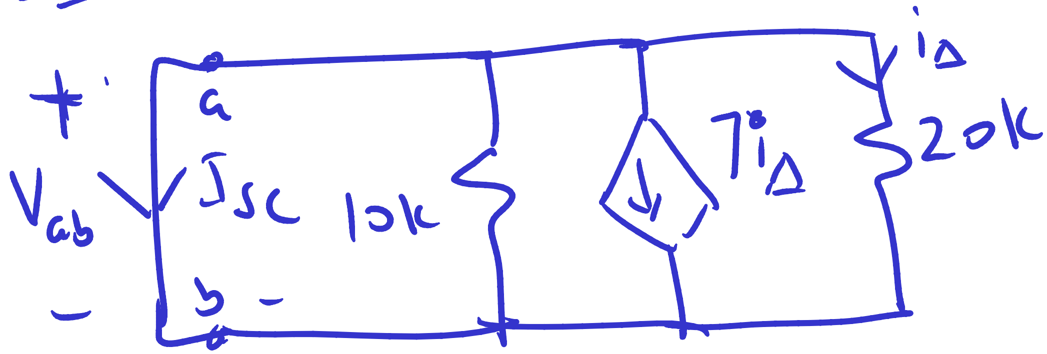
EE100Su08 Lecture #7 (July 9th 2008)

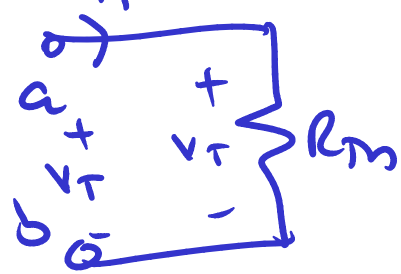
- Outline
  - Questions?
  - Finish Chapter #7
    - RC circuit: total response to a step input
    - RL circuit: total response to a step input
    - Application of Thevenin's Theorem to RC/RL circuits

Question: Thevenin equivalent for a circuit with only dependent sources (and resistors)

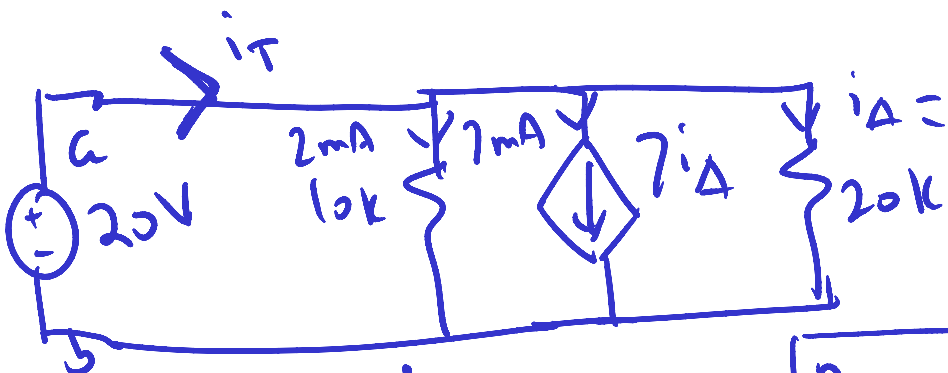
Ex:



Find Thevenin equivalent at ab.



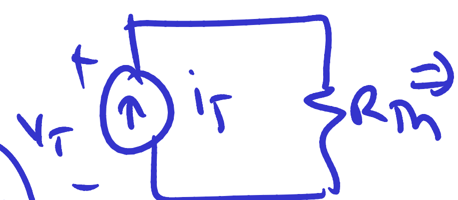
$I_{sc} = ?$ $\quad V_{ab} = 0 \Rightarrow i_\Delta = \frac{V_{ab}}{20k} = 0$



$i_\Delta = 1\,mA$

$R_{Th} = \frac{V_T}{i_T}$

(Or)



$\Rightarrow R_{Th} = \frac{V_T}{i_T}$

$\therefore\ i_T = 10\,mA \Rightarrow \boxed{R_{Th} = \frac{20V}{10mA} = 2k}$

Summary

Capacitor

$$i = C\frac{dv}{dt}; w = \frac{1}{2}Cv^2$$

v cannot change instantaneously

i can change instantaneously

Do not short-circuit a charged capacitor (-> infinite current!)

n cap.'s in series: $$\frac{1}{C_{eq}} = \sum_{i=1}^{n}\frac{1}{C_i}$$

n cap.'s in parallel: $$C_{eq} = \sum_{i=1}^{n} C_i$$

In steady state (not time-varying), a capacitor behaves like an open circuit.

Inductor

$$v = L\frac{di}{dt}; w = \frac{1}{2}Li^2$$

i cannot change instantaneously

v can change instantaneously

Do not open-circuit an inductor with current (-> infinite voltage!)

