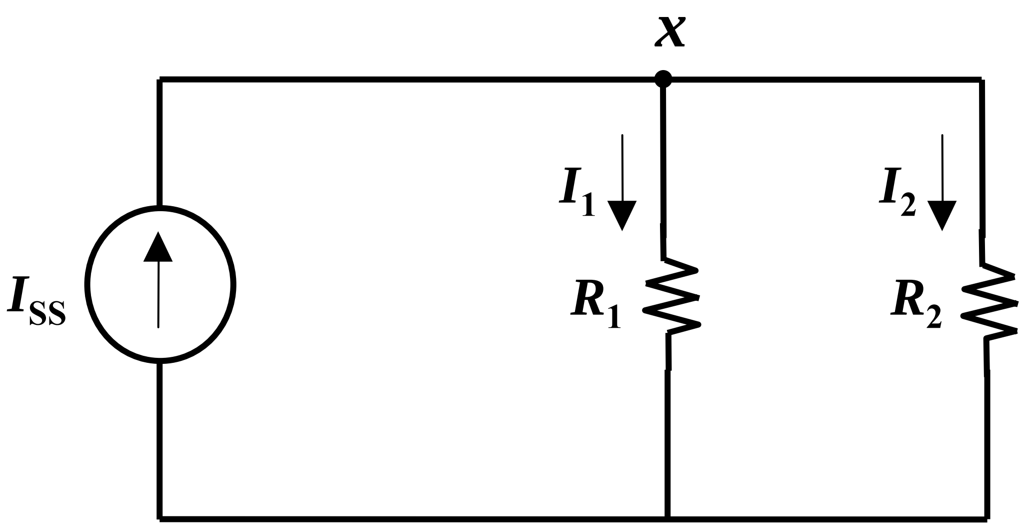
# General Formula for Parallel Resistors

What single resistance $R_{eq}$ is equivalent to three resistors in parallel?




**Equivalent <u>conductance</u> of resistors in parallel is the sum**

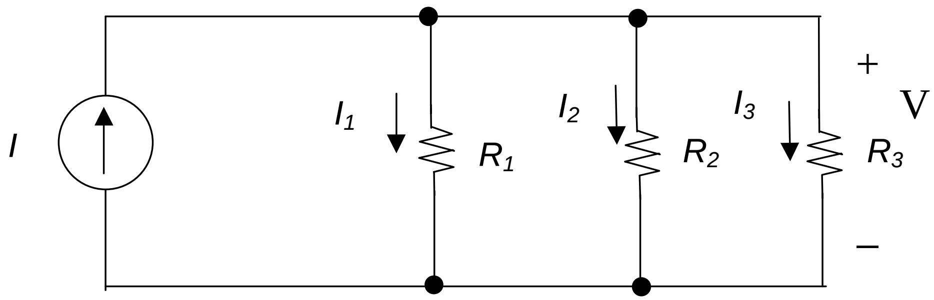
# Current Divider




$$V_x = I_1 R_1 = I_{SS} R_{eq}$$

# Generalized Current Divider Formula

Consider a current divider circuit with >2 resistors in parallel:



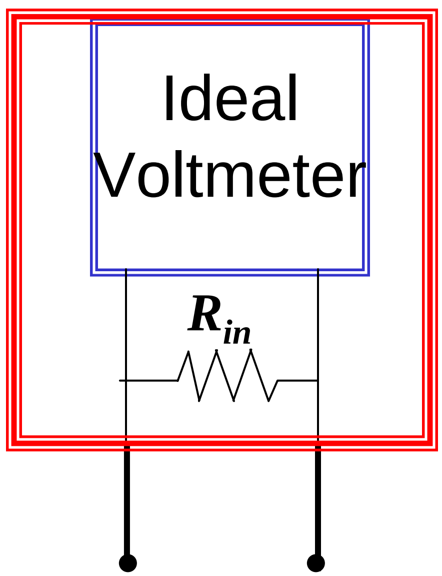

$$V = \frac{I}{\left(\frac{1}{R_1}\right)+\left(\frac{1}{R_2}\right)+\left(\frac{1}{R_3}\right)}$$

$$I_3 = \frac{V}{R_3} = I\left[\frac{1/R_3}{1/R_1 + 1/R_2 + 1/R_3}\right]$$

# Measuring Voltage

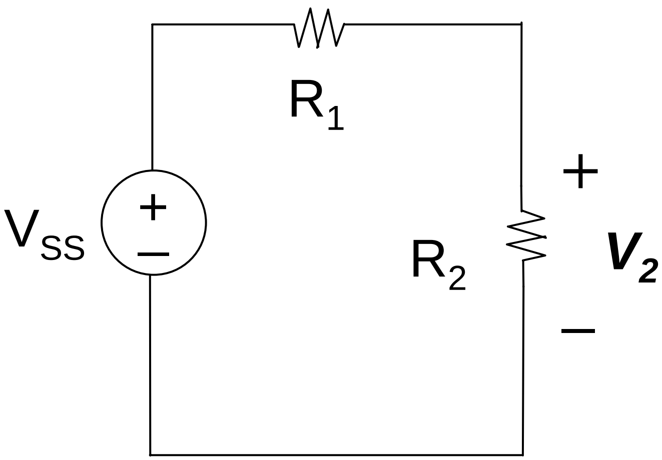
To measure the voltage drop across an element in a real circuit, insert a voltmeter (digital multimeter in voltage mode) **in parallel** with the element.

Voltmeters are characterized by their "voltmeter input resistance" ($R_{in}$). Ideally, this should be very high (typical value 10 MΩ)

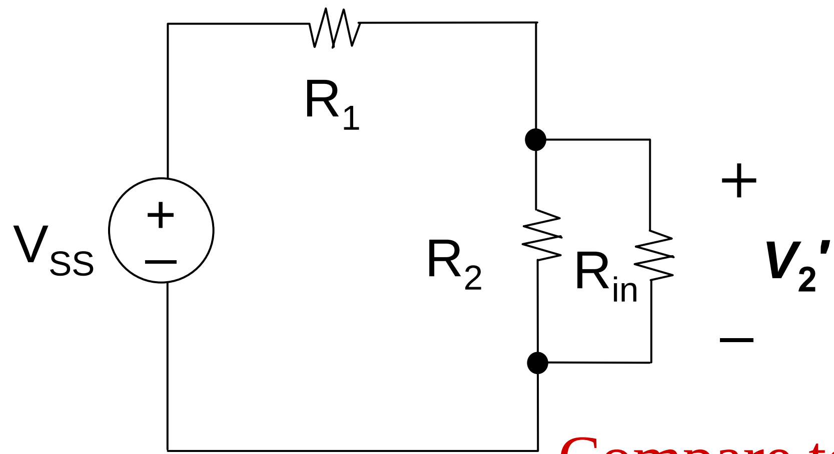


# Effect of Voltmeter

**undisturbed circuit**




**circuit with voltmeter inserted**




Compare to $R_2$

$$V_2 = V_{SS}\left[\frac{R_2}{R_1 + R_2}\right]$$

$$V_2' = V_{SS}\left[\frac{R_2 \| R_{in}}{R_2 \| R_{in} + R_1}\right]$$

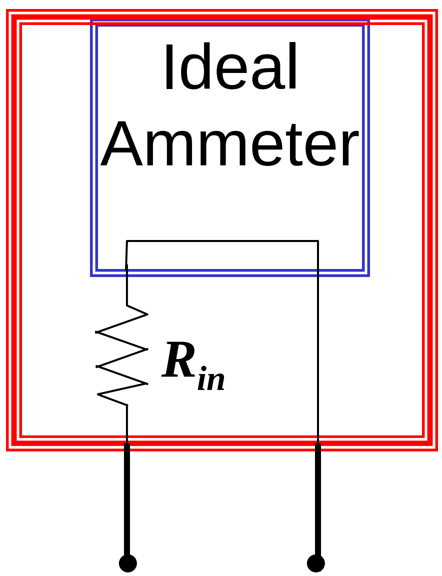
Example: $V_{SS} = 10\,V, R_2 = 100\,K, R_1 = 900\,K \Rightarrow V_2 = 1V$

$R_{in} = 10M,\ V_2' = ?$

# Measuring Current

To measure the current flowing through an element in a real circuit, insert an ammeter (digital multimeter in current mode) **in series** with the element.

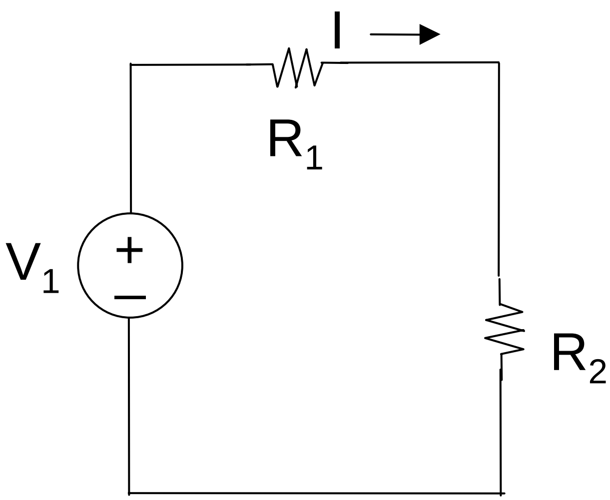
Ammeters are characterized by their "ammeter input resistance" ($R_{in}$). Ideally, this should be very low (typical value 1Ω).

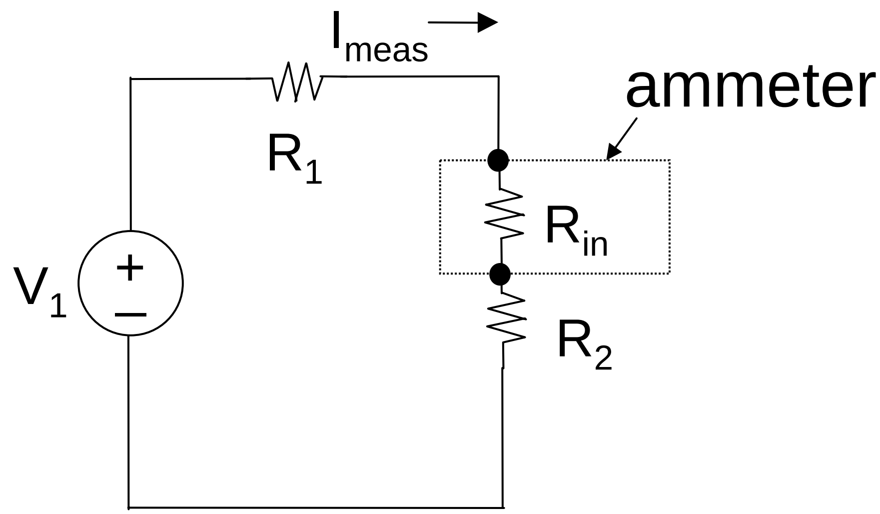


# Effect of Ammeter

Measurement error due to non-zero input resistance:

