Chapters 6 and 7

- Outline
  - The capacitor
  - The inductor
  - RC and RL circuits

The Capacitor

Two conductors (a,b) separated by an insulator:

difference in potential = V_{ab}

=> equal & opposite charge ***Q*** on conductors

$$\boxed{Q = CV_{ab}}$$

(stored charge in terms of voltage)

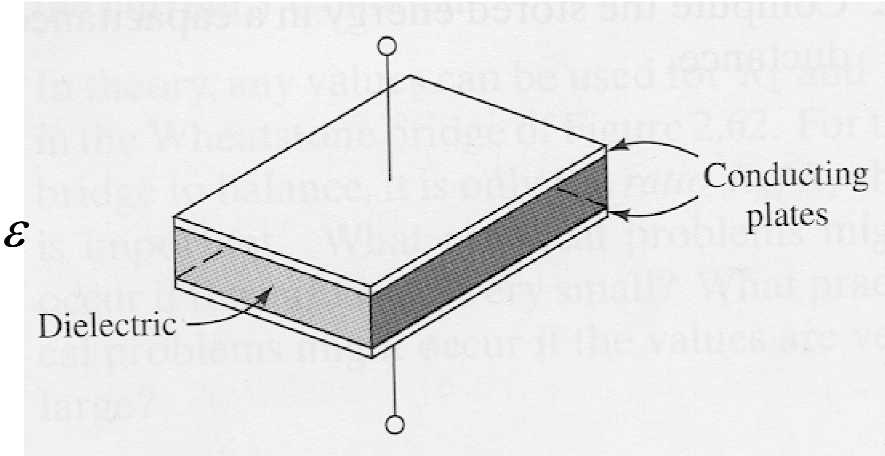
where ***C*** is the **capacitance** of the structure,

➢ positive (+) charge is on the conductor at higher potential

<u>Parallel-plate capacitor</u>:

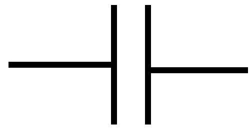
- area of the plates = ***A (m²)***
- separation between plates = ***d (m)***
- ***dielectric permittivity*** of insulator = ε ***(F/m)***

=> capacitance $$\boxed{C = \frac{A\varepsilon}{d}}$$ ***(F)***



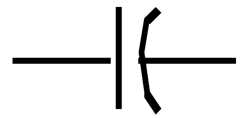
Capacitor

<u>Symbol</u>:

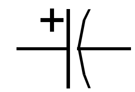


C

or



C



C

Electrolytic (polarized) capacitor

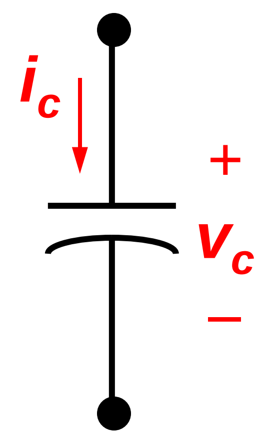
<u>Units</u>: Farads (Coulombs/Volt)

(typical range of values: 1 pF to 1 μF; for "supercapacitors" up to a few F!)

<u>Current-Voltage relationship</u>:

$$i_c = \frac{dQ}{dt} = C\frac{dv_c}{dt} + v_c\frac{dC}{dt}$$

If C (geometry) is unchanging, $i_C = C\, dv_C/dt$



<u>Note</u>: Q (v_c) must be a continuous function of time

Voltage in Terms of Current

$$Q(t) = \int_0^t i_c(t)dt + Q(0)$$

$$v_c(t) = \frac{1}{C}\int_0^t i_c(t)dt + \frac{Q(0)}{C} = \frac{1}{C}\int_0^t i_c(t)dt + v_c(0)$$

Uses: Capacitors are used to store energy for camera flashbulbs, in filters that separate various frequency signals, and they appear as undesired "parasitic" elements in circuits where they usually degrade circuit performance

Stored Energy

CAPACITORS STORE ELECTRIC ENERGY

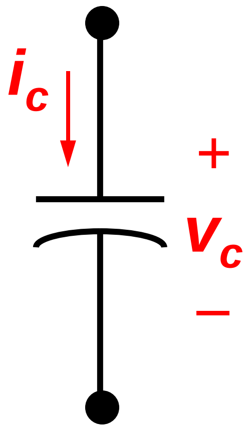
You might think the energy stored on a capacitor is ***QV* = *CV*²**, which has the dimension of Joules. But during charging, the average voltage across the capacitor was only half the final value of ***V*** for a linear capacitor.

Thus, energy is $\frac{1}{2}QV = \boxed{\frac{1}{2}CV^2}$.

<u>Example</u>: A 1 pF capacitance charged to 5 Volts has ½(5V)² (1pF) = 12.5 pJ
(A 5F supercapacitor charged to 5 volts stores 63 J; if it discharged at a constant rate in 1 ms energy is discharged at a 63 kW rate!)

A more rigorous derivation

This derivation holds independent of the circuit!

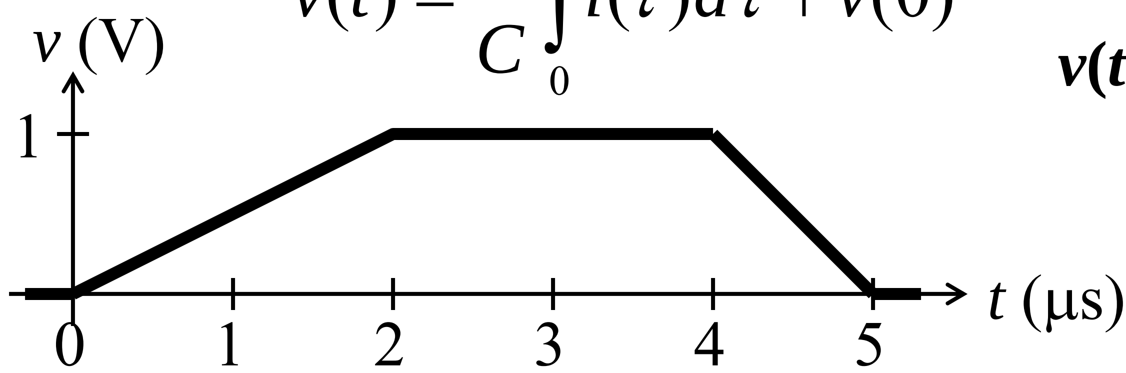


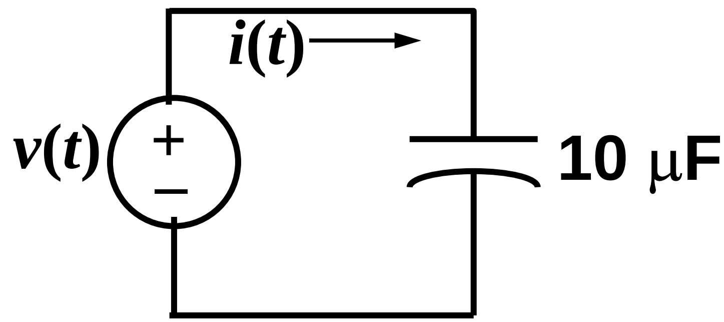
$$w = \int_{t = t_{Initial}}^{t = t_{Final}} v_c \cdot i_c \, dt = \int_{v = V_{Initial}}^{v = V_{Final}} v_c \frac{dQ}{dt} dt = \int_{v = V_{Initial}}^{v = V_{Final}} v_c \, dQ$$

$$w = \int_{v = V_{Initial}}^{v = V_{Final}} C v_c \, dv_c = \frac{1}{2} C V_{Final}{}^2 - \frac{1}{2} C V_{Initial}{}^2$$