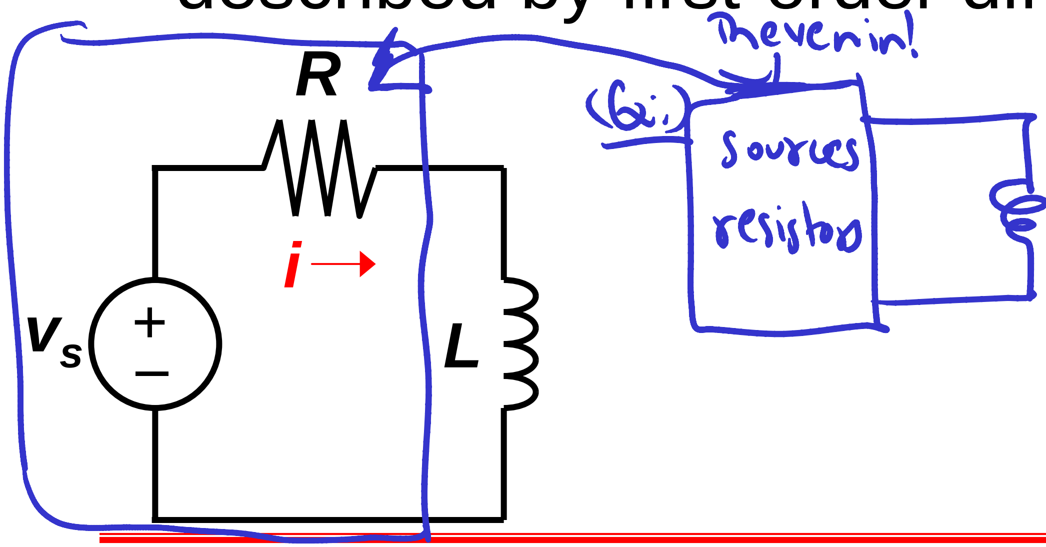
n ind.'s in series: $$L_{eq} = \sum_{i=1}^{n} L_i$$

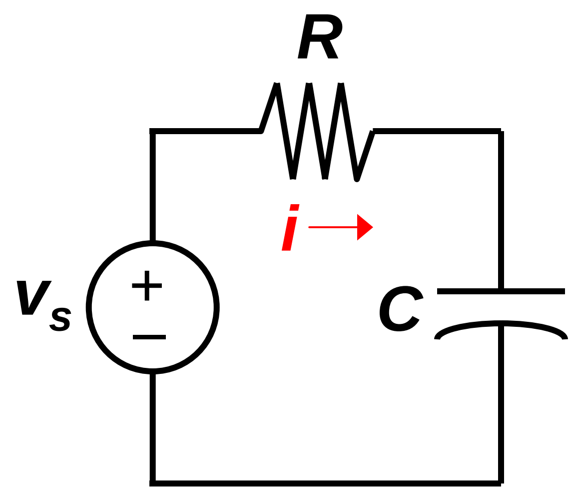
n ind.'s in parallel: $$\frac{1}{L_{eq}} = \sum_{i=1}^{n}\frac{1}{L_i}$$

In steady state, an inductor behaves like a short circuit.

First-Order Circuits

- A circuit that contains only sources, resistors and an inductor is called an ***RL circuit***.
- A circuit that contains only sources, resistors and a capacitor is called an ***RC circuit***.
- RL and RC circuits are called first-order circuits because their voltages and currents are described by first-order differential equations.



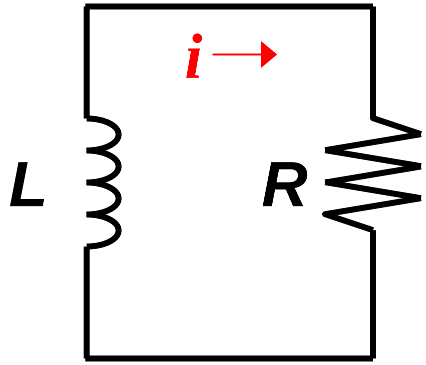


Response of a Circuit

- ***Transient response*** of an RL or RC circuit is
  - Behavior when voltage or current source are **suddenly** applied to or removed from the circuit due to switching.
  - Temporary behavior
- ***Steady-state response (aka. forced response)***
  - Response that persists long after transient has decayed
- ***Natural response*** of an RL or RC circuit is
  - Behavior (*i.e.*, current and voltage) when stored energy in the inductor or capacitor is released to the resistive part of the network (containing no independent sources).

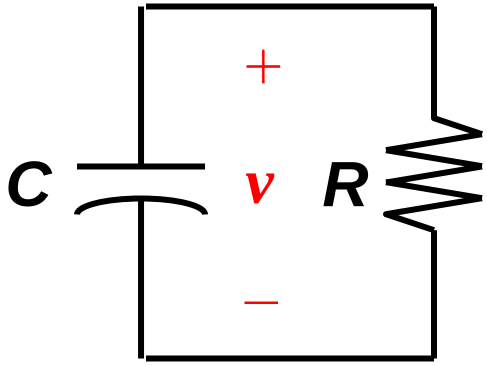
Natural Response Summary

RL Circuit



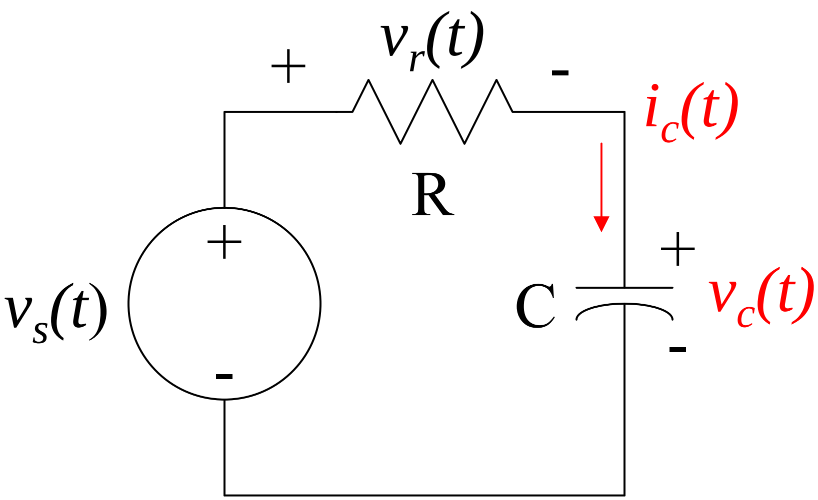
- Inductor current cannot change instantaneously
- In steady state, an inductor behaves like a short circuit.