**undisturbed circuit** | **circuit with ammeter inserted**




$$I = \frac{V_1}{R_1 + R_2}$$

$$I_{meas} = \frac{V_1}{R_1 + R_2 + R_{in}}$$

Example: $V_1$ = 1 V, $R_1$= $R_2$ = 500 Ω, $R_{in}$ = 1Ω

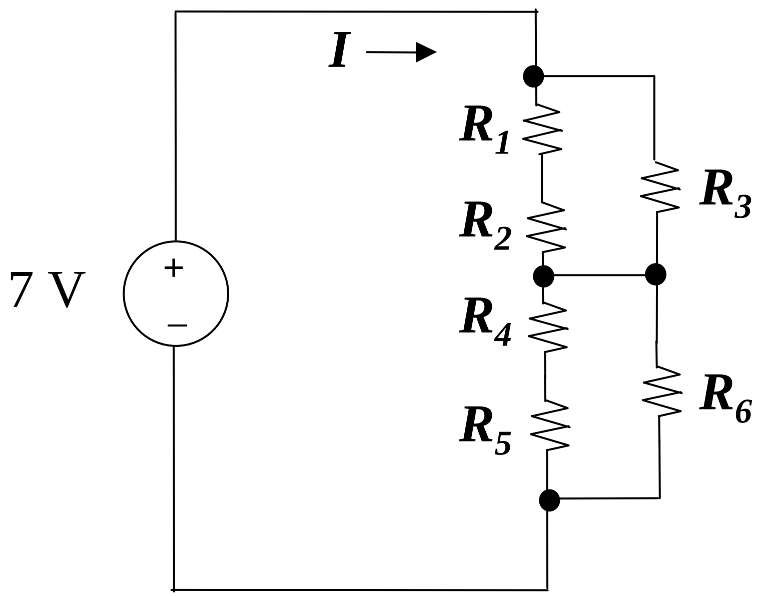
$$I = \frac{1V}{500\Omega + 500\Omega} = 1mA, \quad I_{meas} = ?$$

Compare to $R_2 + R_2$

# Using Equivalent Resistances

Simplify a circuit before applying KCL and/or KVL:

Example: Find $I$




$R_1 = R_2 = 3\ \text{k}\Omega$
$R_3 = 6\ \text{k}\Omega$

$R_4 = R_5 = 5\ \text{k}\Omega$
$R_6 = 10\ \text{k}\Omega$

# Node-Voltage Circuit Analysis Method

1. **Choose a reference node** ("ground")

   *Look for the one with the most connections!*

2. **Define unknown node voltages**

   *those which are not fixed by voltage sources*

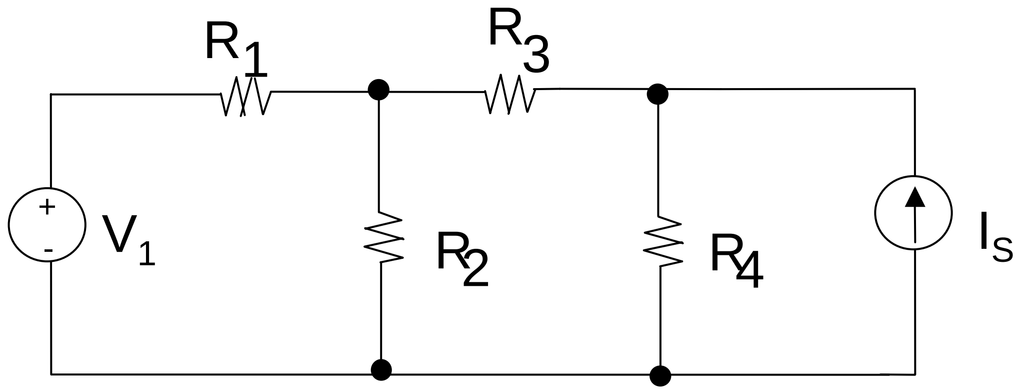
3. **Write KCL at each unknown node**, expressing current in terms of the node voltages (using the *I-V* relationships of branch elements)

   ***Special cases: floating voltage sources***

4. **Solve the set of independent equations**

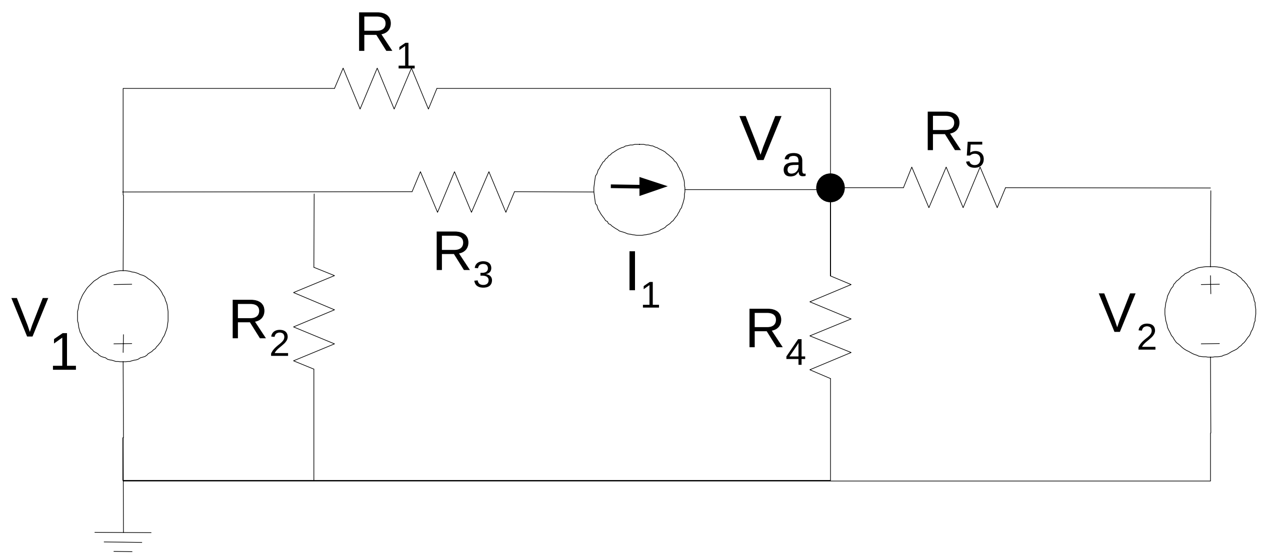
   *N* equations for *N* unknown node voltages

# Nodal Analysis: Example #1




1. Choose a reference node.
2. Define the node voltages (except reference node and the one set by the voltage source).
3. Apply KCL at the nodes with unknown voltage.
4. Solve for unknown node voltages.

# Nodal Analysis: Example #2



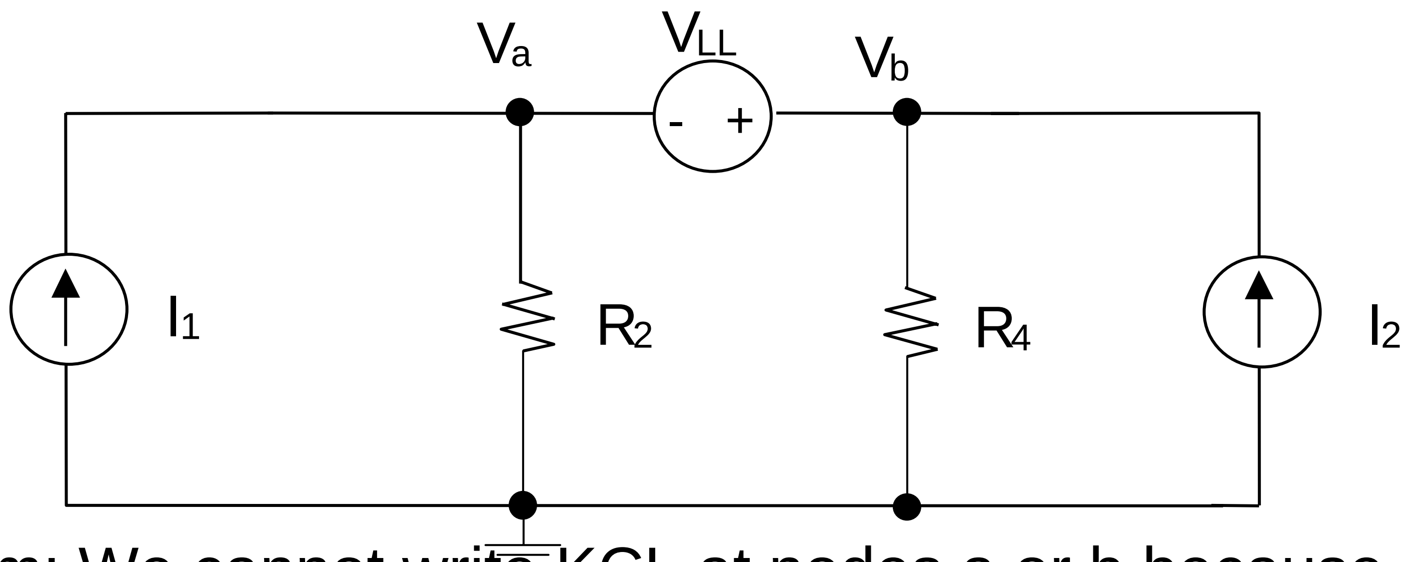

**Challenges:**

**Determine number of nodes needed**

**Deal with different types of sources**

# Nodal Analysis w/ "Floating Voltage Source"

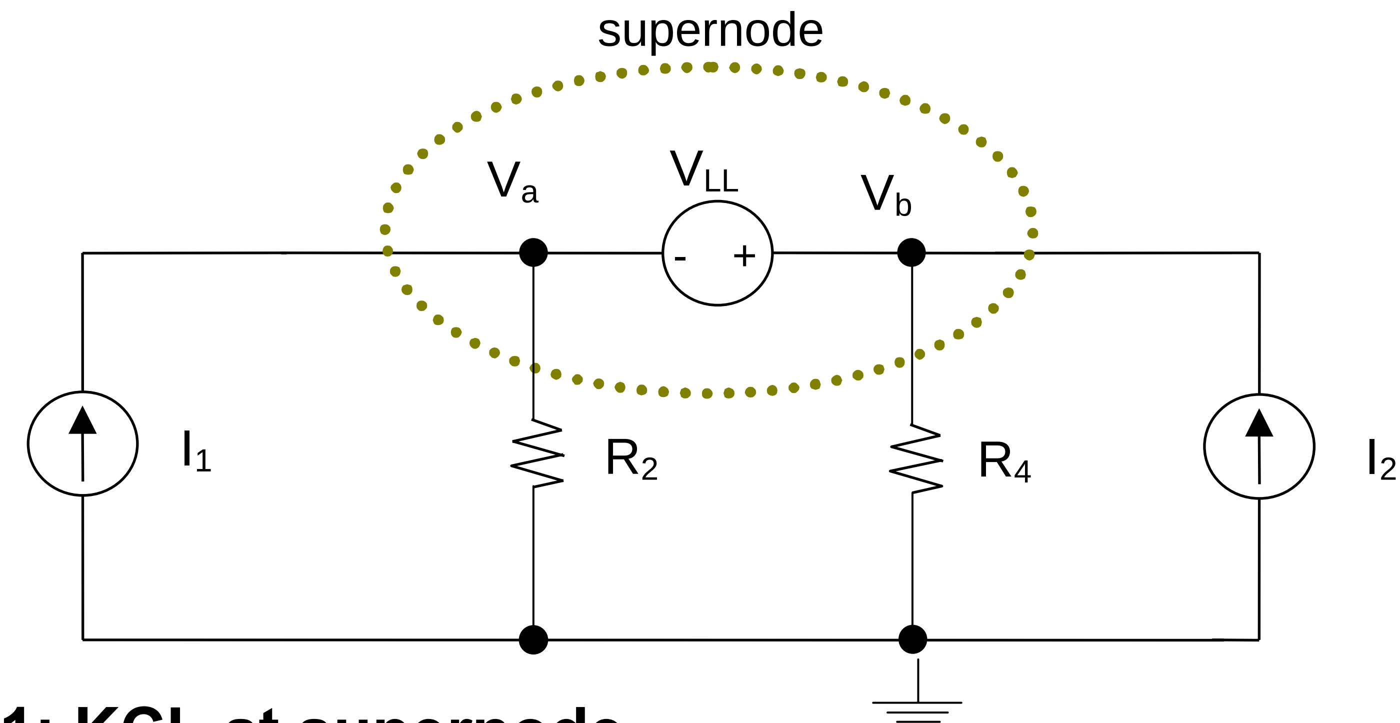
A "floating" voltage source is one for which neither side is connected to the reference node, e.g. $V_{LL}$ in the circuit below:




Problem: We cannot write KCL at nodes a or b because there is no way to express the current through the voltage source in terms of $V_a$-$V_b$.