Example: Current, Power & Energy for a Capacitor

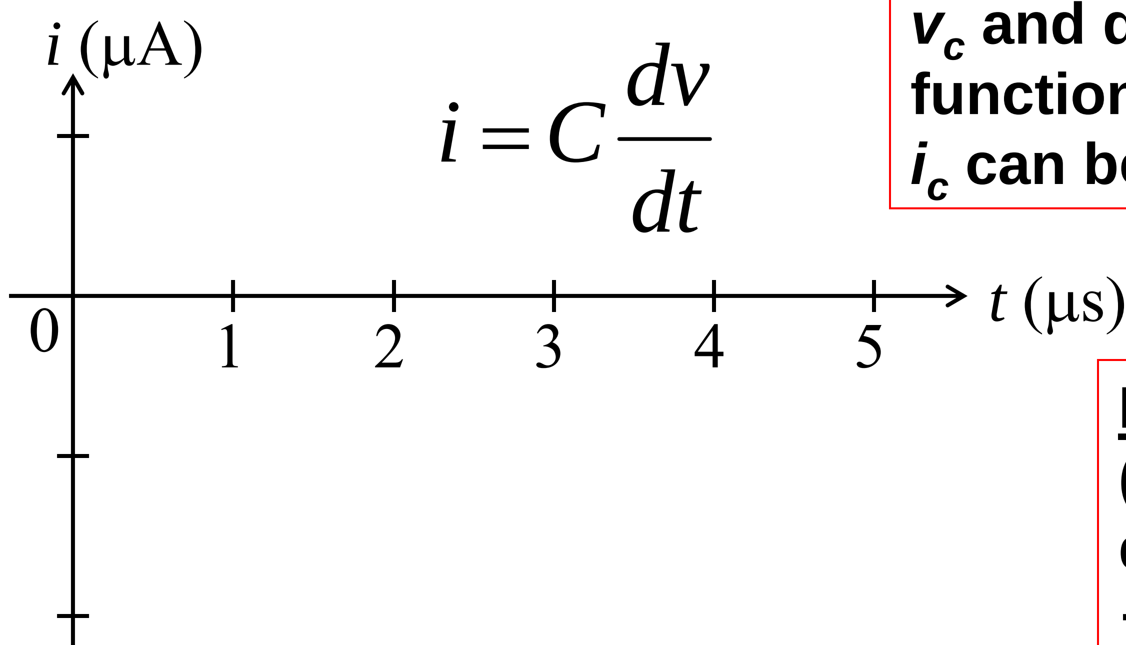
$$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau + v(0)$$



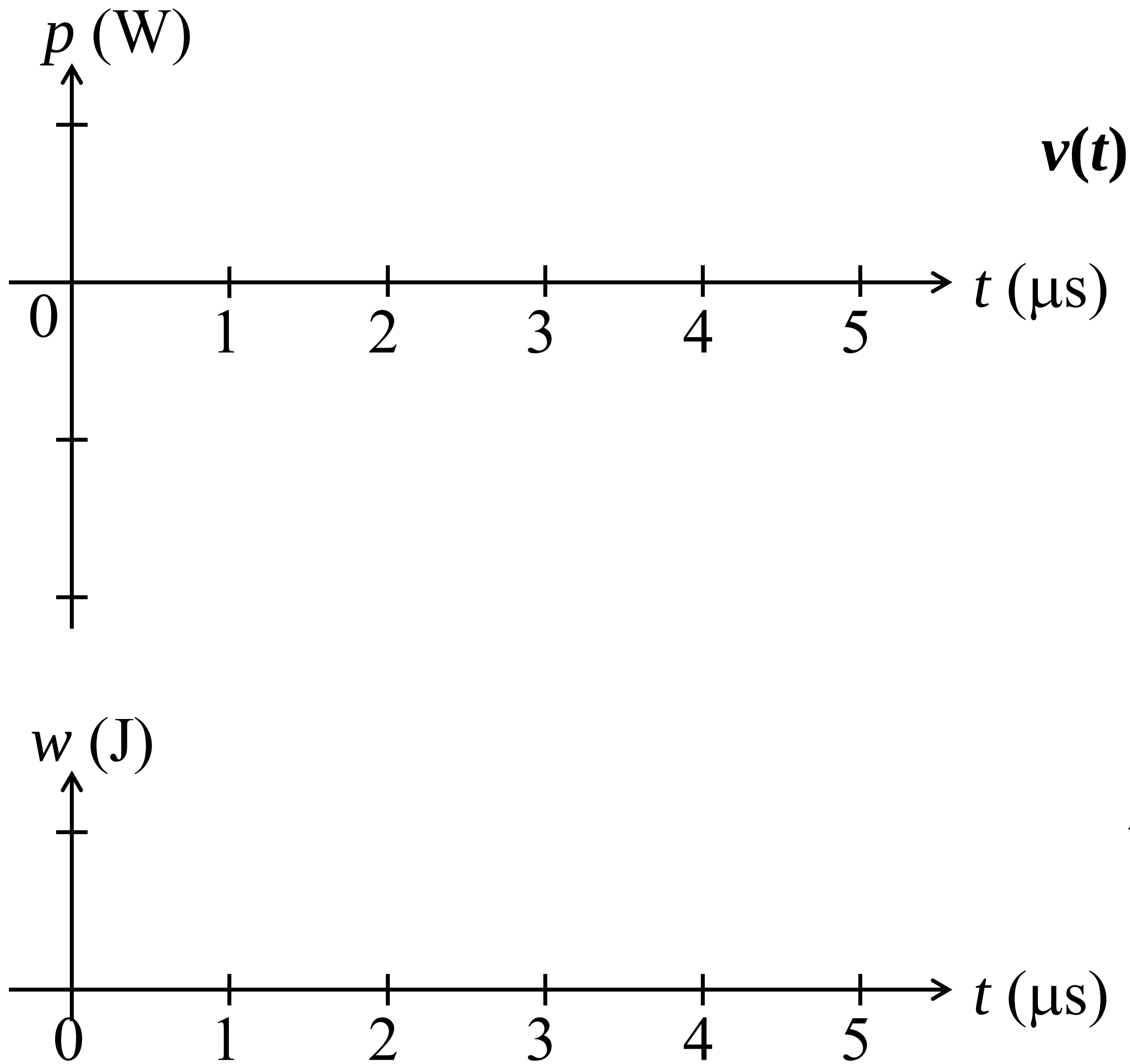


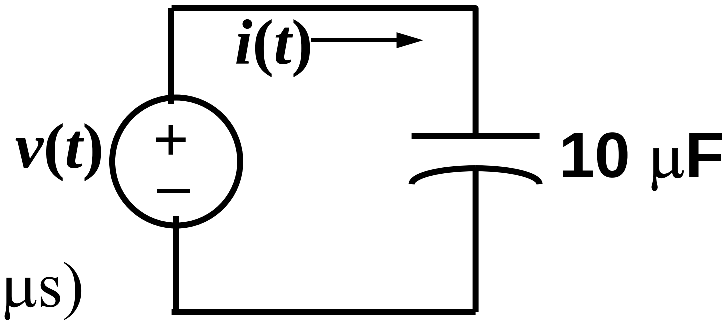
v_c and q must be continuous functions of time; however, i_c can be discontinuous.

$$i = C\frac{dv}{dt}$$



<u>Note</u>: In "steady state" (dc operation), time derivatives are zero → *C* is an open circuit