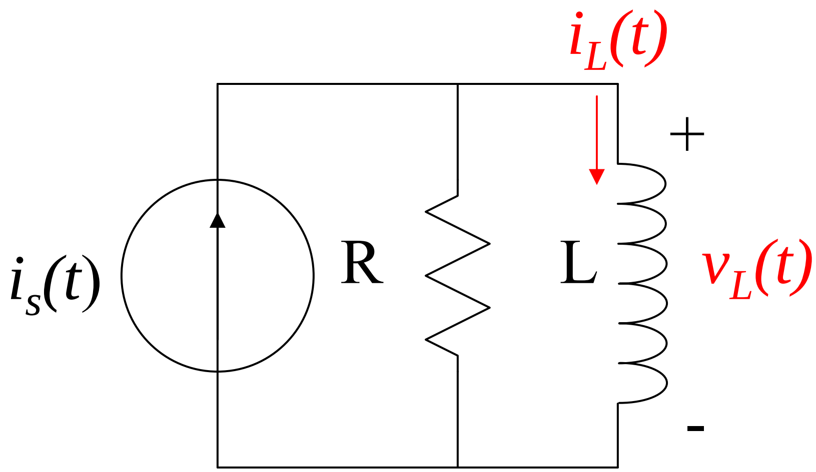
RC Circuit



- Capacitor voltage cannot change instantaneously
- In steady state, a capacitor behaves like an open circuit

First Order Circuits





KVL around the loop:

$v_r(t) + v_c(t) = v_s(t)$

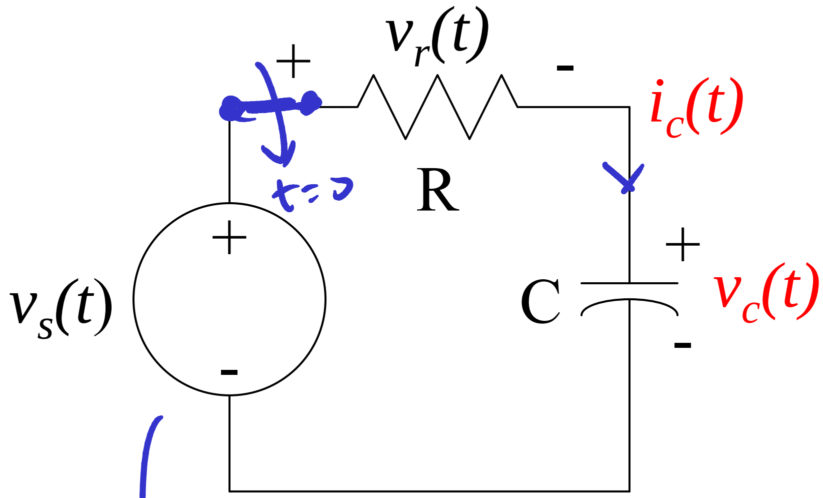
$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

KCL at the node:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)dx = i_s(t)$$

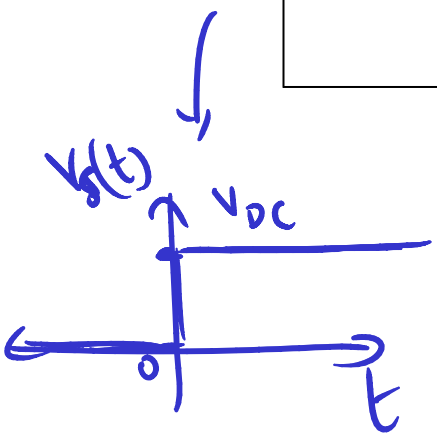
$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

Solution of RC circuit:



(Q:) Find $v_c(t)$, $t>0$

$v_c(0) = v_{initial}$



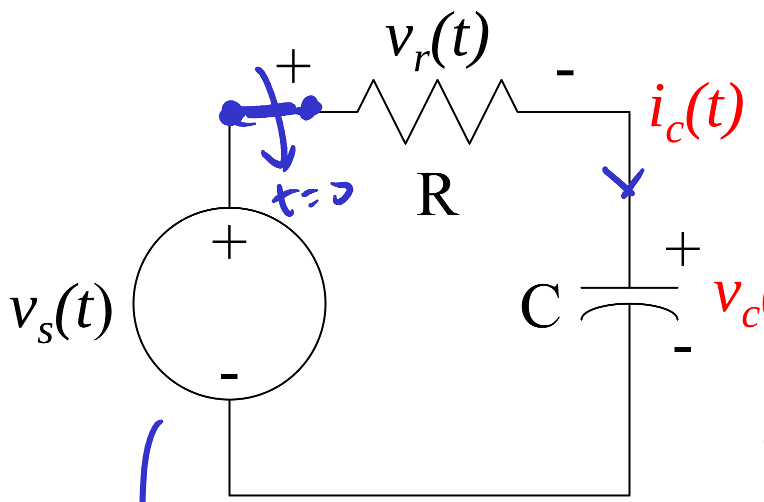
KVL: $v_s = v_r + v_c$

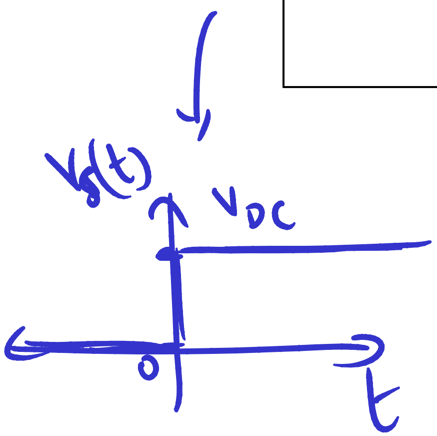
$\Rightarrow v_s(t) = iR + v_c(t)$

$\Rightarrow v_s(t) = C\frac{dv_c}{dt}R + v_c(t)$

$\Rightarrow \boxed{v_s(t) = RC\frac{dv_c}{dt} + v_c}$ $v_c(0) = v_{initial}$

Solution of RC circuit:





$$\boxed{V_s(t) = RC\frac{dv_c}{dt} + V_c(t)}$$

$V_c(0) = v_{initial}$

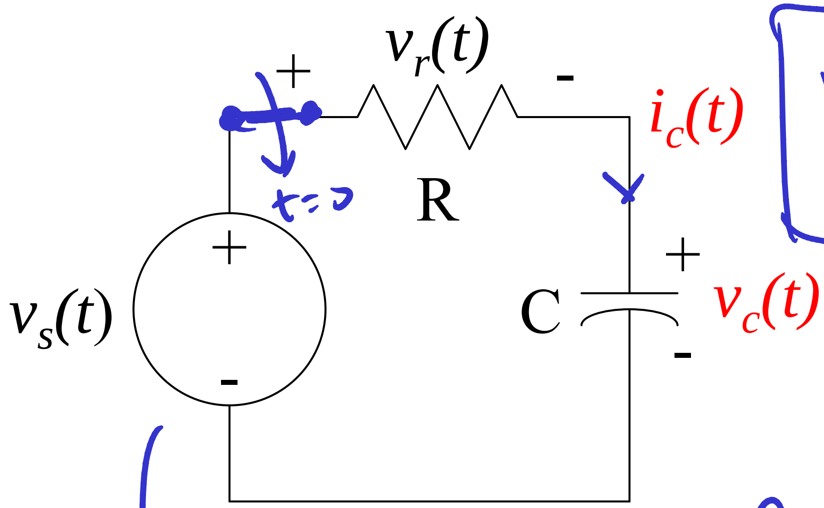
Lets try & guess the solution: suppose $V_s(t) = 0$

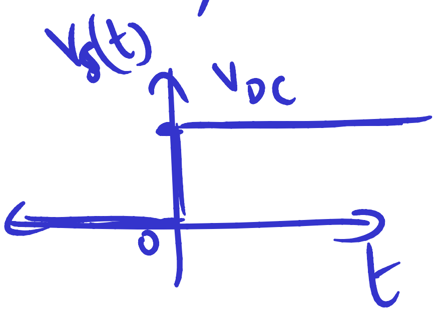
$$RC\frac{dv_c}{dt} = -V_c$$

Try: $V_c(t) = V_o e^{-t/RC}$ $[V_c(0) = V_o]$

$$\Rightarrow \frac{dV_c}{dt} = V_o e^{-t/RC} \cdot -\frac{1}{RC} = -\frac{V_o}{RC} e^{-t/RC}$$

Solution of RC circuit:





$$\boxed{V_s(t) = RC\frac{dv_c}{dt} + V_c(t)}$$

$V_c(0) = v_{initial}$

$$V_c(t) = A + Be^{-t/\alpha} \quad (1)$$

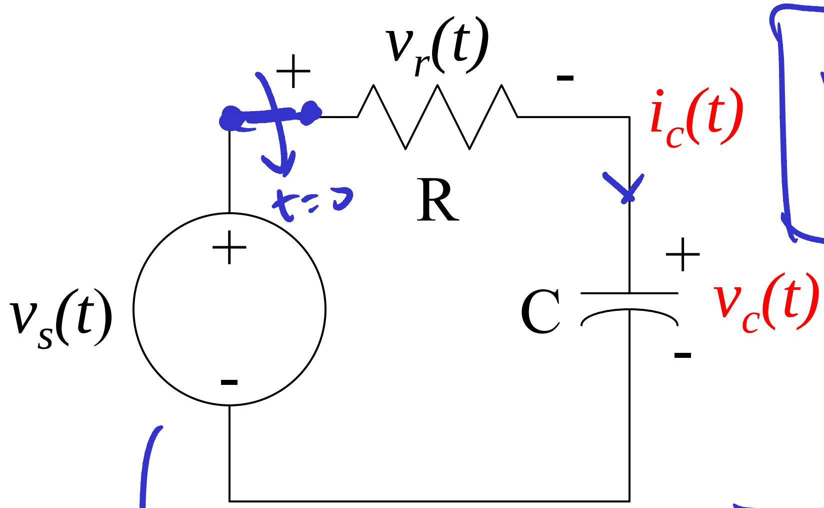
Goal: Find A, B, α

(1) must satisfy initial conditions:

$$V_C(0) = \boxed{V_{initial} = A + B}$$

(2) Consider $\lim_{t \to \infty} V_c(t) = A$

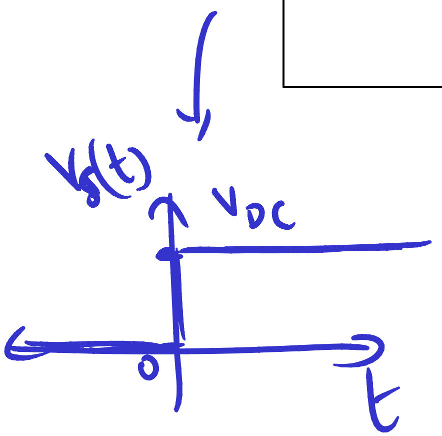
Solution of RC circuit:



$$\boxed{V_s(t) = RC\frac{dv_c}{dt} + V_c(t)}$$

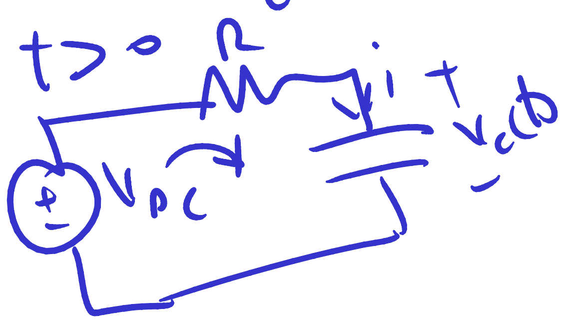
$V_c(0) = v_{initial}$

$$V_c(t) = A + Be^{-t/\tau} \quad (1)$$



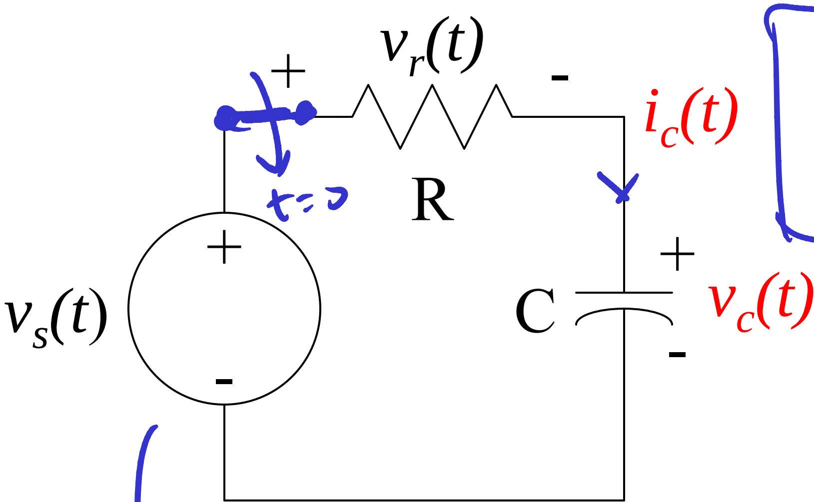
$V_c(0) \Rightarrow A+B = V_{initial}$ [specified]

$V_c(t\to\infty) \Rightarrow$ capacitor charges up to V_{DC} [constant]



$$\Rightarrow i(t\to\infty) = C\frac{dv_c}{dt} = C\frac{d}{dt}(V_{DC}) = 0$$

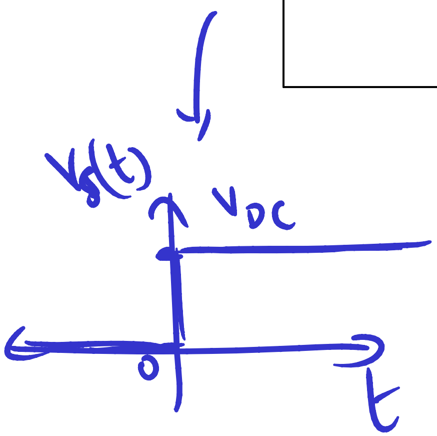
Solution of RC circuit:



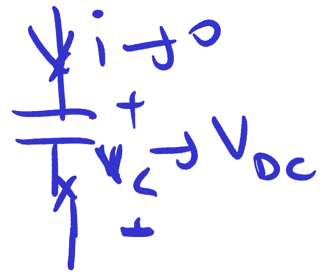
$$\boxed{V_s(t) = RC\frac{dv_c}{dt} + v_c(t)}$$

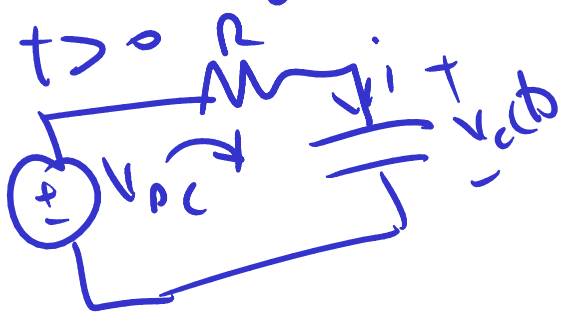
$v_c(0) = v_{initial}$

$$v_c(t) = A + Be^{-t/\tau} \quad \text{—(1)}$$

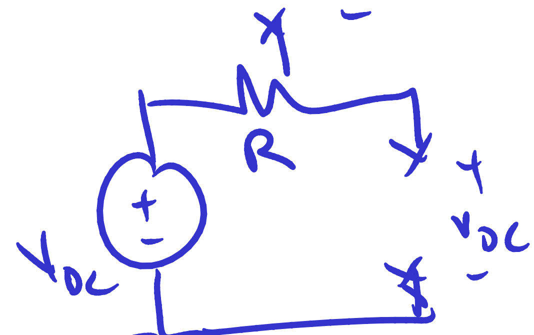


$t \to \infty$:





∴ As $t \to \infty$:



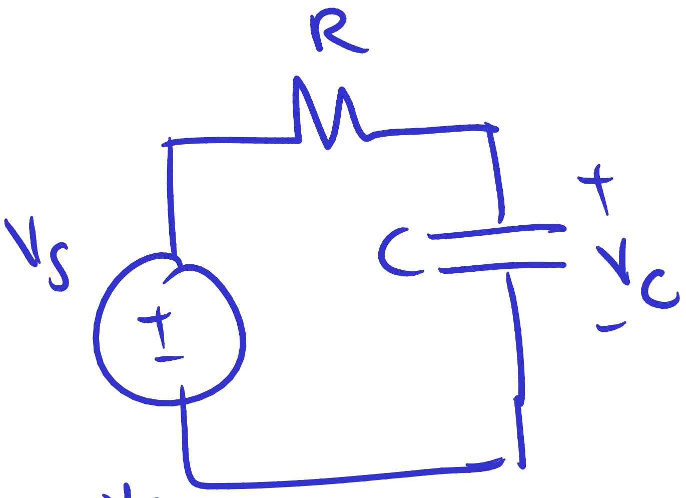
∴ $v_c(t \to \infty) = V_{DC}$

$\Rightarrow \boxed{A = V_{DC}}$

$$V_c(t) = A + Be^{-t/\alpha}$$

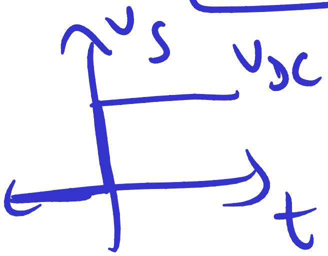
$$V_c(0) = A + B = V_{initial}$$

$$V_c(t \to \infty) = A = V_{DC}$$



$$\therefore B = V_{initial} - A$$

$$= V_{initial} - V_{DC}$$



$$\therefore V_c(t) = V_{DC} + (V_{initial} - V_{DC})e^{-t/\alpha}$$

$$RC\frac{dv_c}{dt} + V_c = V_S = V_{DC} \quad (t \geq 0)$$

$$RC\frac{dv_c}{dt} + V_c = V_{DC}, \qquad V_c(t) = V_{DC} + (V_{initial} - V_{DC})\, e^{-t/\alpha}$$

$$\Rightarrow RC\left[(V_{initial} - V_{DC})\, e^{-t/\alpha} \cdot -\frac{1}{\alpha}\right] + \left\{V_{DC} + (V_{initial} - V_{DC})\, e^{-t/\alpha}\right\} = V_{DC}$$

$$\Rightarrow \boxed{\alpha = RC}$$

$$\therefore \boxed{V_c(t) = V_{DC} + (V_{initial} - V_{DC})\, e^{-\frac{t}{RC}} \quad t \geq 0 \quad \text{Volts}}$$

$$\boxed{\alpha = RC}$$

$$\therefore \boxed{V_c(t) = V_{DC} + (V_{initial} - V_{DC})\, e^{-\frac{t}{RC}} \quad t \geq 0 \quad \text{Volts}}$$

Note: (1) $[\alpha]$ = seconds (time)

(2) $RC = \frac{v}{i} \cdot \frac{q}{v}$

$$\Rightarrow [RC] = \left[\frac{\cancel{v}}{i}\right] \cdot \left[\frac{q}{\cancel{v}}\right] = \left(\frac{q}{\frac{dq}{dt}}\right) = \text{time!}$$

Significance: $\alpha \triangleq$ Time constant $= \tau = RC$

$$\boxed{\tau = RC}$$

$$\therefore \boxed{V_C(t) = V_{DC} + (V_{initial} - V_{DC})\, e^{-\frac{t}{\tau}} \text{ Volts} \quad t \geq 0}$$

Consider: $t = 5\tau$

$$\therefore V_C(t = 5\tau) = V_{DC} + (V_{initial} - V_{DC})\, e^{-5}$$

$$\approx V_{DC}$$

$\Rightarrow$ 5 time constants, circuit has reached steady-state!

Some observations:

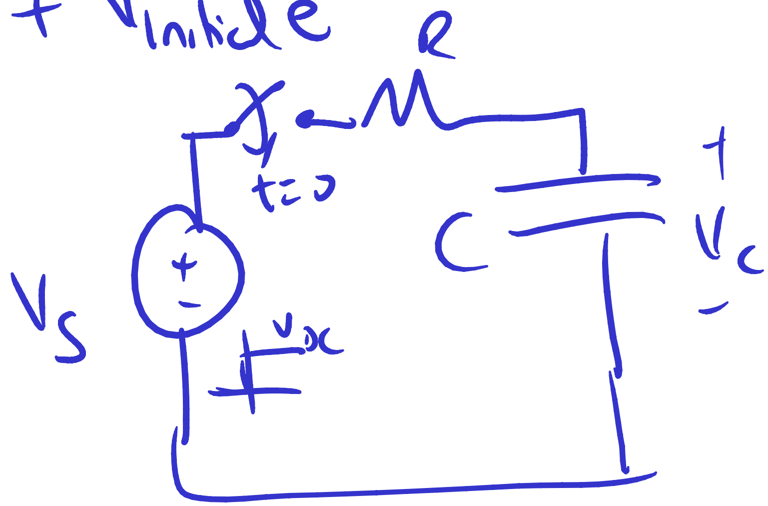
(1) $\boxed{\tau = RC}$

$$\therefore \boxed{V_c(t) = \underbrace{V_{DC}}_{①} + \underbrace{(V_{initial} - V_{DC})e^{-\frac{t}{\tau}}}_{②} \quad t \geq 0 \quad \text{Volts}}$$

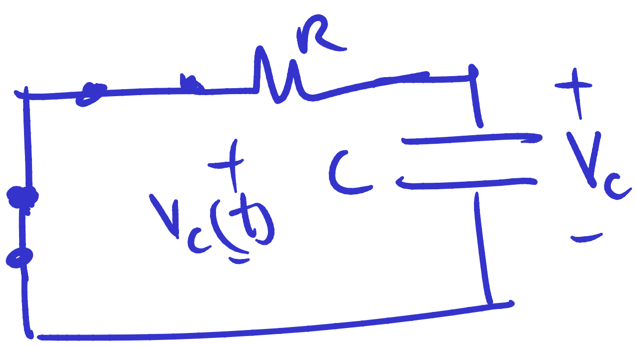
Notice: ② is the transient response, ① is the steady-state response.