Solution: Define a "supernode" – that chunk of the circuit containing nodes a and b. Express KCL for this supernode. Incorporate voltage source constraint into KCL equation.

# Nodal Analysis: Example #3




**<u>Eq'n 1</u>: KCL at supernode**

**Substitute property of voltage source:**

# Formal Circuit Analysis Methods

## NODAL ANALYSIS

("Node-Voltage Method")

0) Choose a reference node

1) Define unknown node voltages

2) Apply KCL to each unknown node, expressing current in terms of the node voltages

   => N equations for
   N unknown node voltages

3) Solve for node voltages

   => determine branch currents

## MESH ANALYSIS

("Mesh-Current Method")

1) Select M independent mesh currents such that at least one mesh current passes through each branch*

   M = #branches - #nodes + 1

2) Apply KVL to each mesh, expressing voltages in terms of mesh currents

   => M equations for
   M unknown mesh currents

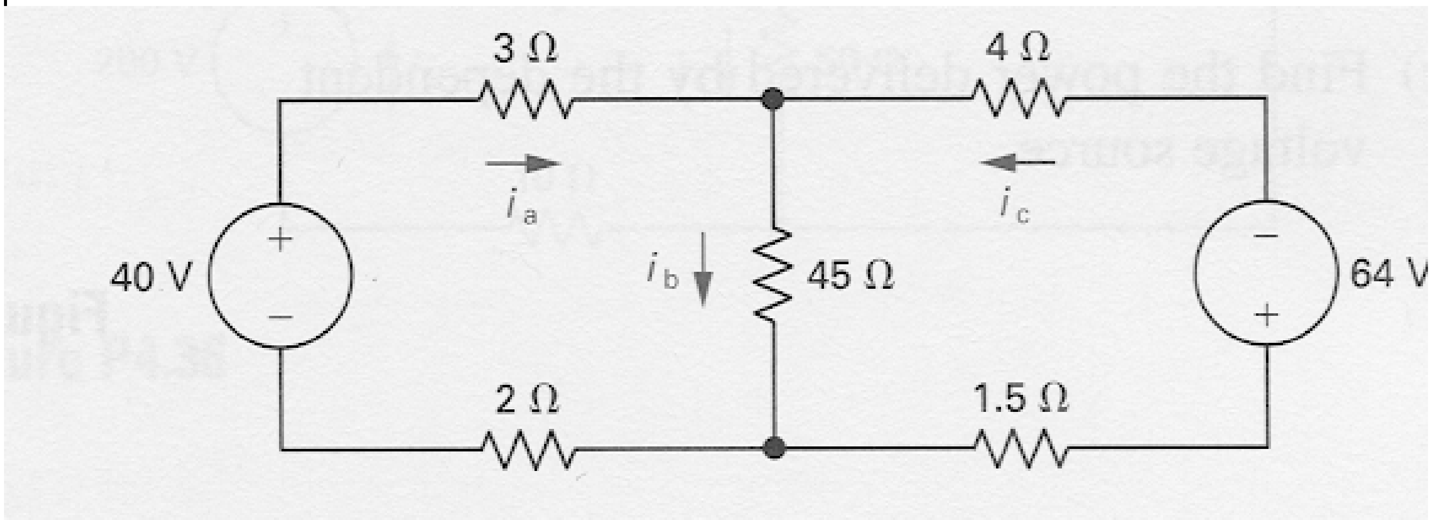
3) Solve for mesh currents

   => determine node voltages

*Simple method for *planar* circuits

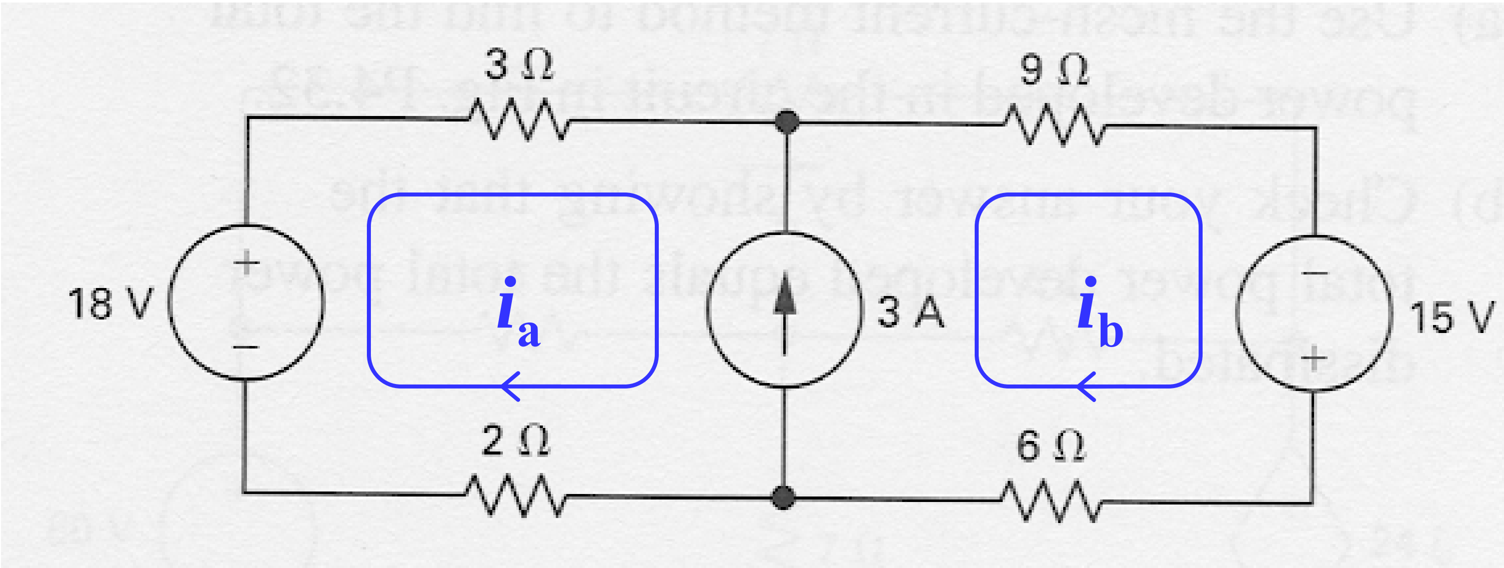
A mesh current is not necessarily identified with a branch current.

# Mesh Analysis: Example #1




1. Select M mesh currents.
2. Apply KVL to each mesh.
3. Solve for mesh currents.

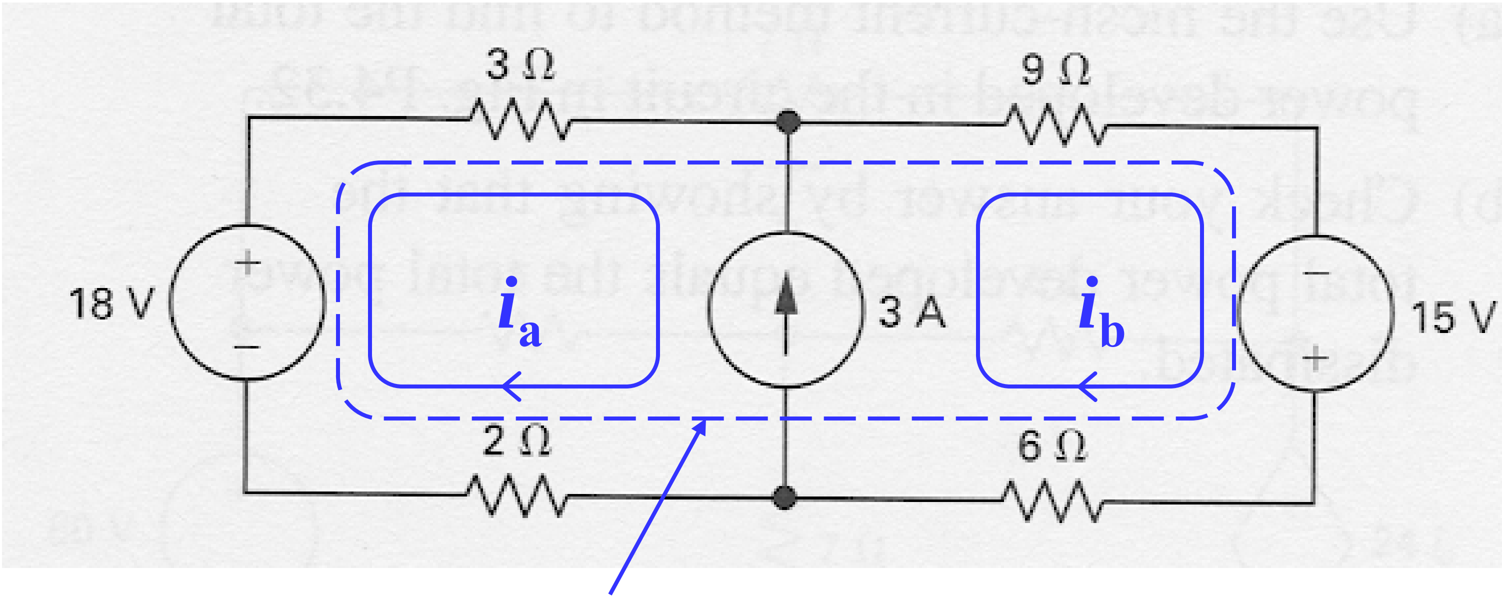
# Mesh Analysis with a Current Source




Problem: We cannot write KVL for meshes a and b because there is no way to express the voltage drop across the current source in terms of the mesh currents.

Solution: Define a "supermesh" – a mesh which avoids the branch containing the current source. Apply KVL for this supermesh.

# Mesh Analysis: Example #2




**<u>Eq'n 1</u>: KVL for supermesh**

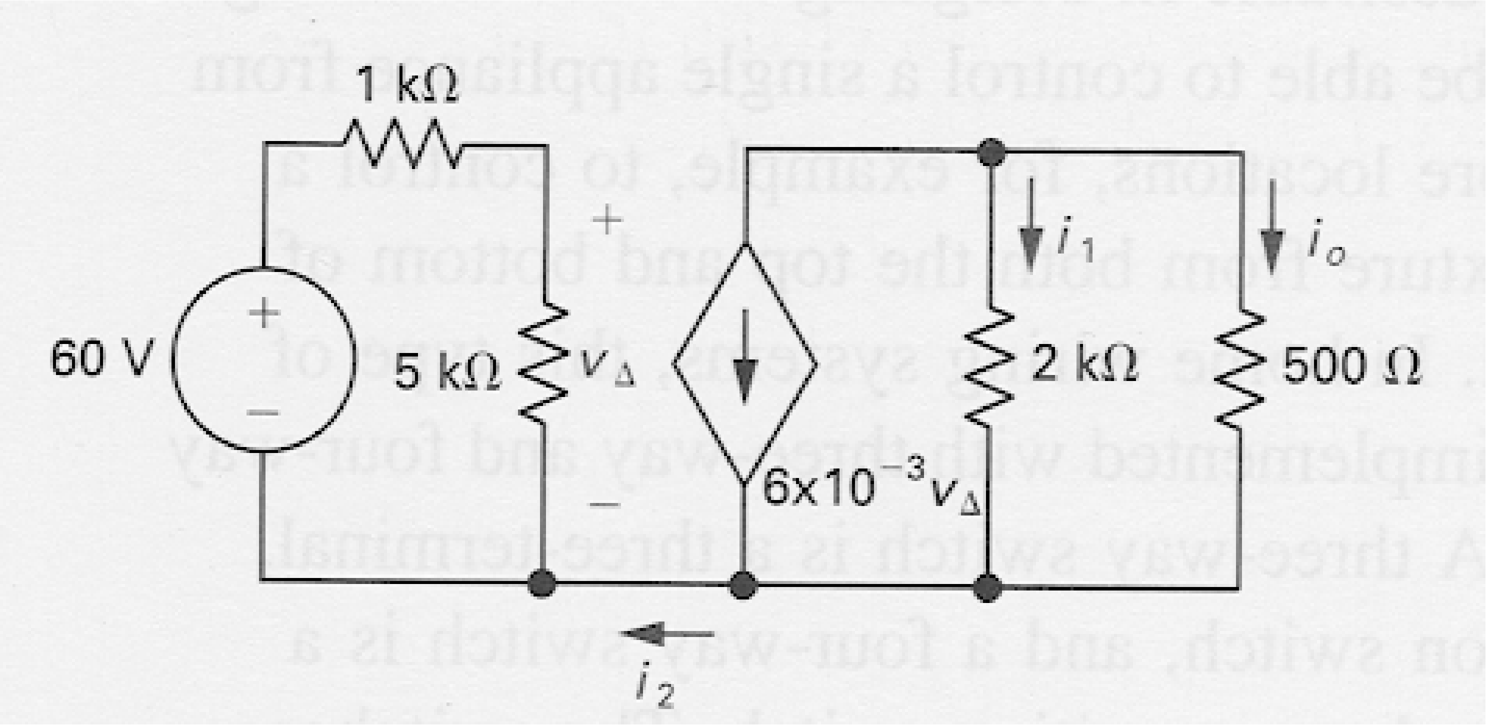
**<u>Eq'n 2</u>: Constraint due to current source:**