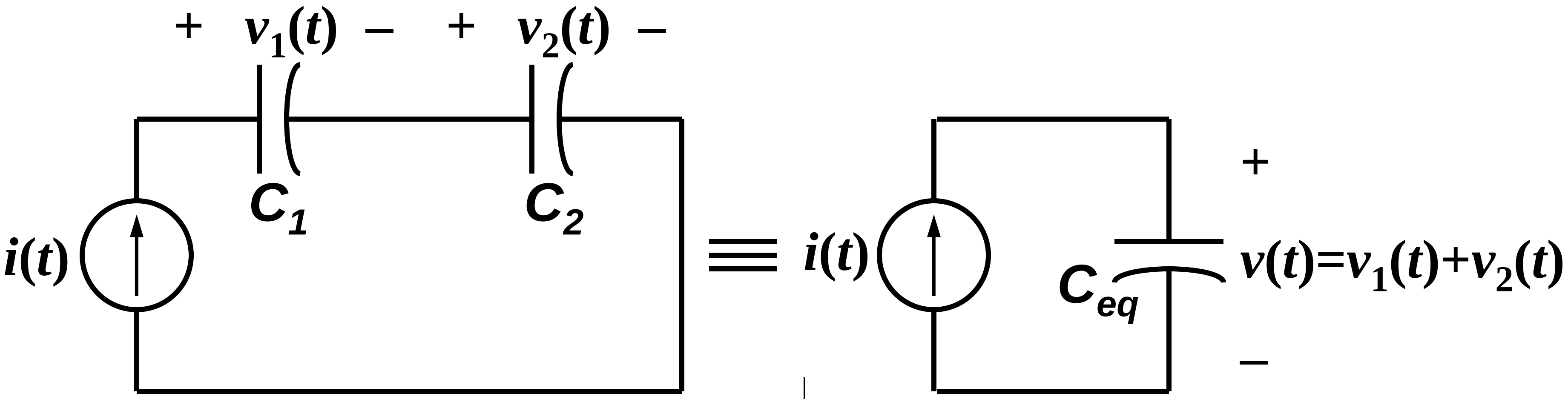




$$p = vi$$

$$w = \int_0^t p\,d\tau = \frac{1}{2}Cv^2$$

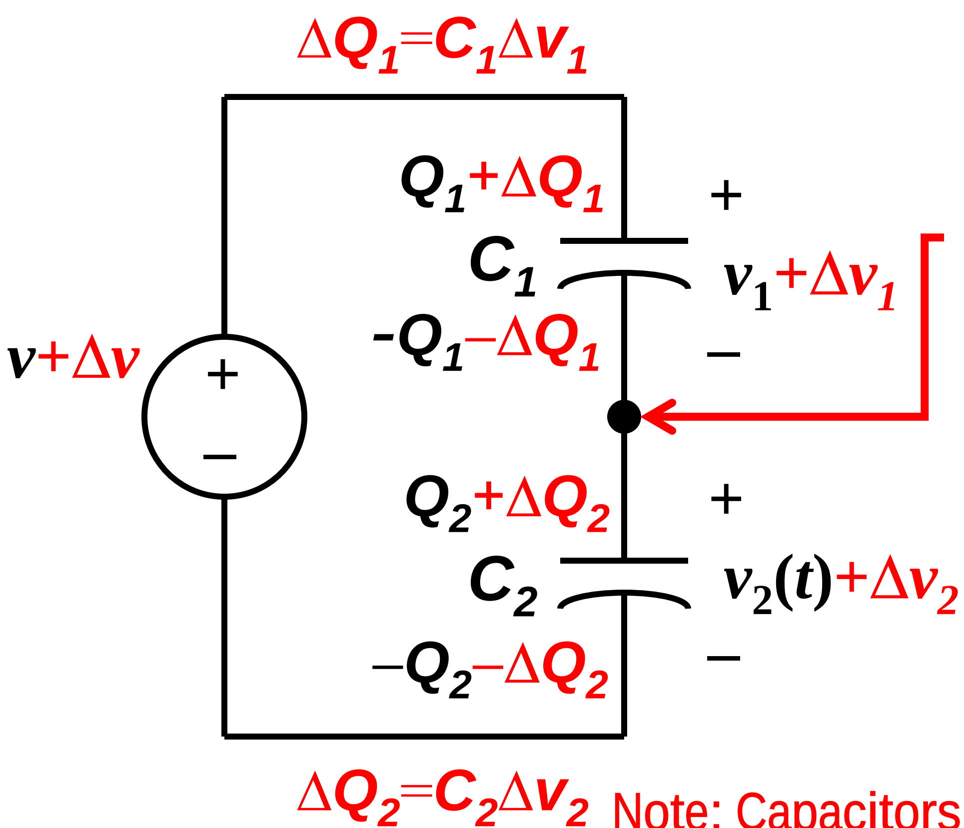
Capacitors in Series



$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$$

Capacitive Voltage Divider

Q: Suppose the voltage applied across a series combination of capacitors is changed by Δv. How will this affect the voltage across each individual capacitor?



$$\Delta v = \Delta v_1 + \Delta v_2$$

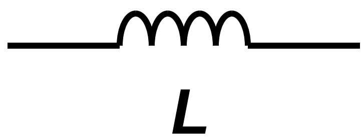
Note that no net charge can can be introduced to this node. Therefore, $-\Delta Q_1 + \Delta Q_2 = 0$

$$\Rightarrow C_1 \Delta v_1 = C_2 \Delta v_2$$

$$\boxed{\Delta v_2 = \frac{C_1}{C_1 + C_2} \Delta v}$$

Note: Capacitors in series have the same incremental charge.

Inductor

Symbol: 

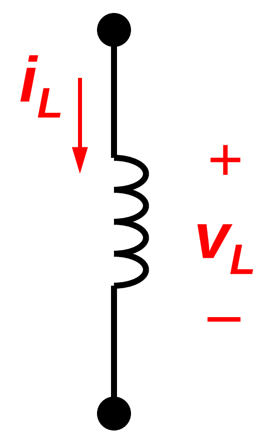
Units: Henrys (Volts • second / Ampere)

(typical range of values: μH to 10 H)

Current in terms of voltage:

$$di_L = \frac{1}{L} v_L(t)dt$$

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L(\tau)d\tau + i(t_0)$$



Note: i_L **must be a continuous function of time**

Stored Energy

INDUCTORS STORE MAGNETIC ENERGY

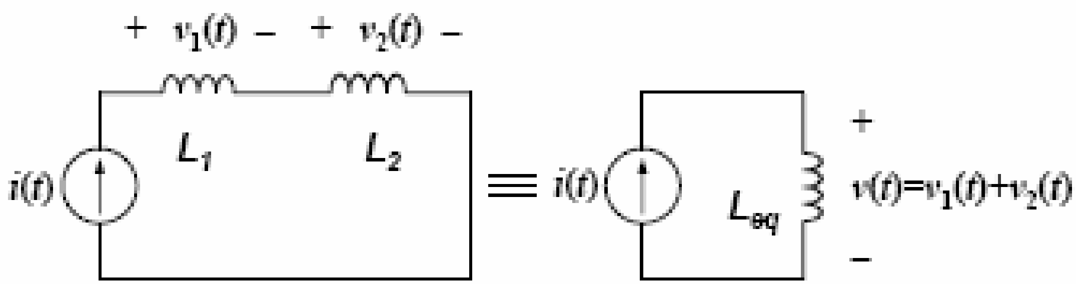
Consider an inductor having an initial current $i(t_0) = i_0$

$$p(t) = v(t)i(t) =$$

$$w(t) = \int_{t_0}^{t} p(\tau)d\tau =$$

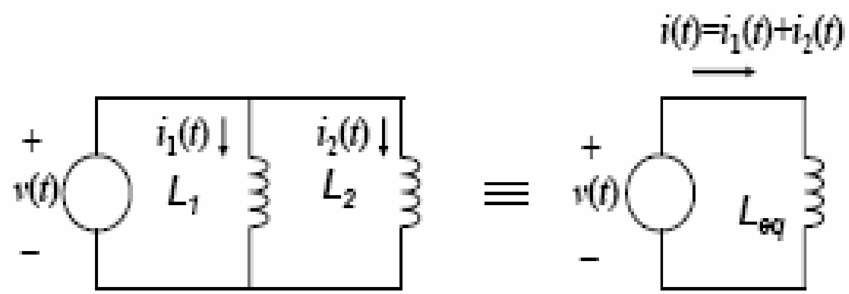
$$w(t) = \frac{1}{2}Li^2 - \frac{1}{2}Li_0{}^2$$

Inductors in Series and Parallel



Common Current

$$\boxed{L_{eq} = L_1 + L_2}$$



Common Voltage

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}}$$

Summary

Capacitor

$$i = C\frac{dv}{dt}; w = \frac{1}{2}Cv^2$$

v cannot change instantaneously

i can change instantaneously

Do not short-circuit a charged capacitor (-> infinite current!)

n cap.'s in series: $\frac{1}{C_{eq}} = \sum_{i=1}^{n}\frac{1}{C_i}$

n cap.'s in parallel: $C_{eq} = \sum_{i=1}^{n}C_i$

In steady state (not time-varying), a capacitor behaves like an open circuit.

Inductor

$$v = L\frac{di}{dt}; w = \frac{1}{2}Li^2$$

i cannot change instantaneously

v can change instantaneously

Do not open-circuit an inductor with current (-> infinite voltage!)

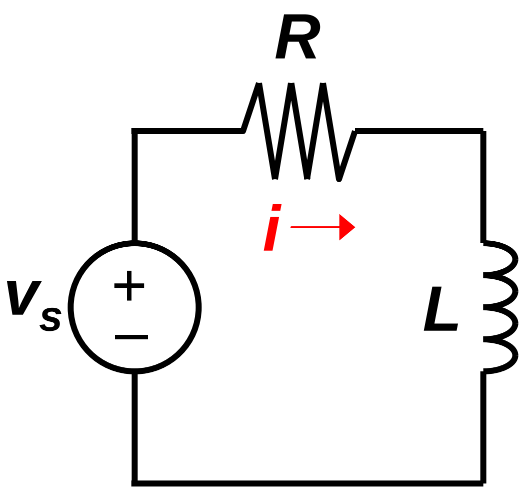
n ind.'s in series: $L_{eq} = \sum_{i=1}^{n}L_i$

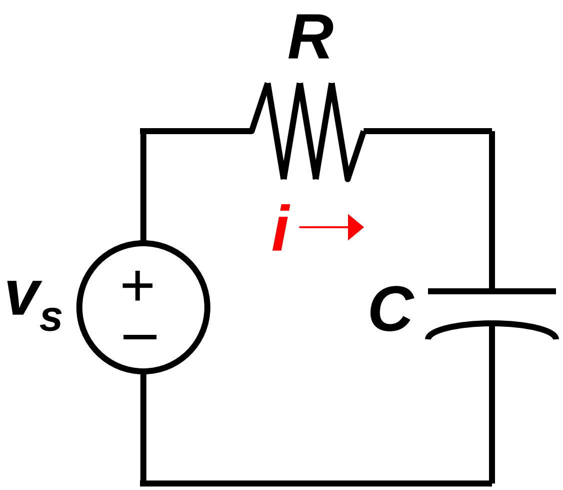
n ind.'s in parallel: $\frac{1}{L_{eq}} = \sum_{i=1}^{n}\frac{1}{L_i}$

In steady state, an inductor behaves like a short circuit.