(2) $V_c = V_{DC}(1 - e^{-t/\tau}) + V_{initial}e^{-t/\tau}$ — (1)

Circuit:



Suppose $V_{DC} = 0 \quad t \geq 0$

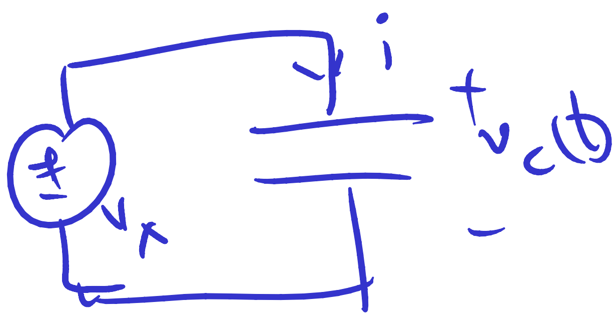


$V_c(t) = V_{initial}e^{-t/\tau}$

(From (1), $V_{DC} = 0$)

Observations (contd.)

(3) Can you physically change the voltage across a capacitor instantaneously?



$$i = C \frac{dv_c}{dt}$$

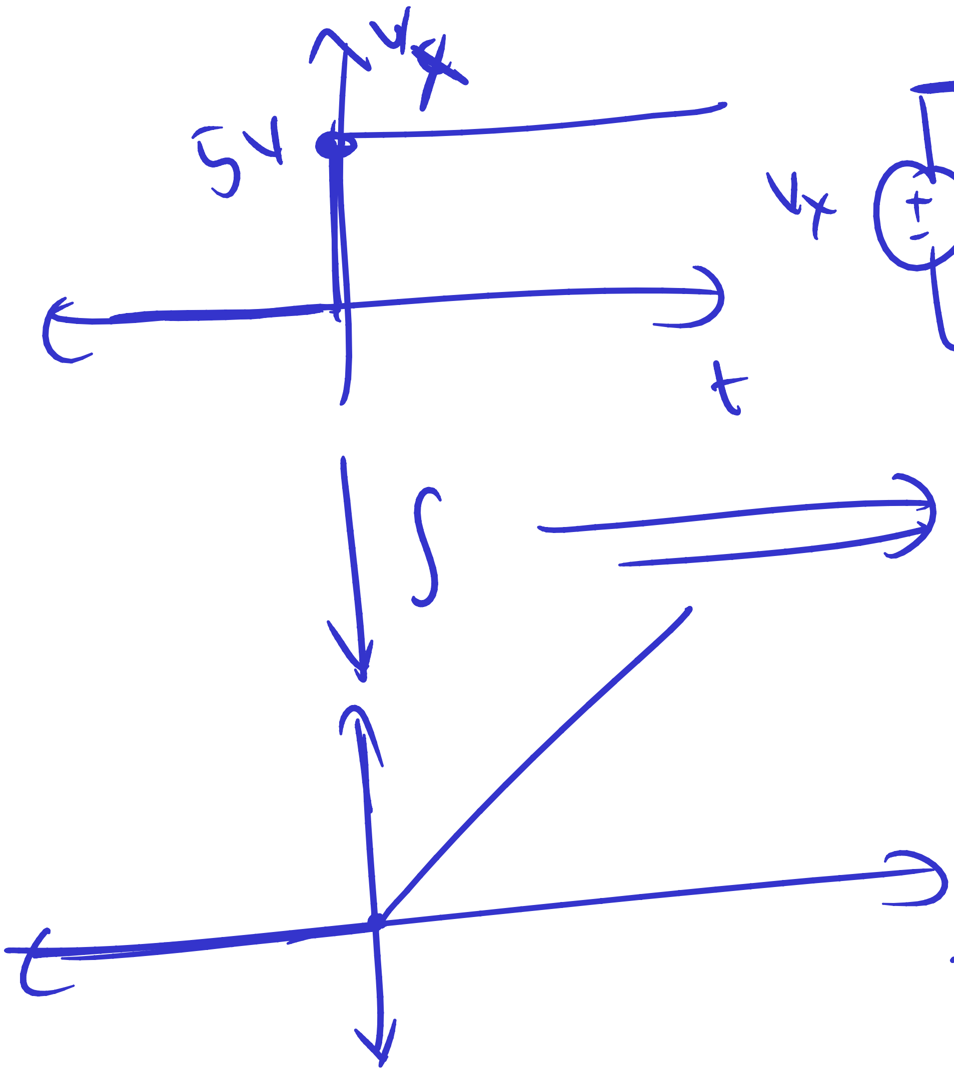
$$i = C \frac{\Delta v}{\Delta t}$$

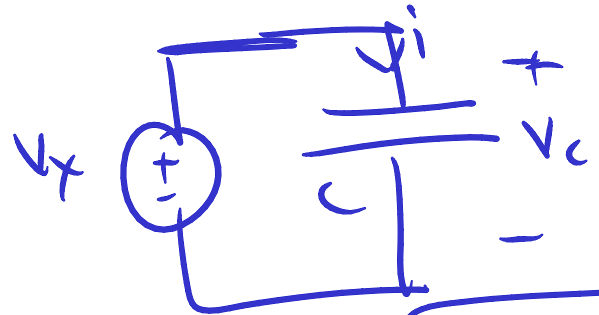
$\Rightarrow i \to \infty$ ✗ Bad idea

$\Rightarrow$ Capacitors maintain voltage across its terminals across discontinuous

Observations (contd.)