# Mesh Analysis with Dependent Sources

- Exactly analogous to Node Analysis
- Dependent Voltage Source: (1) Formulate and write KVL mesh eqns. (2) Include and express dependency constraint in terms of mesh currents
- Dependent Current Source: (1) Use supermesh. (2) Include and express dependency constraint in terms of mesh currents

# Circuit w/ Dependent Source Example

**Find $i_2$, $i_1$ and $i_o$**



# Superposition

A ***linear circuit*** is one constructed only of linear elements (linear resistors, **and linear capacitors and inductors,** linear dependent sources) and independent sources. Linear

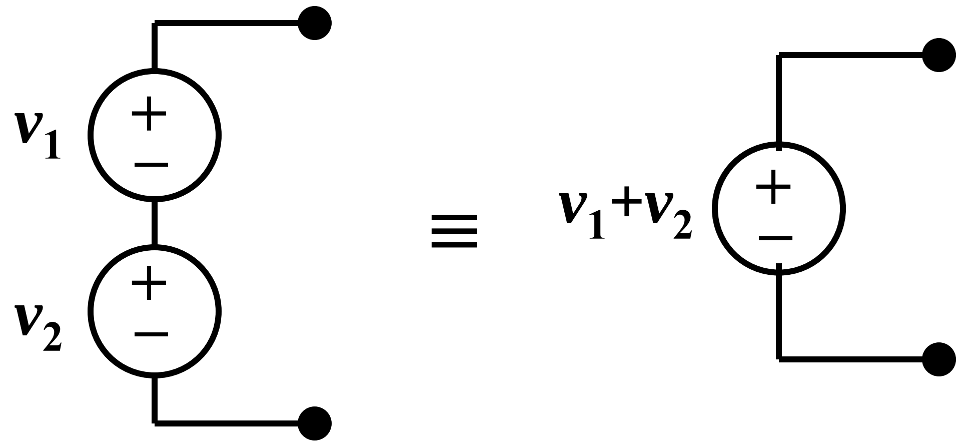
means I-V charcteristic of elements/sources are straight lines when plotted

Principle of Superposition:

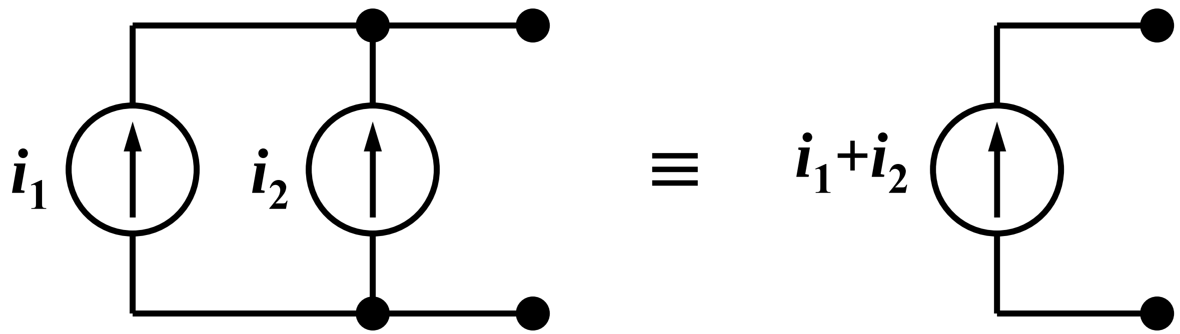
- In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as the algebraic sum of the individual contributions of each source acting alone.

# Source Combinations

- Voltage sources in series can be replaced by an equivalent voltage source:




- Current sources in parallel can be replaced by an equivalent current source:



# Superposition

Procedure:

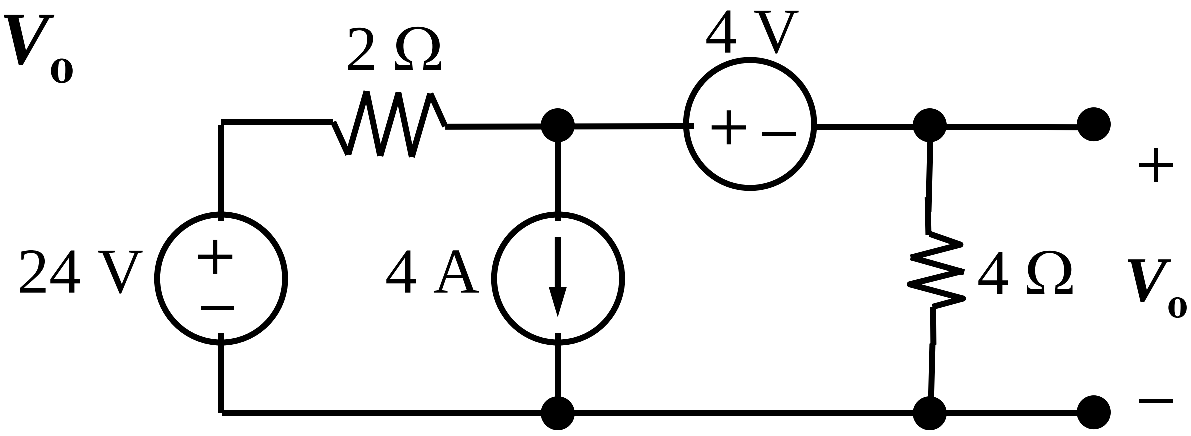
1. Determine contribution due to one independent source
   - Set all other sources to 0: Replace independent voltage source by short circuit, independent current source by open circuit
2. Repeat for each independent source
3. Sum individual contributions to obtain desired voltage or current

# Open Circuit and Short Circuit

- Open circuit → i=0 ; Cut off the branch
- Short circuit → v=0 ; replace the element by wire
- Turn off an independent voltage source means
  - V=0
  - Replace by wire
  - Short circuit
- Turn off an independent current source means
  - i=0
  - Cut off the branch
  - open circuit

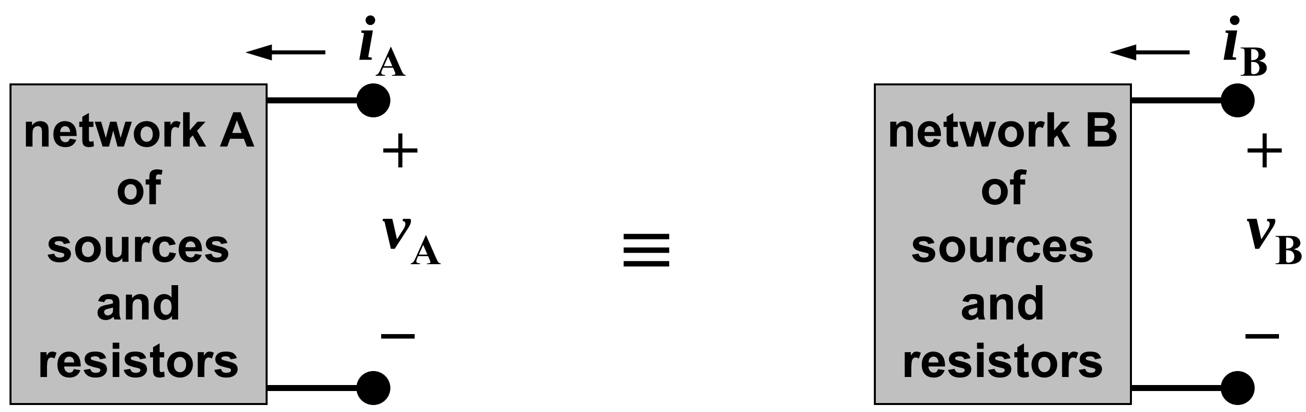
# Superposition Example

- Find $V_o$



# Equivalent Circuit Concept

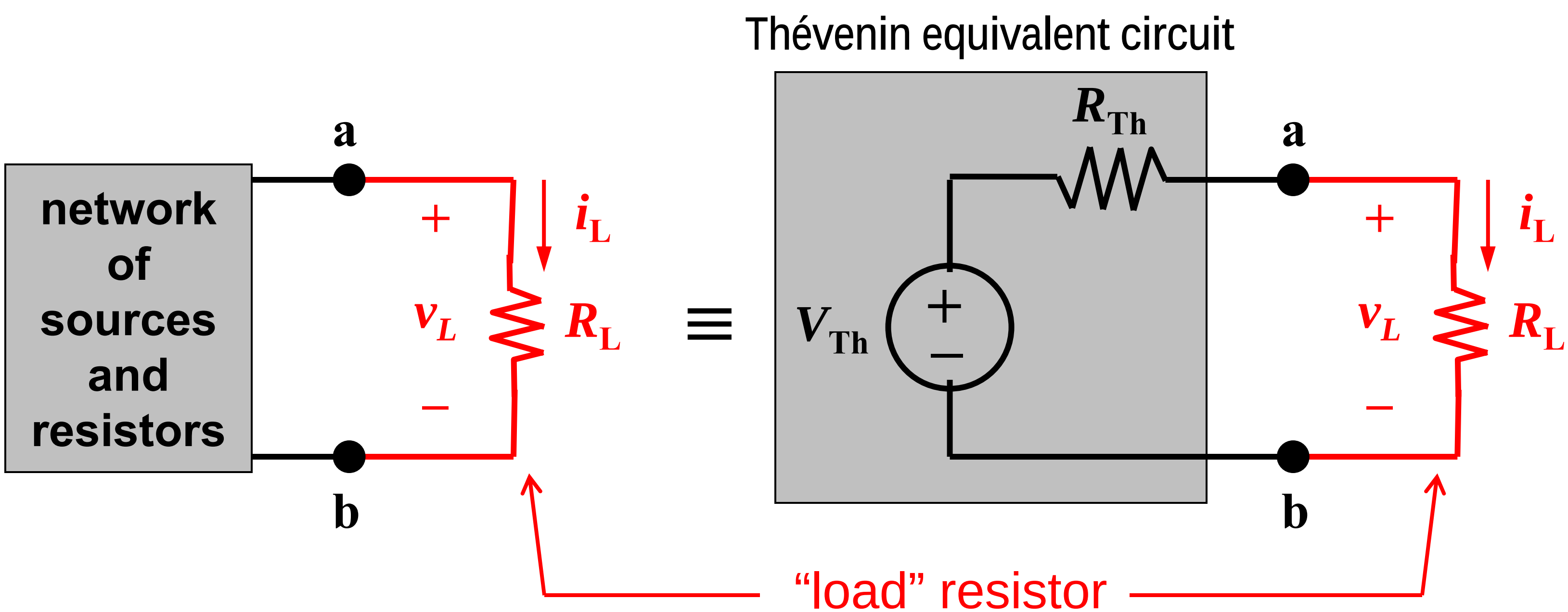
- A network of voltage sources, current sources, and resistors can be replaced by an ***equivalent circuit*** which has identical terminal properties (*I-V* characteristics) without affecting the operation of the rest of the circuit.




$$i_A(v_A) = i_B(v_B)$$

# Thévenin Equivalent Circuit

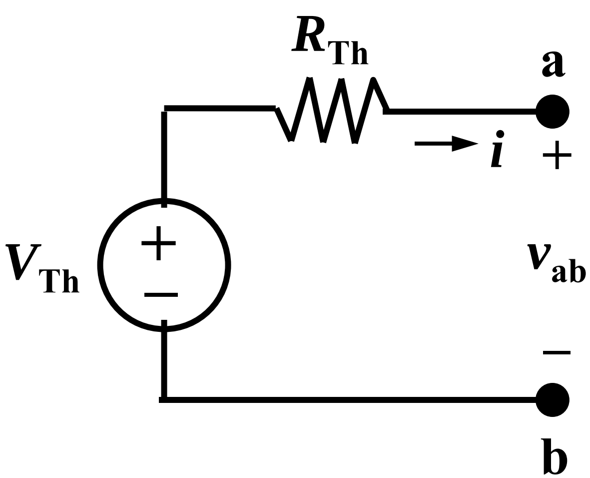
- Any* *linear* 2-terminal (1-port) network of indep. voltage sources, indep. current sources, and linear resistors can be replaced by an equivalent circuit consisting of an independent voltage source in series with a resistor without affecting the operation of the rest of the circuit.