First-Order Circuits

- A circuit that contains only sources, resistors and an inductor is called an ***RL circuit***.
- A circuit that contains only sources, resistors and a capacitor is called an ***RC circuit***.
- RL and RC circuits are called first-order circuits because their voltages and currents are described by first-order differential equations.



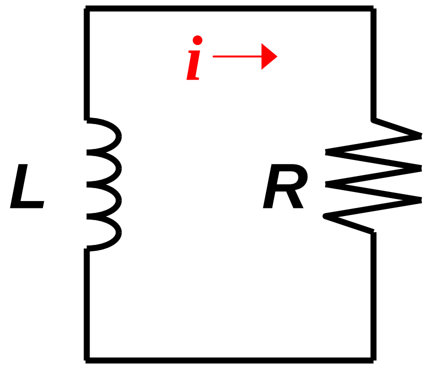


Response of a Circuit

- ***Transient response*** of an RL or RC circuit is
  - Behavior when voltage or current source are **suddenly** applied to or removed from the circuit due to switching.
  - Temporary behavior
- ***Steady-state response (aka. forced response)***
  - Response that persists long after transient has decayed
- ***Natural response*** of an RL or RC circuit is
  - Behavior (*i.e.*, current and voltage) when stored energy in the inductor or capacitor is released to the resistive part of the network (containing no independent sources).

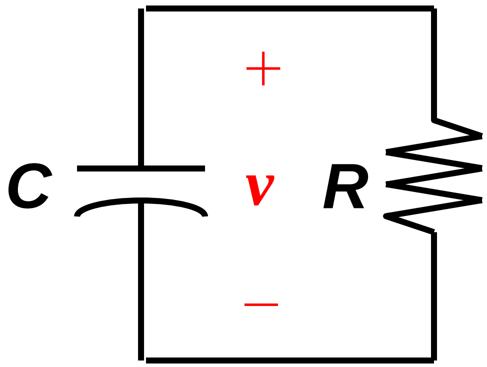
Natural Response Summary

RL Circuit



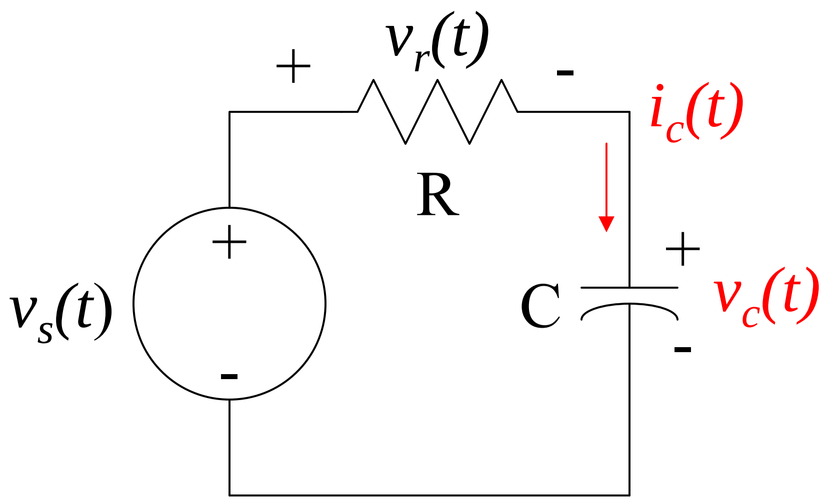
- Inductor current cannot change instantaneously
- In steady state, an inductor behaves like a short circuit.

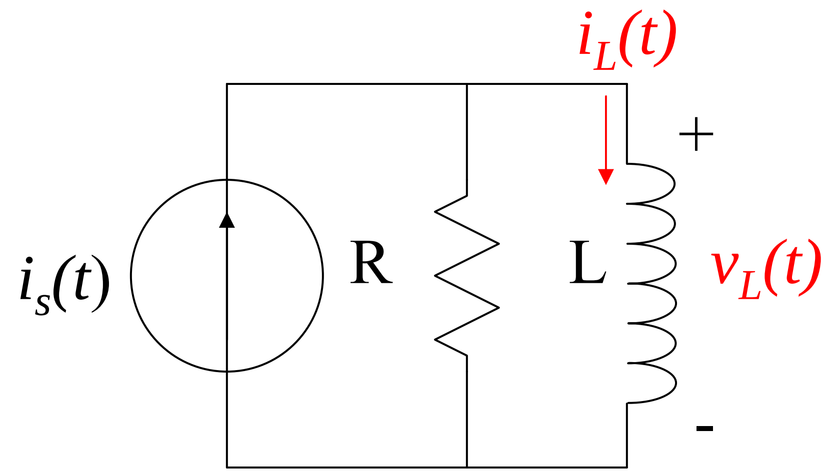
RC Circuit



- Capacitor voltage cannot change instantaneously
- In steady state, a capacitor behaves like an open circuit

First Order Circuits





KVL around the loop:

$v_r(t) + v_c(t) = v_s(t)$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

KCL at the node:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)dx = i_s(t)$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

Procedure for Finding Transient Response

1. **Identify the variable of interest**
   - For RL circuits, it is usually the inductor current $i_L(t)$
   - For RC circuits, it is usually the capacitor voltage $v_c(t)$
2. **Determine the initial value (at $t = t_0^-$ and t_0^+) of the variable**
   - Recall that $i_L(t)$ and $v_c(t)$ are continuous variables:

     $$i_L(t_0^+) = i_L(t_0^-) \quad \text{and} \quad v_c(t_0^+) = v_c(t_0^-)$$

   - Assuming that the circuit reached steady state before t_0, use the fact that an inductor behaves like a short circuit in steady state or that a capacitor behaves like an open circuit in steady state

Procedure (cont'd)

3. **Calculate the final value of the variable**
   (its value as $t \rightarrow \infty$)
   - Again, make use of the fact that an inductor behaves like a short circuit in steady state ($t \rightarrow \infty$) or that a capacitor behaves like an open circuit in steady state ($t \rightarrow \infty$)

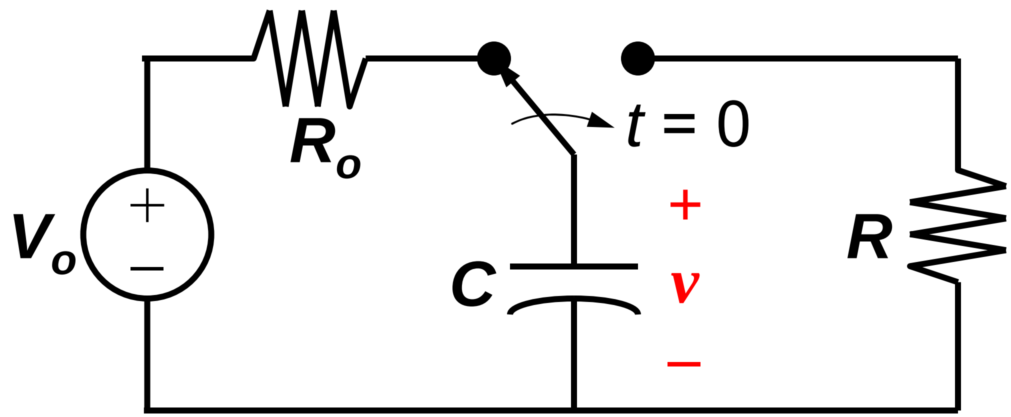
4. **Calculate the time constant for the circuit**

   $\tau = L/R$ for an RL circuit, where R is the Thévenin equivalent resistance "seen" by the inductor

   $\tau = RC$ for an RC circuit where R is the Thévenin equivalent resistance "seen" by the capacitor