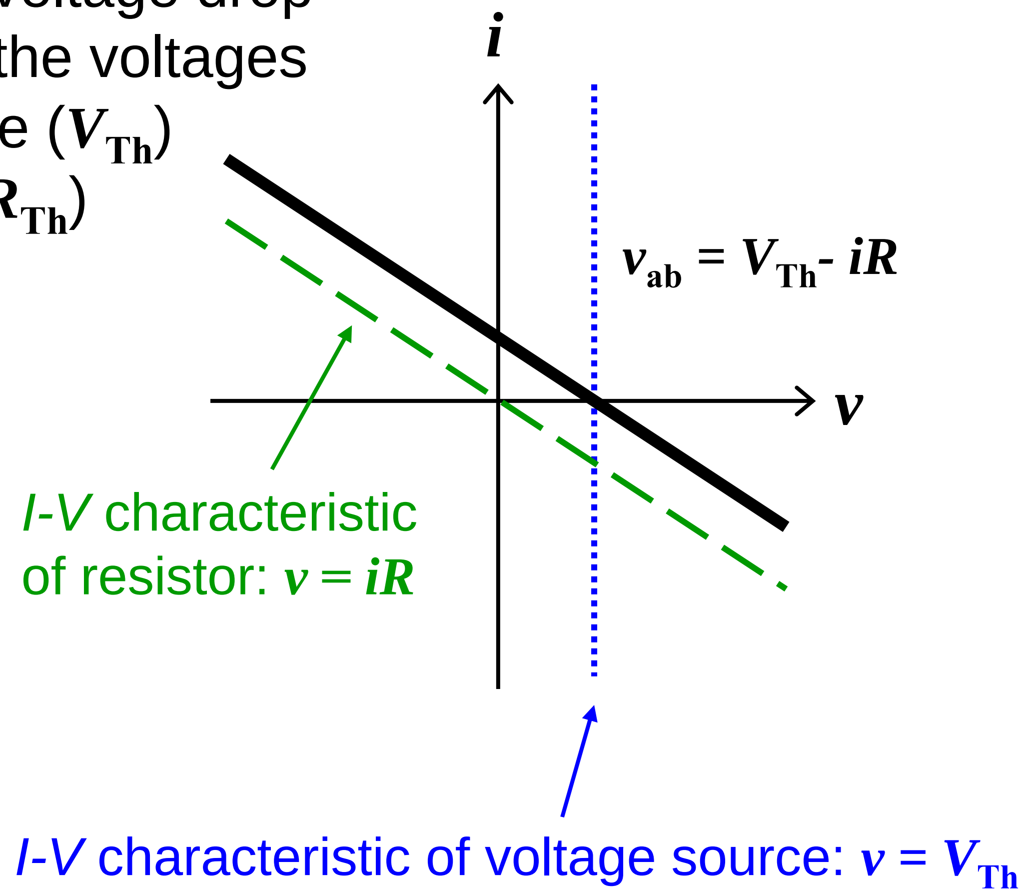


# *I-V* Characteristic of Thévenin Equivalent

- **The *I-V* characteristic for the series combination of elements is obtained by adding their voltage drops:**

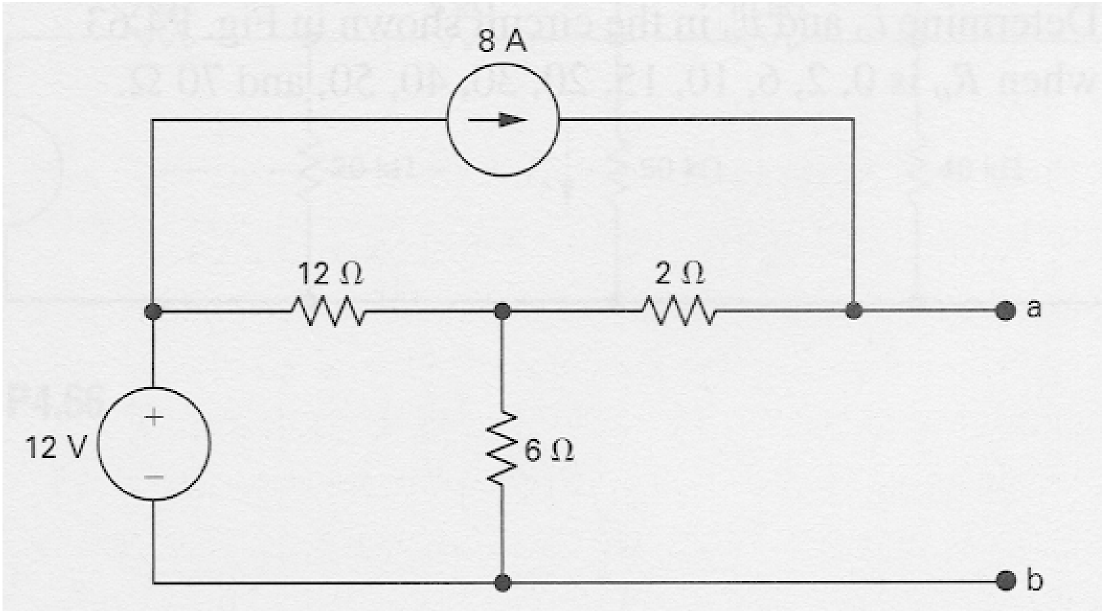
For a given current $i$, the voltage drop $v_{ab}$ is equal to the sum of the voltages dropped across the source ($V_{Th}$) and across the resistor ($iR_{Th}$)



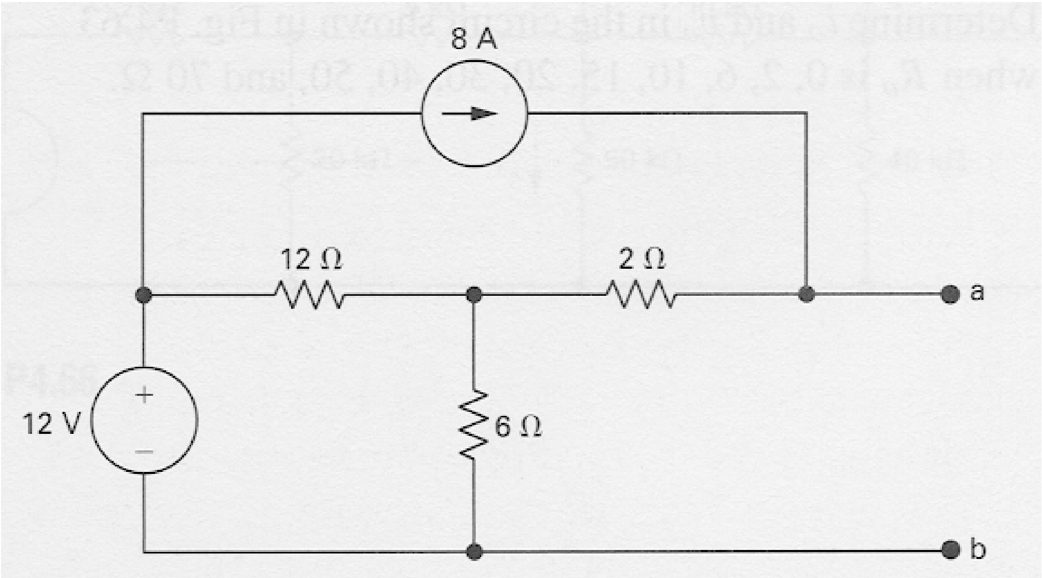



# Thévenin Equivalent Example

Find the Thevenin equivalent with respect to the terminals a,b:



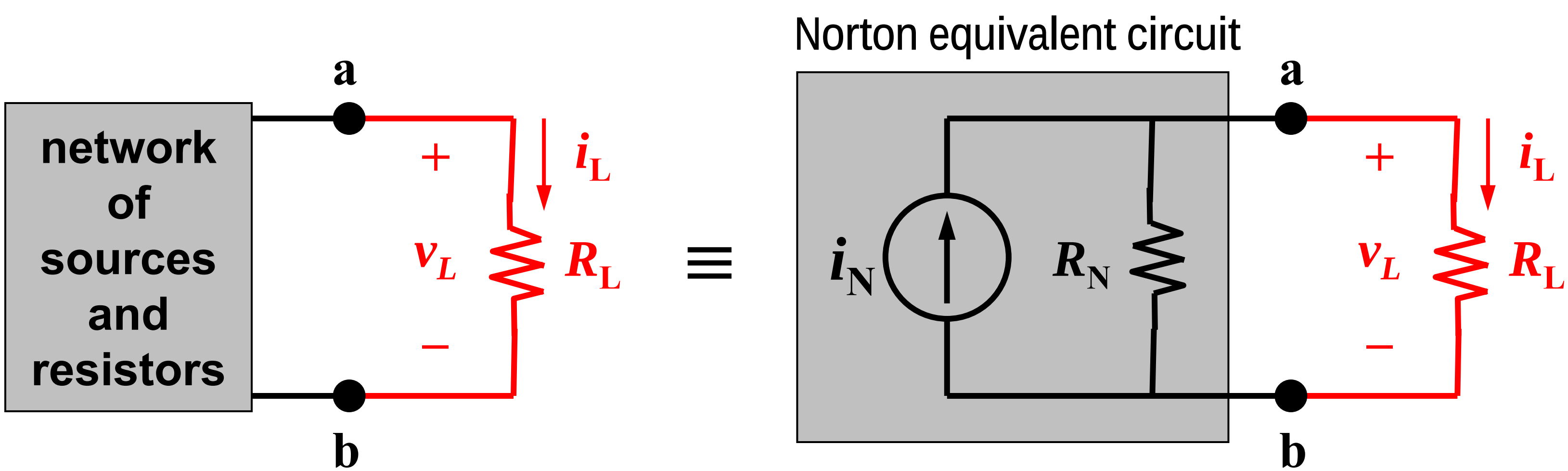
# $R_{Th}$ Calculation Example #1




**Set all independent sources to 0:**

# Norton Equivalent Circuit

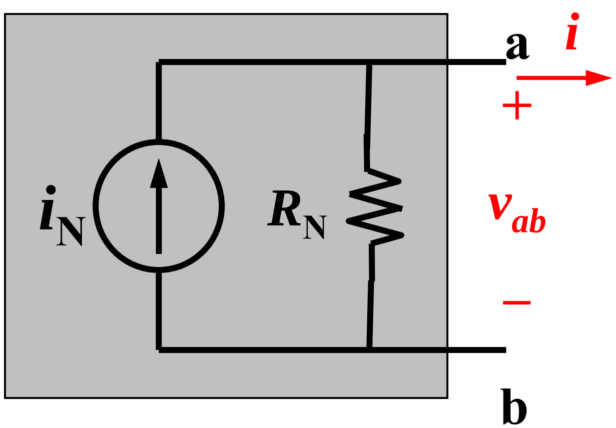
- Any* *linear* 2-terminal (1-port) network of indep. voltage sources, indep. current sources, and linear resistors can be replaced by an equivalent circuit consisting of an independent current source in parallel with a resistor without affecting the operation of the rest of the circuit.

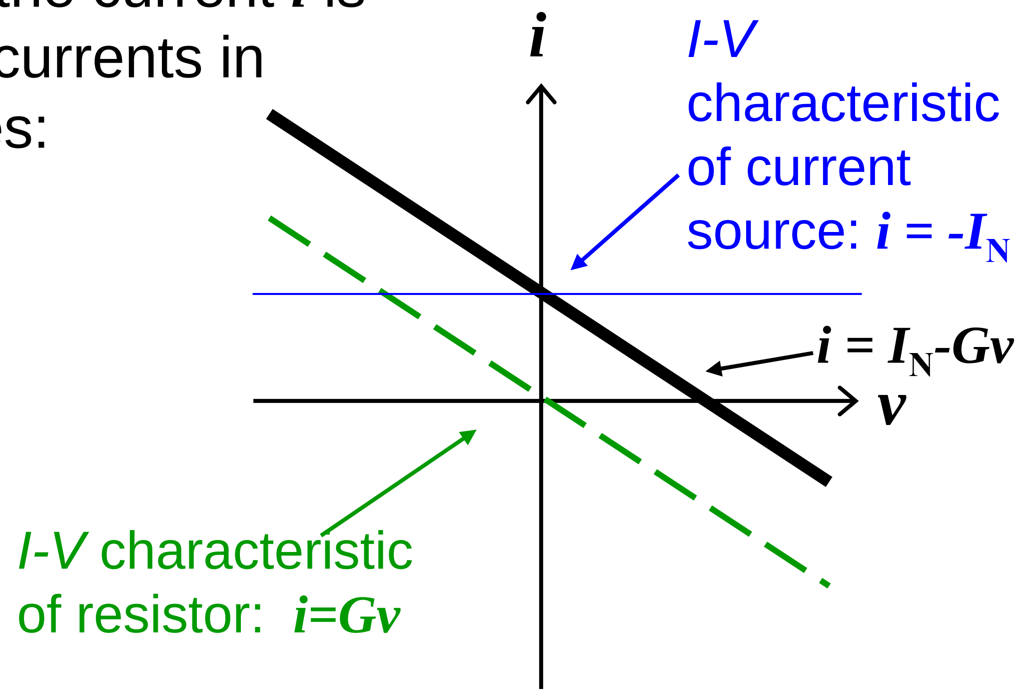


# *I-V* Characteristic of Norton Equivalent

- **The *I-V* characteristic for the parallel combination of elements is obtained by adding their currents:**

For a given voltage $\boldsymbol{v}_{\mathbf{ab}}$, the current $\boldsymbol{i}$ is equal to the sum of the currents in each of the two branches:



# Finding $I_N$ and $R_N = R_{Th}$

Analogous to calculation of Thevenin Eq. Ckt:

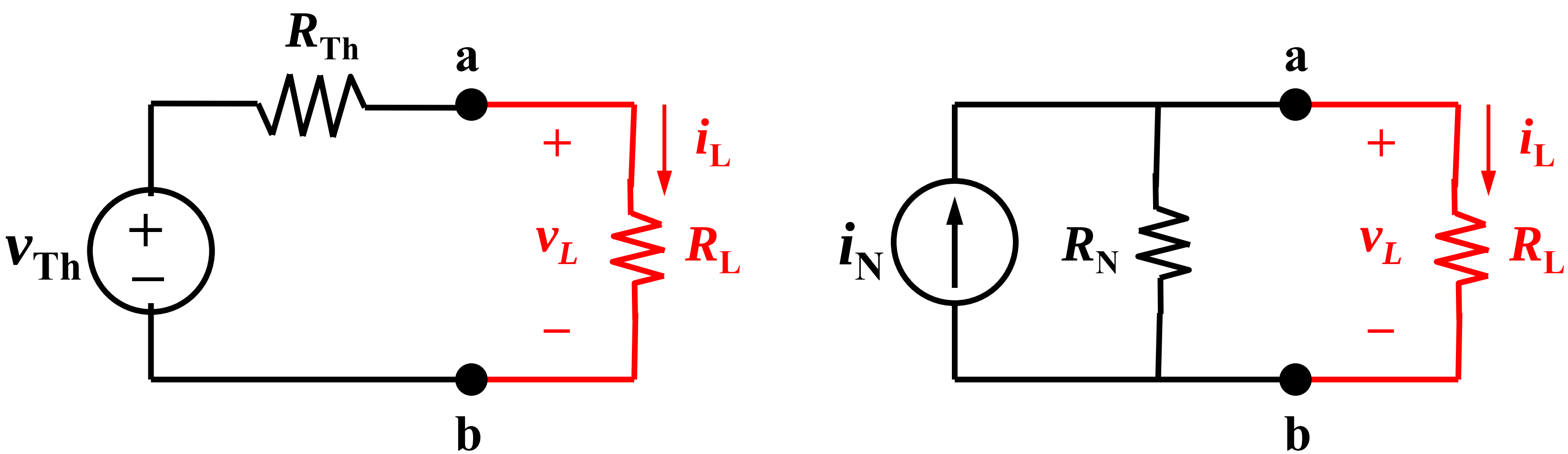
1) Find o.c voltage and s.c. current

$$I_N \equiv i_{sc} = V_{Th}/R_{Th}$$

2) Or, find s.c. current and Norton (Thev) resistance

# Finding $I_N$ and $R_N$

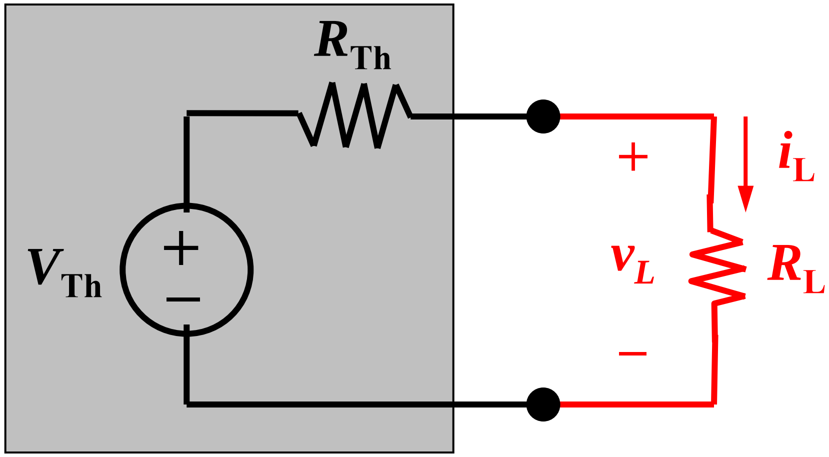
- We can derive the Norton equivalent circuit from a Thévenin equivalent circuit simply by making a source transformation:




$$R_N = R_{Th} = \frac{v_{oc}}{i_{sc}}; \quad i_N = \frac{v_{Th}}{R_{Th}} = i_{sc}$$

# Maximum Power Transfer Theorem

Thévenin equivalent circuit




Power absorbed by load resistor:

$$p = i_L^2 R_L = \left( \frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

To find the value of $R_L$ for which $p$ is maximum, set $\dfrac{dp}{dR_L}$ to 0:

$$\frac{dp}{dR_L} = V_{Th}^2 \left[ \frac{(R_{Th} + R_L)^2 - R_L \times 2(R_{Th} + R_L)}{(R_{Th} + R_L)^4} \right] = 0$$

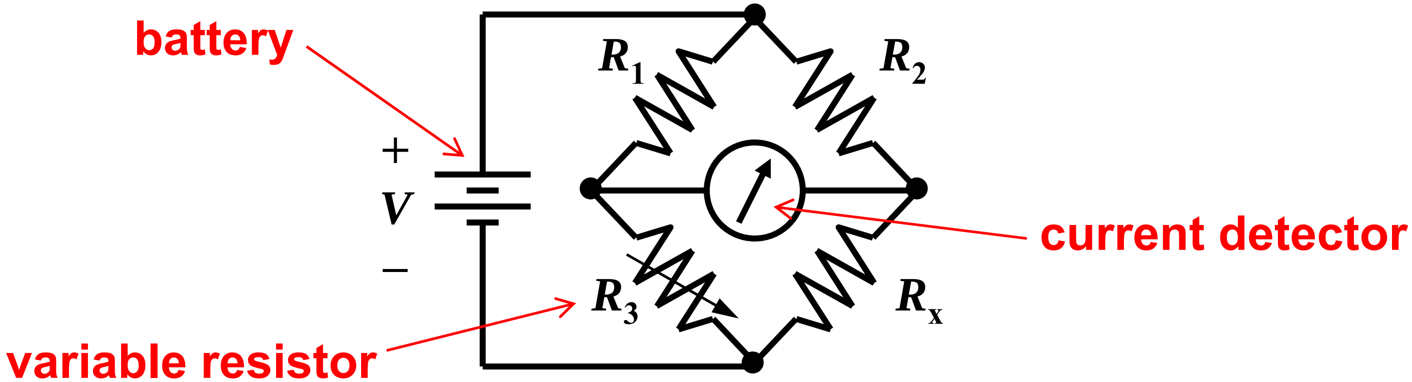
$$\Rightarrow (R_{Th} + R_L)^2 - R_L \times 2(R_{Th} + R_L) = 0$$

$$\Rightarrow \boxed{R_{Th} = R_L}$$

A resistive load receives maximum power from a circuit if the load resistance equals the Thévenin resistance of the circuit.