Natural Response of an RC Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

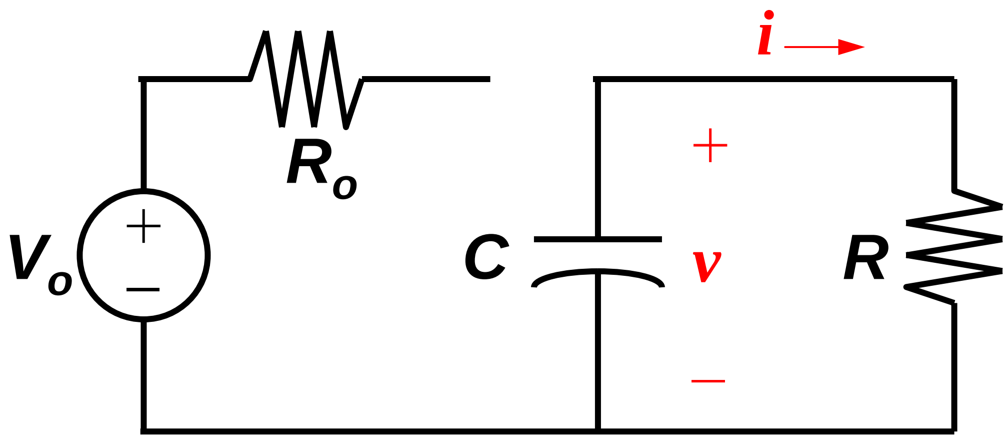
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- The voltage on the capacitor at $t = 0^-$ is V_o

Solving for the Voltage ($t \geq 0$)

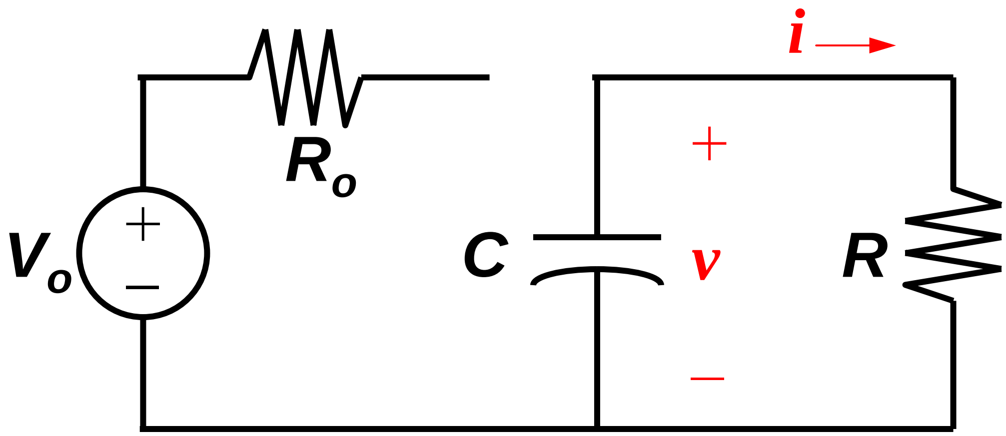
- For t > 0, the circuit reduces to



- Applying KCL to the RC circuit:

- Solution: $$v(t) = v(0)e^{-t/RC}$$

Solving for the Current (t > 0)



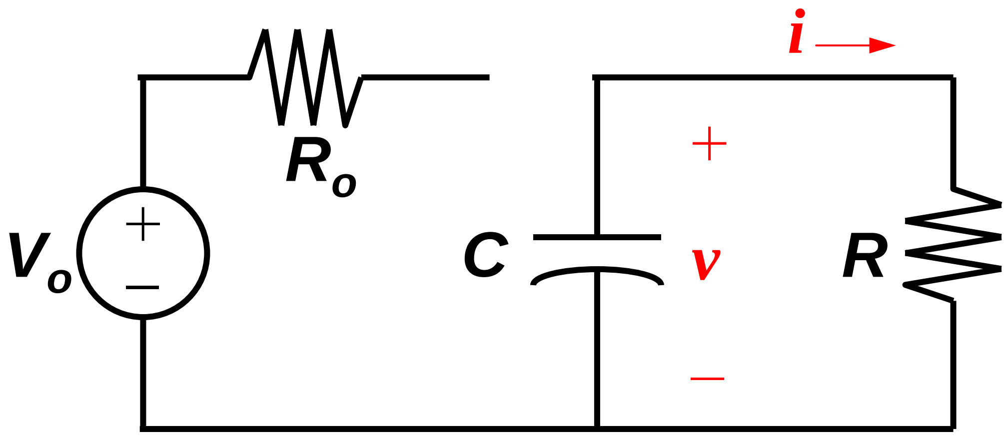
$$v(t) = V_o e^{-t/RC}$$

- Note that the current changes abruptly:
  $i(0^-) = 0$

$$\text{for } \; t > 0, \; i(t) = \frac{v}{R} = \frac{V_o}{R} e^{-t/RC}$$

$$\Rightarrow \; i(0^+) = \frac{V_o}{R}$$

Solving for Power and Energy Delivered (t > 0)



$$v(t) = V_o e^{-t/RC}$$

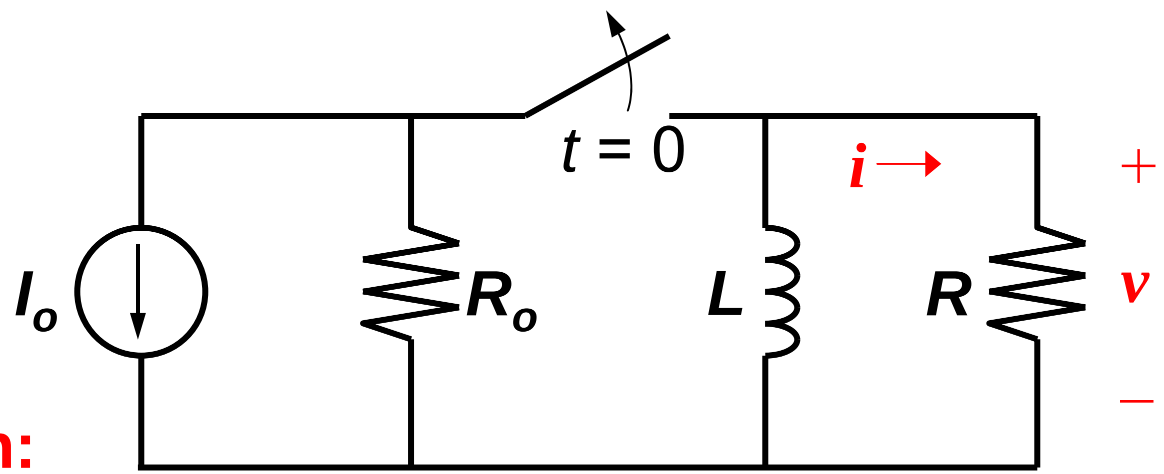
$$p = \frac{v^2}{R} = \frac{V_o^2}{R} e^{-2t/RC}$$

$$w = \int_0^t p(x)dx = \int_0^t \frac{V_o^2}{R} e^{-2x/RC} dx$$

$$= \frac{1}{2} C V_o^2 \left(1 - e^{-2t/RC}\right)$$

Natural Response of an RL Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

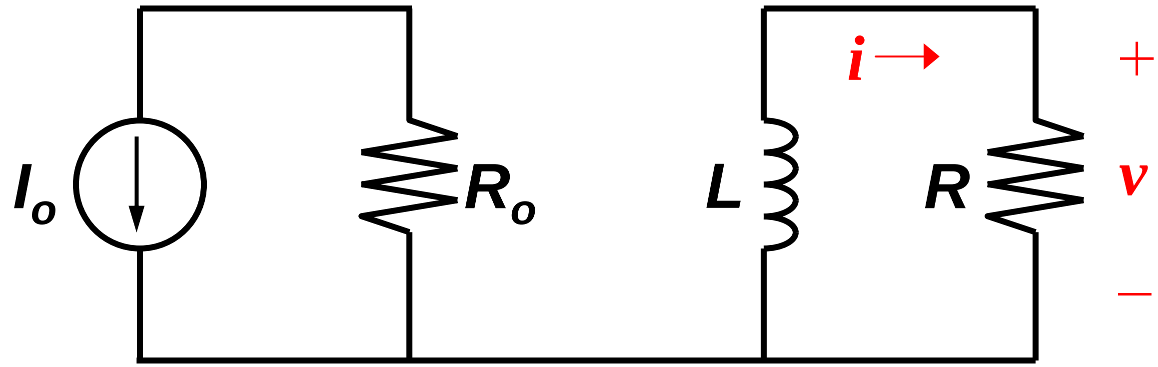
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- t<0 the entire system is at steady-state; and the inductor is → like short circuit
- The current flowing in the inductor at $t = 0^-$ is I_o and V across is 0.