# The Wheatstone Bridge

- Circuit used to precisely measure resistances in the range from 1 Ω to 1 MΩ, with ±0.1% accuracy
  - $R_1$ and $R_2$ are resistors with known values
  - $R_3$ is a variable resistor (typically 1 to 11,000Ω)
  - $R_x$ is the resistor whose value is to be measured

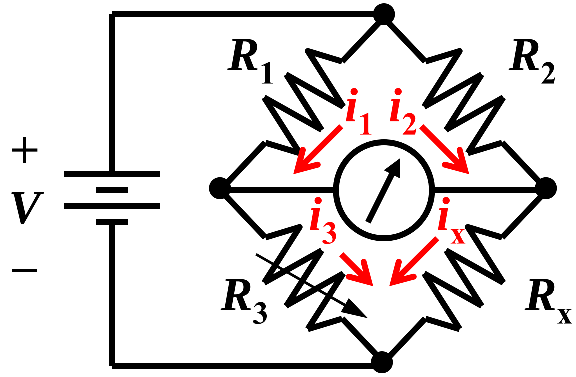


# Finding the value of $R_x$

- Adjust $R_3$ until there is no current in the detector

Then, $\boxed{R_x = \dfrac{R_2}{R_1} R_3}$

<u>Derivation</u>:

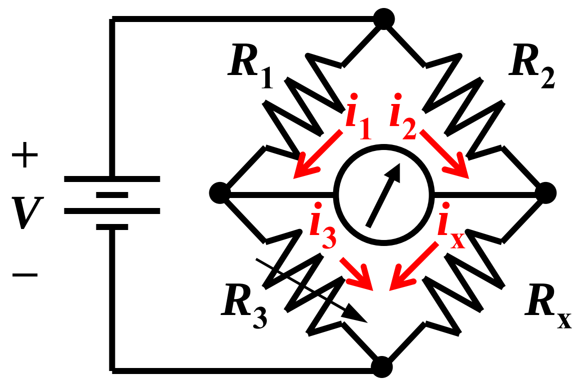



Typically, $R_2 / R_1$ can be varied
from 0.001 to 1000 in decimal steps

# Finding the value of $R_x$

- Adjust $R_3$ until there is no current in the detector

Then, $\boxed{R_x = \dfrac{R_2}{R_1} R_3}$




Typically, $R_2 / R_1$ can be varied from 0.001 to 1000 in decimal steps

Derivation:

KCL => $i_1 = i_3$ and $i_2 = i_x$

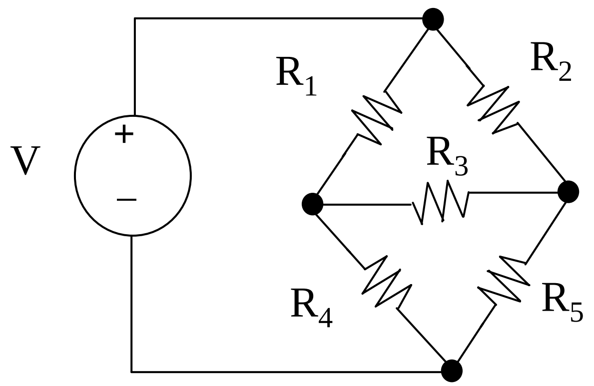
KVL => $i_3 R_3 = i_x R_x$ and $i_1 R_1 = i_2 R_2$

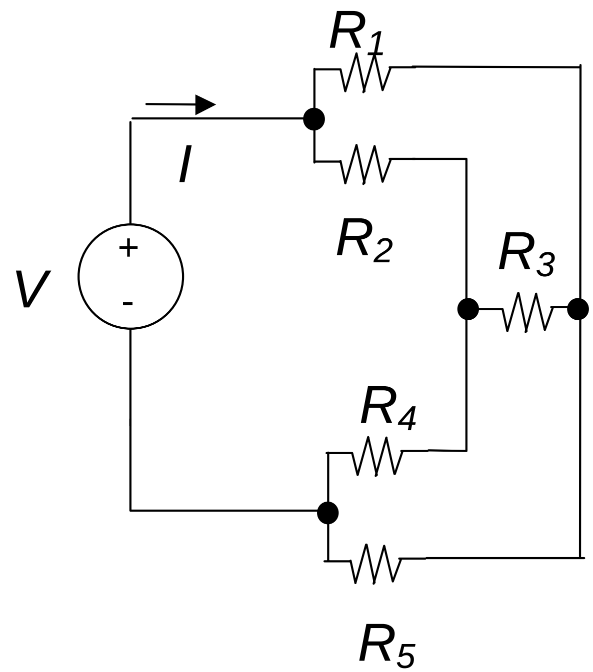
$$i_1 R_3 = i_2 R_x$$

$$\frac{R_3}{R_1} = \frac{R_x}{R_2}$$

# Identifying Series and Parallel Combinations

Some circuits *must* be analyzed (not amenable to simple inspection)



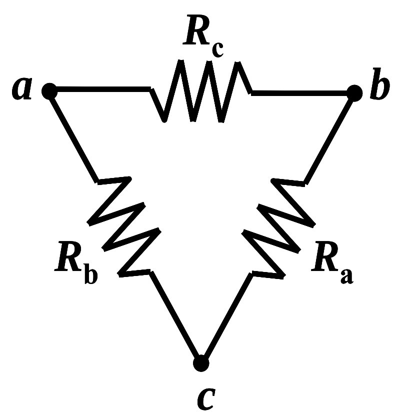



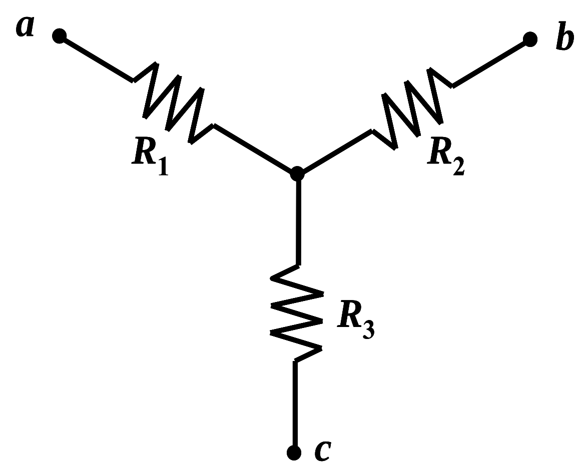

Special cases:

$R_3 = 0$ **OR** $R_3 = \infty$

# Y-Delta Conversion

- These two resistive circuits are equivalent for voltages and currents <u>external</u> to the Y and Δ circuits. Internally, the voltages and currents are different.



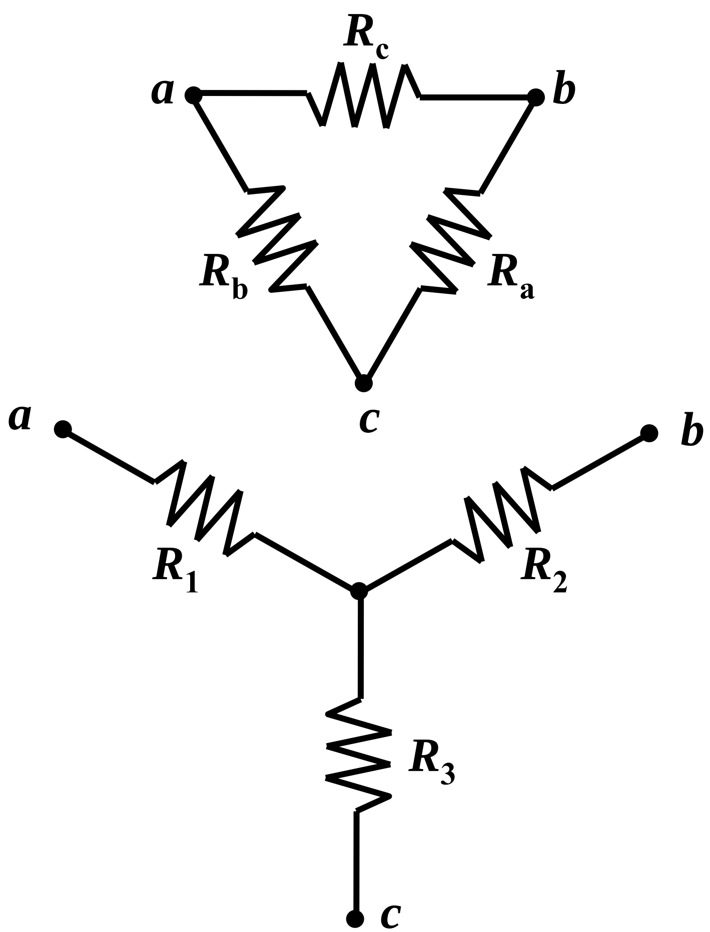




$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \qquad R_2 = \frac{R_a R_c}{R_a + R_b + R_c} \qquad R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

**Brain Teaser Category: Important for motors and electrical utilities.**

# Delta-to-Wye (Pi-to-Tee) Equivalent Circuits

- In order for the Delta interconnection to be equivalent to the Wye interconnection, the resistance between corresponding terminal pairs must be the same




$$R_{ab} = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} = R_1 + R_2$$

$$R_{bc} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} = R_2 + R_3$$

$$R_{ca} = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c} = R_1 + R_3$$

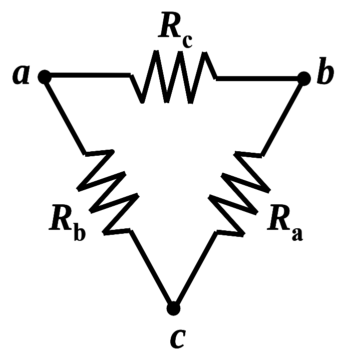
# Δ-Y and Y-Δ Conversion Formulas

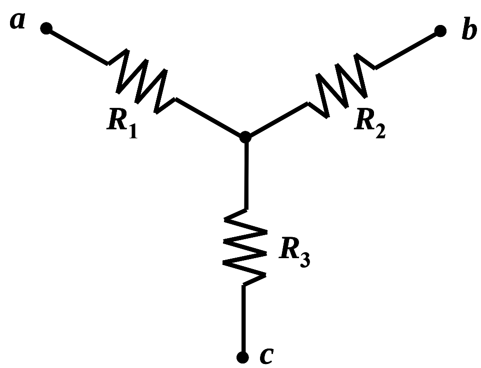
**Delta-to-Wye conversion**

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$




**Wye-to-Delta conversion**

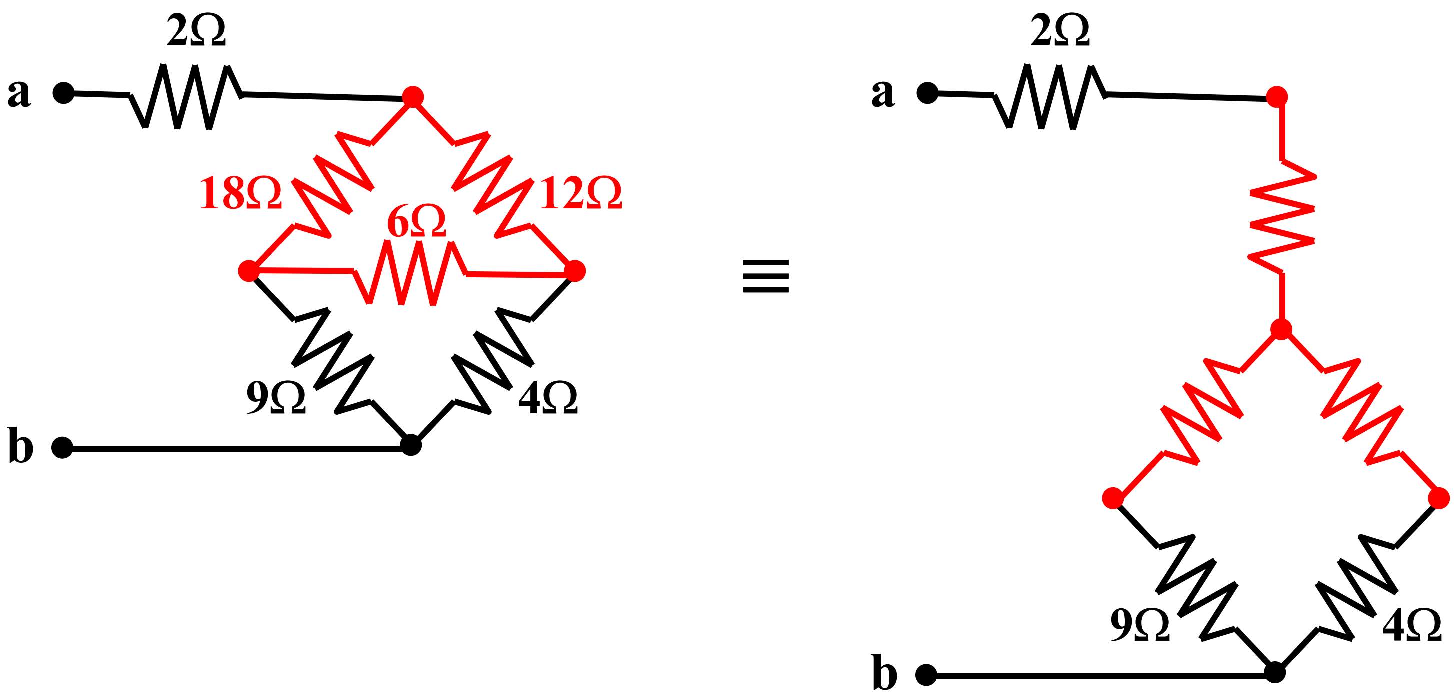
$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

$$R_b = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

# Circuit Simplification Example

Find the equivalent resistance $\boldsymbol{R}_{\mathbf{ab}}$:



# Dependent Sources

- Node-Voltage Method
  - Dependent current source:
    - treat as independent current source in organizing node eqns
    - substitute constraining dependency in terms of defined node voltages.
  - Dependent voltage source:
    - treat as independent voltage source in organizing node eqns
    - Substitute constraining dependency in terms of defined node voltages.
- Mesh Analysis
  - Dependent Voltage Source:
    - Formulate and write KVL mesh eqns.
    - Include and express dependency constraint in terms of mesh currents
  - Dependent Current Source:
    - Use supermesh.
    - Include and express dependency constraint in terms of mesh currents

# Comments on Dependent Sources

A dependent source establishes a voltage or current whose value depends on the value of a voltage or current at a specified location in the circuit.