Solving for the Current ($t \geq 0$)

- For t > 0, the circuit reduces to

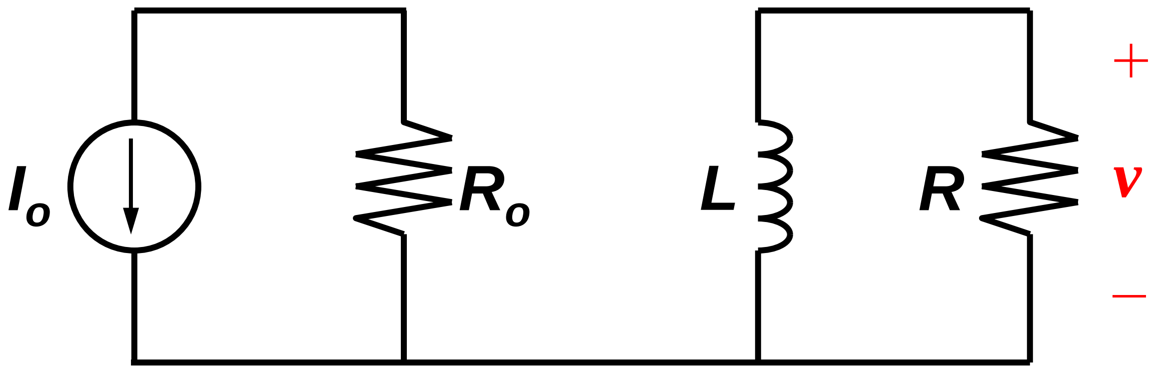


- Applying KVL to the LR circuit:
- $v(t) = i(t)R$
- At $t=0^+$, $i = I_{0,}$
- At arbitrary t>0, $i = i(t)$ and $v(t) = -L\frac{di(t)}{dt}$

- Solution: $i(t) = i(0)e^{-(R/L)t} = I_0 e^{-(R/L)t}$

Solving for the Voltage (t > 0)

$$i(t) = I_o e^{-(R/L)t}$$



- Note that the voltage changes abruptly:

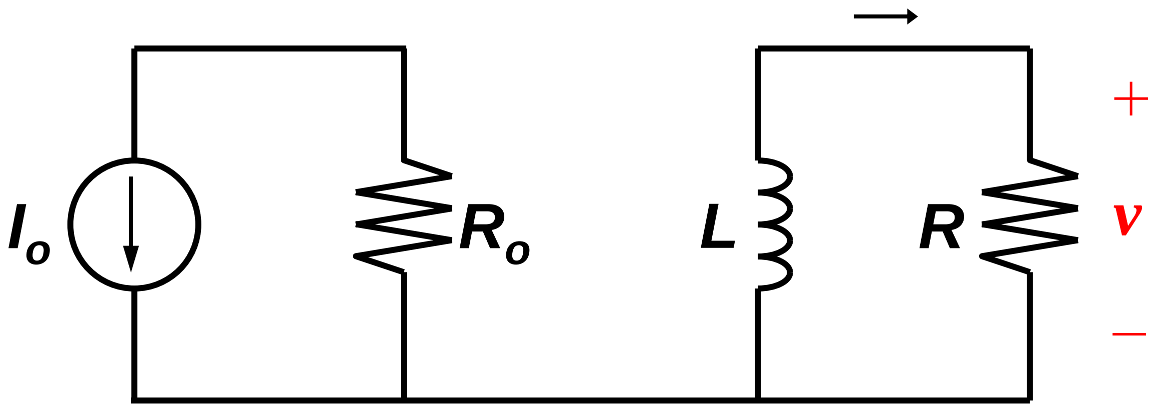
$$v(0^-) = 0$$

$$\text{for } t > 0,\; v(t) = iR = I_o R e^{-(R/L)t}$$

$$\Rightarrow\; v(0^+) = I_0 R$$

Solving for Power and Energy Delivered (t > 0)

$$i(t) = I_o e^{-(R/L)t}$$



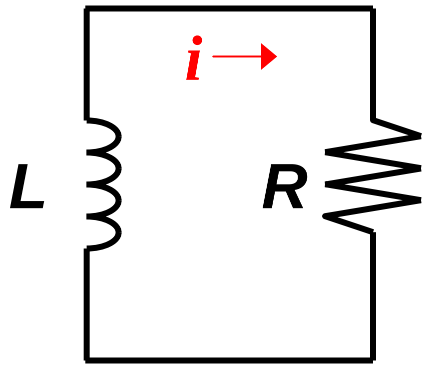
$$p = i^2 R = I_o^2 R e^{-2(R/L)t}$$

$$w = \int_0^t p(x)dx = \int_0^t I_o^2 R e^{-2(R/L)x} dx$$

$$= \frac{1}{2} L I_o^2 \left(1 - e^{-2(R/L)t}\right)$$

Natural Response Summary

RL Circuit



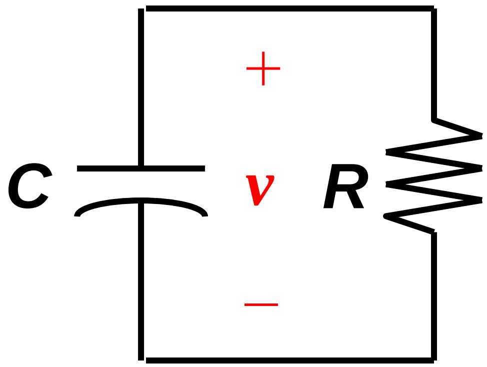
- Inductor current cannot change instantaneously

$$i(0^-) = i(0^+)$$

$$i(t) = i(0)e^{-t/\tau}$$

- time constant $\tau = \frac{L}{R}$

RC Circuit



- Capacitor voltage cannot change instantaneously

$$v(0^-) = v(0^+)$$

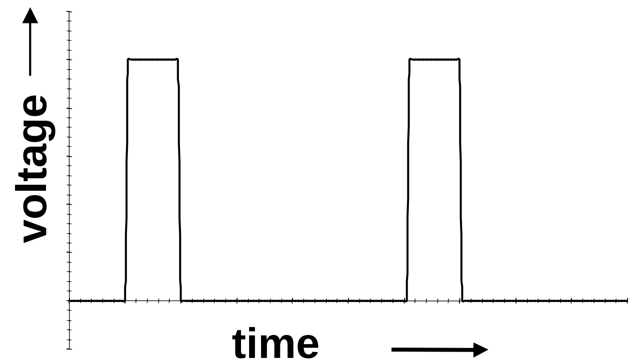
$$v(t) = v(0)e^{-t/\tau}$$

- time constant $\tau = RC$

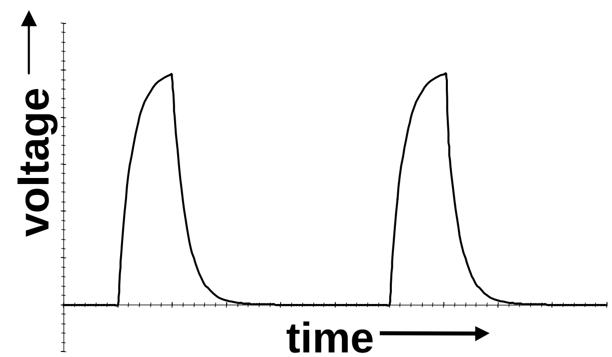
Digital Signals

We compute with pulses.

We send beautiful pulses in:



But we receive lousy-looking pulses at the output:

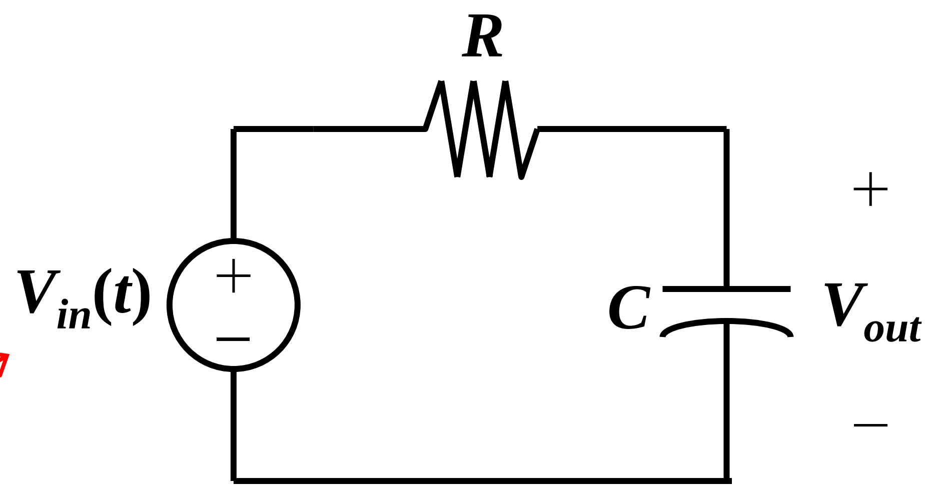


Capacitor charging effects are responsible!

- Every node in a real circuit has capacitance; it's the charging of these capacitances that limits circuit performance (speed)

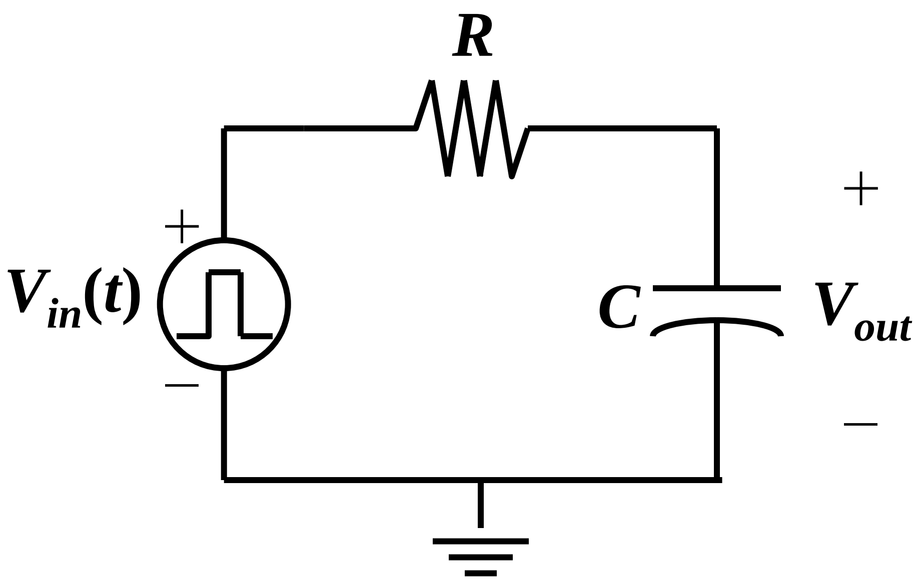
Circuit Model for a Logic Gate

- Recall (from Lecture 1) that electronic building blocks referred to as "logic gates" are used to implement logical functions (NAND, NOR, NOT) in digital ICs
  - Any logical function can be implemented using these gates.
- A logic gate can be modeled as a simple RC circuit:



switches between "low" (logic 0) and "high" (logic 1) voltage states

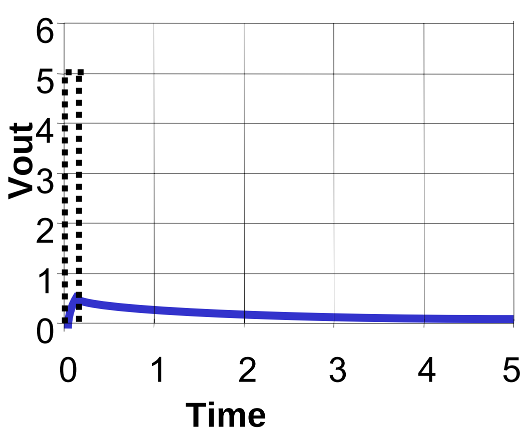
Pulse Distortion



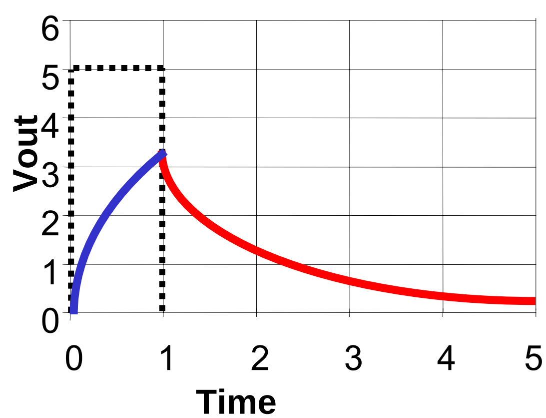
The input voltage pulse width must be large enough; otherwise the output pulse is distorted.

(We need to wait for the output to reach a recognizable logic level, before changing the input again.)

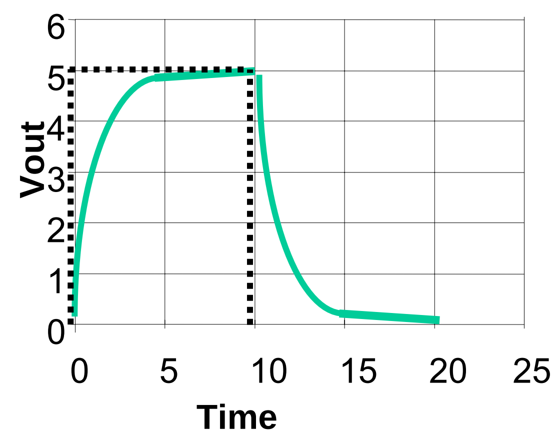
Pulse width = 0.1*RC*



Pulse width = *RC*

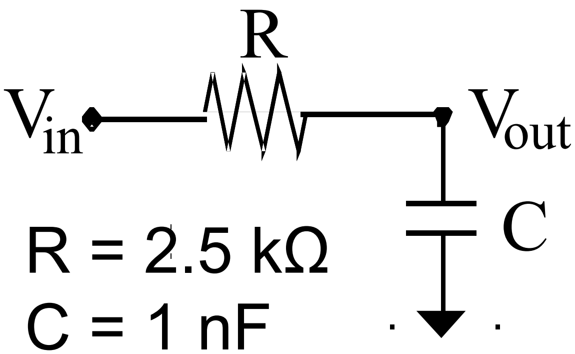


Pulse width = 10*RC*



Example

Suppose a voltage pulse of width 5 μs and height 4 V is applied to the input of this circuit beginning at t = 0:



R = 2.5 kΩ
C = 1 nF

$$\tau = RC = 2.5\ \mu s$$

- First, V_{out} will increase exponentially toward 4 V.
- When V_{in} goes back down, V_{out} will decrease exponentially back down to 0 V.

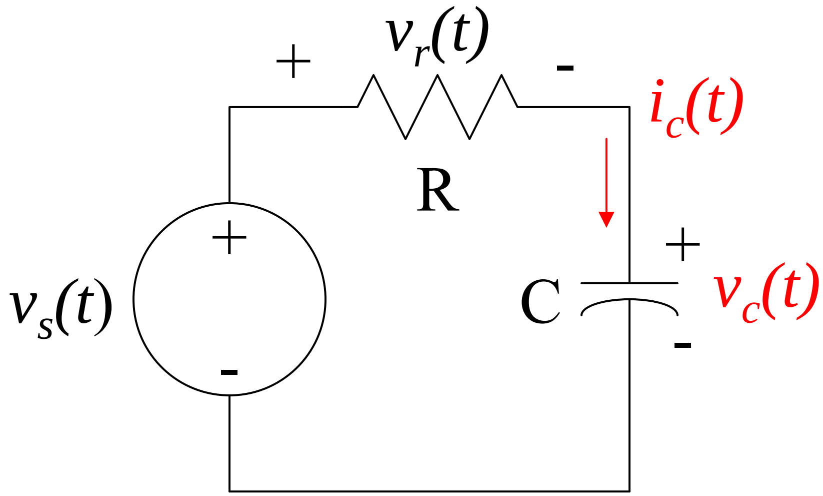
What is the peak value of V_{out}?

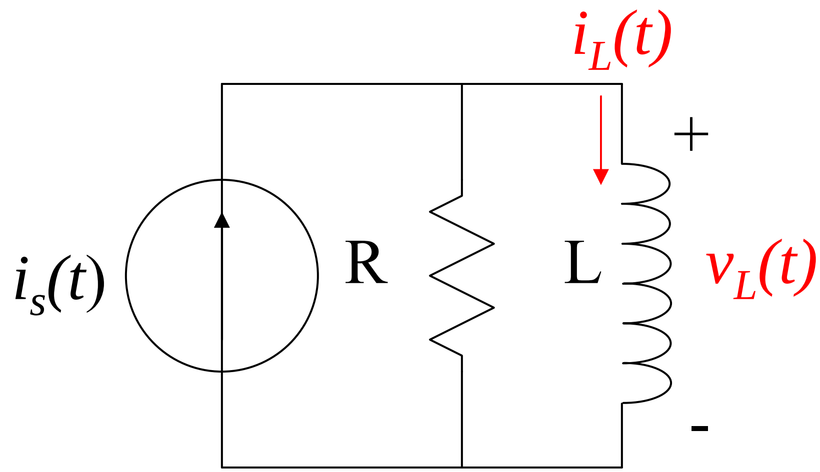
The output increases for 5 μs, or 2 time constants.

→ It reaches $1\text{-}e^{-2}$ or 86% of the final value.