(device model, used to model behavior of transistors & amplifiers)

To specify a dependent source, we must identify:

1. the controlling voltage or current (must be calculated, in general)
2. the relationship between the controlling voltage or current and the supplied voltage or current
3. the reference direction for the supplied voltage or current

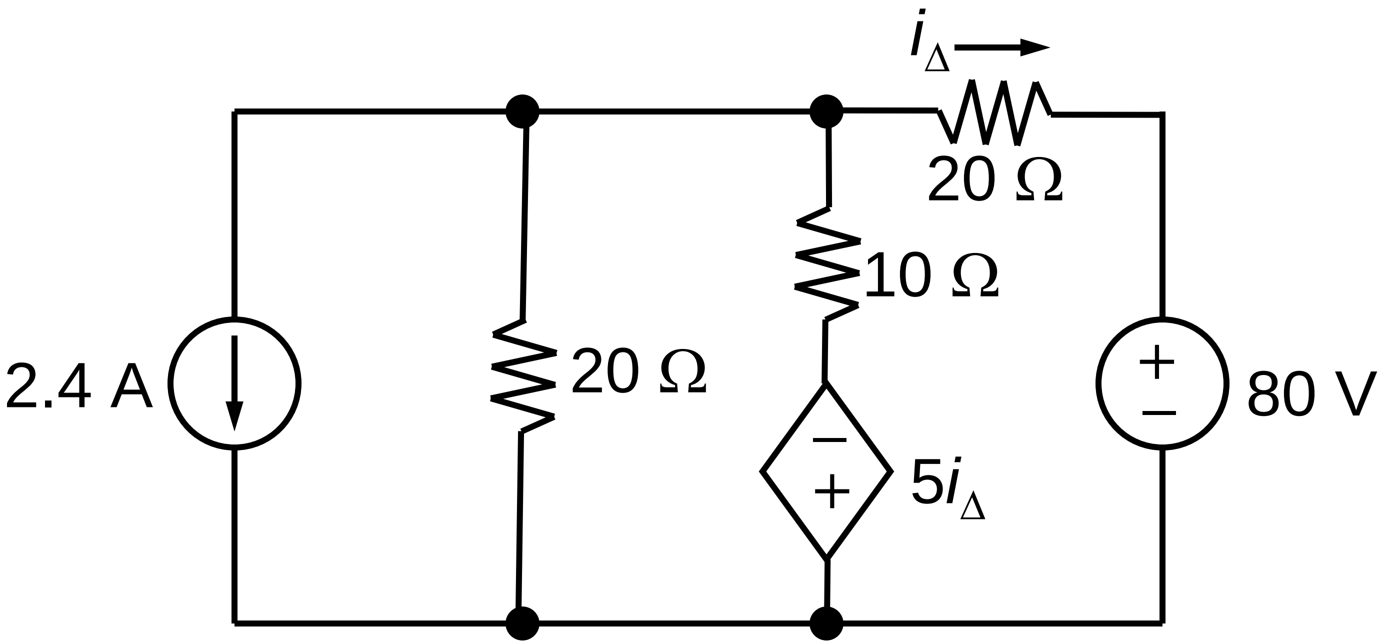
**The relationship between the dependent source and its reference cannot be broken!**

– Dependent sources cannot be turned off for various purposes (*e.g.* to find the Thévenin resistance, or in analysis using Superposition).

# Node-Voltage Method and Dependent Sources

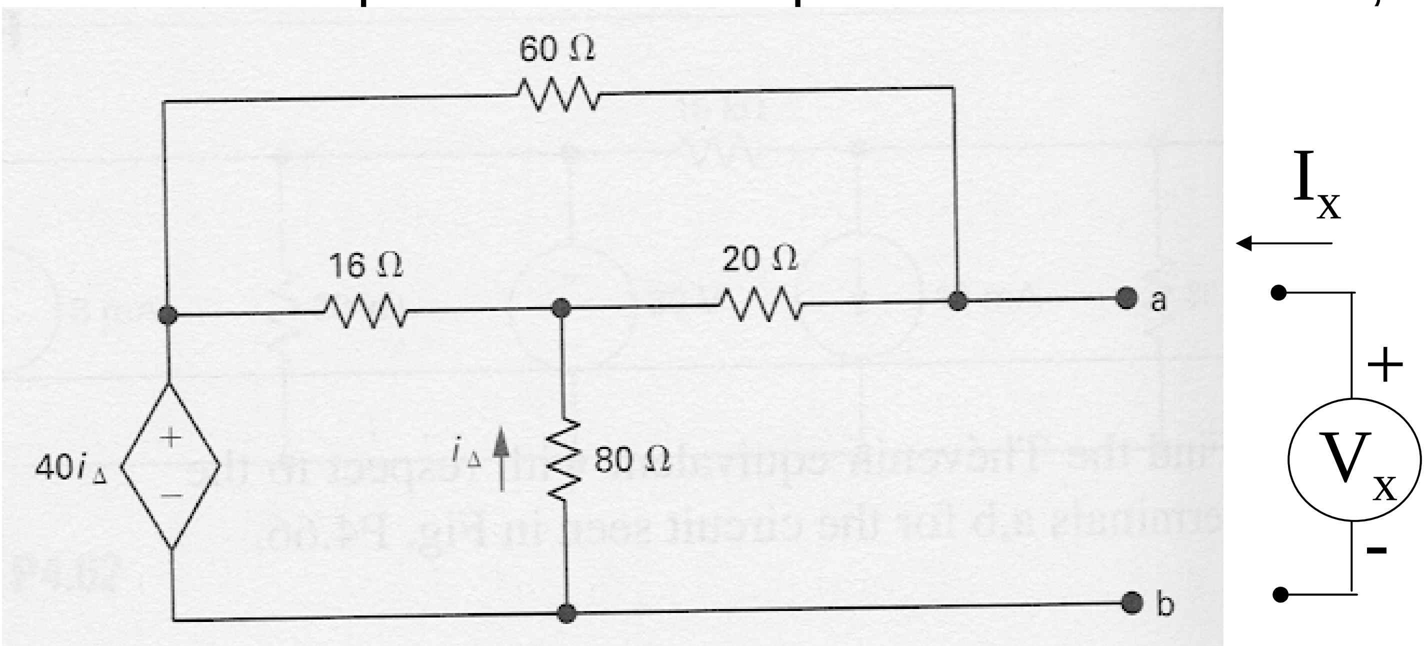
- If a circuit contains dependent sources, what to do?

<u>Example</u>:



# $R_{Th}$ Calculation Example #2

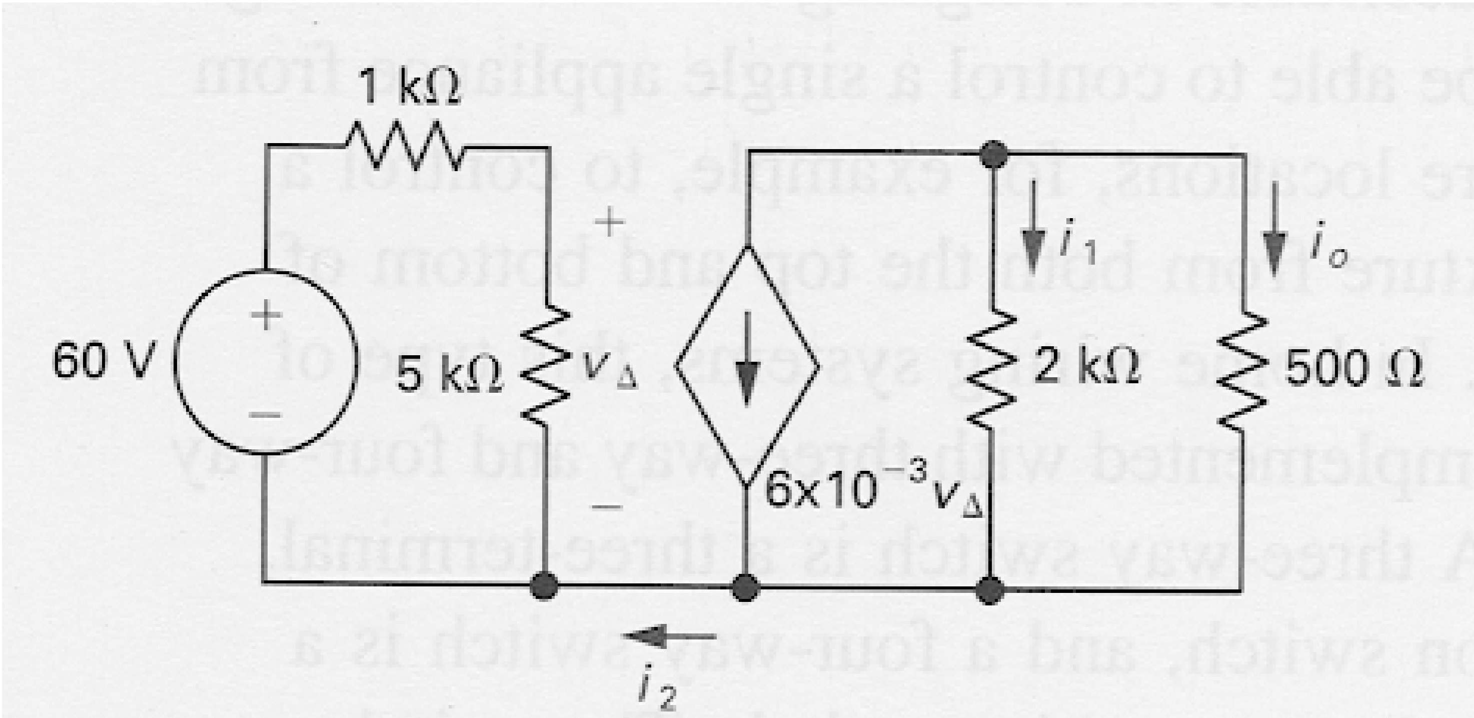
Find the Thevenin equivalent with respect to the terminals a,b:




Since there is no independent source and we cannot arbitrarily turn off the dependence source, we can add a voltage source $V_x$ across terminals a-b and measure the current through this terminal $I_x$ . $R_{th} = V_x / I_x$

# Circuit w/ Dependent Source Example

**Find $i_2$, $i_1$ and $i_o$**



## Summary of Techniques for Circuit Analysis -1

- Resistor network
  - Parallel resistors
  - Series resistors
  - Y-delta conversion
  - “Add” current source and find voltage (or vice versa)
- Superposition
  - Leave one independent source on at a time
  - Sum over all responses
  - Voltage off → SC
  - Current off → OC

## Summary of Techniques for Circuit Analysis -2

- Node Analysis
  - Node voltage is the unknown
  - Solve for KCL
  - Floating voltage source using super node
- Mesh Analysis
  - Loop current is the unknown
  - Solve for KVL
  - Current source using super mesh
- Thevenin and Norton Equivalent Circuits
  - Solve for OC voltage
  - Solve for SC current