0.86 x 4 V = 3.44 V is the peak value

First Order Circuits: Forced Response





KVL around the loop:

$v_r(t) + v_c(t) = v_s(t)$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

KCL at the node:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)dx = i_s(t)$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

Complete Solution

- Voltages and currents in a 1st order circuit satisfy a differential equation of the form

$$x(t) + \tau \frac{dx(t)}{dt} = f(t)$$

  - *f(t)* is called the forcing function.
- The complete solution is the sum of particular solution (forced response) and complementary solution (natural response).

$$x(t) = x_p(t) + x_c(t)$$

  - Particular solution satisfies the forcing function
  - Complementary solution is used to satisfy the initial conditions.
  - The initial conditions determine the value of *K.*

$$x_p(t) + \tau \frac{dx_p(t)}{dt} = f(t)$$

$$x_c(t) + \tau \frac{dx_c(t)}{dt} = 0$$

Homogeneous equation

$$x_c(t) = Ke^{-t/\tau}$$

The Time Constant

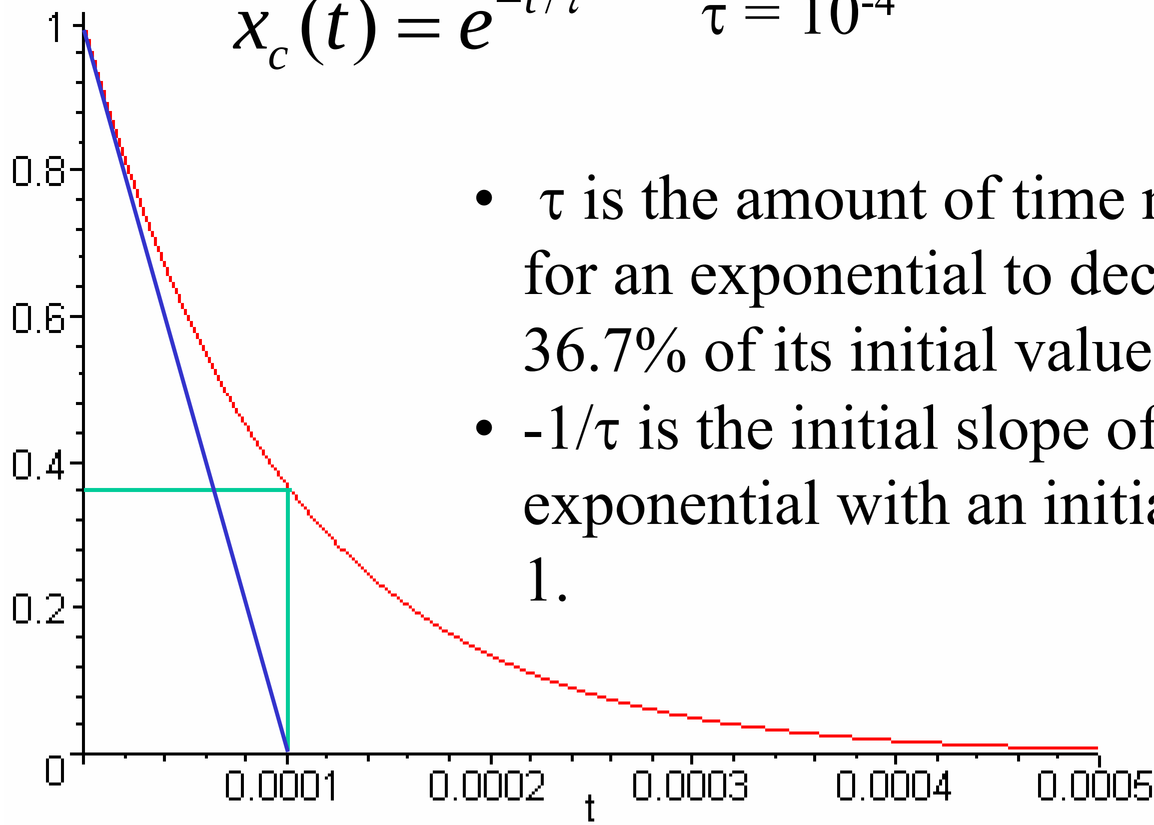
- The complementary solution for any 1st order circuit is

$$x_c(t) = Ke^{-t/\tau}$$

- For an RC circuit, $\tau = RC$
- For an RL circuit, $\tau = L/R$

What Does $X_c(t)$ Look Like?

$$x_c(t) = e^{-t/\tau} \qquad \tau = 10^{-4}$$



- τ is the amount of time necessary for an exponential to decay to 36.7% of its initial value.
- $-1/\tau$ is the initial slope of an exponential with an initial value of 1.

The Particular Solution

- The particular solution $x_p(t)$ is usually a weighted sum of $f(t)$ and its first derivative.
- If $f(t)$ is constant, then $x_p(t)$ is constant.
- If $f(t)$ is sinusoidal, then $x_p(t)$ is sinusoidal.

The Particular Solution: F(t) Constant

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F$$

Guess a solution

$$x_P(t) = A + Bt$$

$$(A + Bt) + \tau \frac{d(A + Bt)}{dt} = F$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$(A + Bt) + \tau B = F$$

$$(A + \tau B - F) + (B)t = 0$$

$$(B) = 0$$

$$B = 0$$

$$(A + \tau B - F) = 0$$

$$A = F$$

The Particular Solution: F(t) Sinusoid

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

Guess a solution $\quad x_P(t) = A\sin(wt) + B\cos(wt)$

$$(A\sin(wt) + B\cos(wt)) + \tau \frac{d(A\sin(wt) + B\cos(wt))}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$(A - \tau\omega B - F_A)\sin(\omega t) + (B + \tau\omega A - F_B)\cos(\omega t) = 0$$

$$(A - \tau\omega B - F_A) = 0 \qquad (B + \tau\omega A - F_B) = 0$$

$$A = \frac{F_A + \tau\omega F_B}{(\tau\omega)^2 + 1} \qquad B = -\frac{\tau\omega F_A - F_B}{(\tau\omega)^2 + 1}$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$x_P(t) = \frac{1}{\sqrt{(\tau\omega)^2 + 1}}\left[\frac{\tau\omega}{\sqrt{(\tau\omega)^2 + 1}}\sin(\omega t) + \frac{1}{\sqrt{(\tau\omega)^2 + 1}}\cos(\omega t)\right]$$

$$= \frac{1}{\sqrt{(\tau\omega)^2 + 1}}\cos(\omega t - \theta); \quad where\ \theta = \tan^{-1}(\tau\omega)$$

The Particular Solution: F(t) Exp.

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_1 e^{-\alpha t} + F_2$$

Guess a solution

$$x_P(t) = A + Be^{-\alpha t}$$

$$(A + Be^{-\alpha t}) + \tau \frac{d(A + Be^{-\alpha t})}{dt} = F_1 e^{-\alpha t} + F_2$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$(A + Be^{-\alpha t}) - \alpha \tau B e^{-\alpha t} = F_1 e^{-\alpha t} + F_2$$

$$(A - F_2) + (B - \alpha \tau - F_1) e^{-\alpha t} = 0$$

$$(B - \alpha \tau - F_1) = 0$$

$$B = \alpha \tau + F1$$

$$(A - F_2) = 0$$

$$A = F_2$$

The Total Solution: F(t) Sinusoid

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$x_P(t) = A\sin(wt) + B\cos(wt) \qquad A = \frac{F_A + \tau\omega F_B}{(\tau\omega)^2 + 1} \qquad B = -\frac{\tau\omega F_A - F_B}{(\tau\omega)^2 + 1}$$

$$x_C(t) = Ke^{-t/\tau}$$

$$x_T(t) = A\sin(wt) + B\cos(wt) + Ke^{-t/\tau}$$

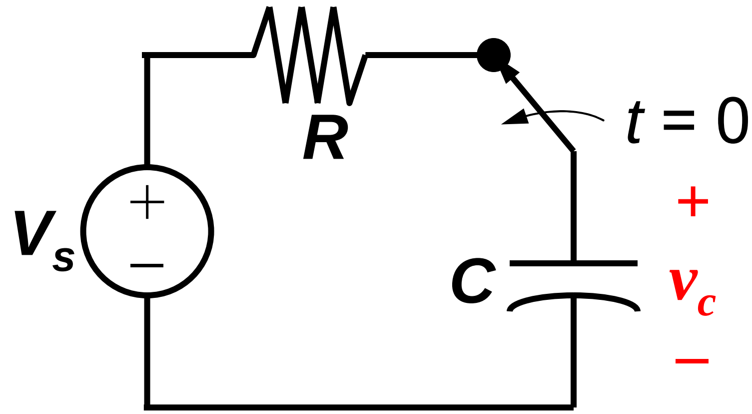
Only K is unknown and is determined by the initial condition at t =0

Example: $x_T(t=0) = V_C(t=0)$

$$x_T(0) = A\sin(0) + B\cos(0) + Ke^{-0/\tau} = V_C(t=0)$$

$$x_T(0) = B + K = V_C(t=0) \qquad K = V_C(t=0) - B$$

Example



- Given $v_c(0^-)=1$, $V_s=2\cos(\omega t)$, $\omega=200$.
- *Find* $i(t)$, $v_c(t)=?$