(3)





$$i = C\frac{dv_c}{dt}$$

Current thro capacitor can change instantaneously

Procedure for Finding RC/RL Response

1. **Identify the variable of interest**
   - For RL circuits, it is usually the inductor current $i_L(t)$
   - For RC circuits, it is usually the capacitor voltage $v_c(t)$
2. **Determine the initial value (at $t = t_0^-$ and t_0^+) of the variable**
   - Recall that $i_L(t)$ and $v_c(t)$ are continuous variables:

     $$i_L(t_0^+) = i_L(t_0^-) \quad \text{and} \quad v_c(t_0^+) = v_c(t_0^-)$$

   - Assuming that the circuit reached steady state before t_0 , use the fact that an inductor behaves like a short circuit in steady state or that a capacitor behaves like an open circuit in steady state

Procedure (cont'd)

3. **Calculate the final value of the variable** (its value as $t \rightarrow \infty$)
   - Again, make use of the fact that an inductor behaves like a short circuit in steady state ($t \rightarrow \infty$) or that a capacitor behaves like an open circuit in steady state ($t \rightarrow \infty$)

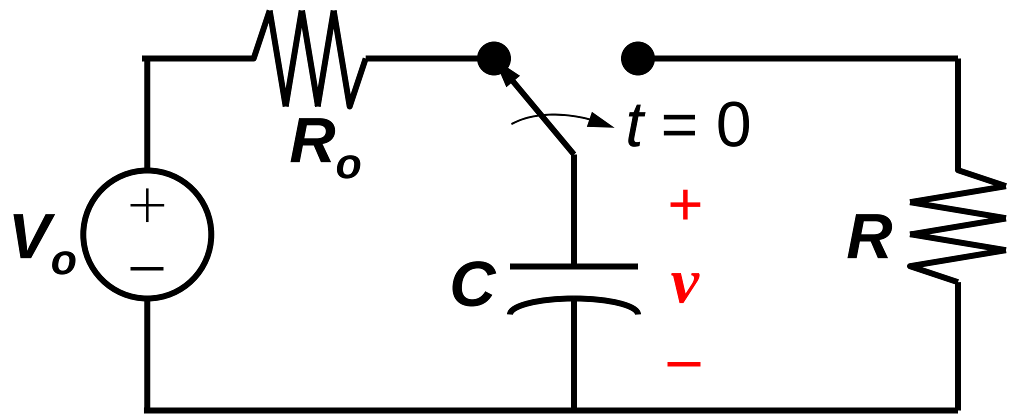
4. **Calculate the time constant for the circuit**

   $\tau = L/R$ for an RL circuit, where R is the Thévenin equivalent resistance "seen" by the inductor

   $\tau = RC$ for an RC circuit where R is the Thévenin equivalent resistance "seen" by the capacitor

Natural Response of an RC Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

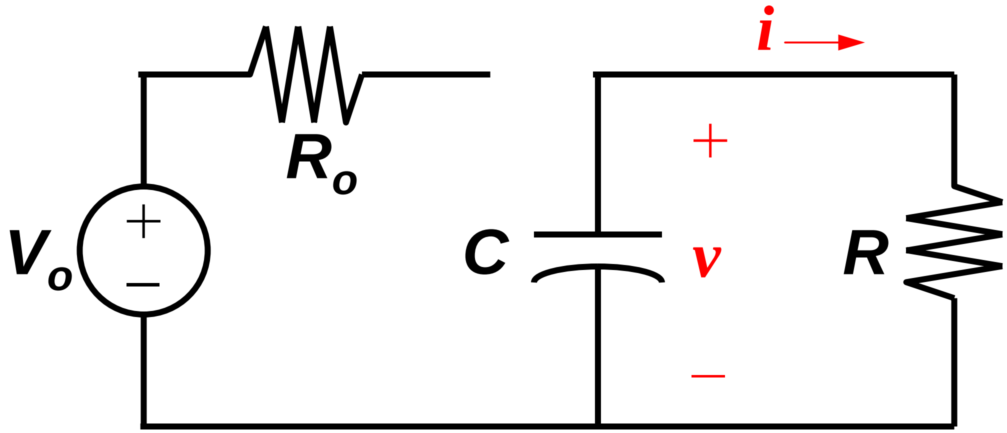
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- The voltage on the capacitor at $t = 0^-$ is V_o

Solving for the Voltage ($t \geq 0$)

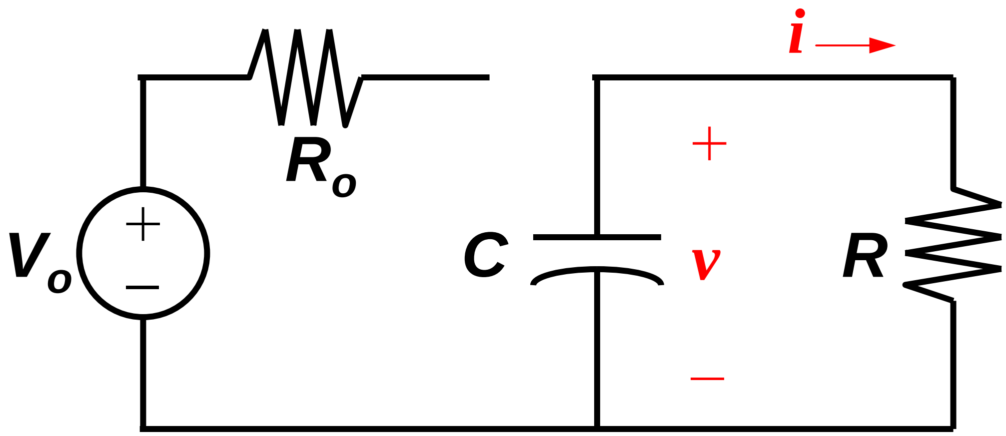
- For t > 0, the circuit reduces to



- Applying KCL to the RC circuit:

- Solution: $$v(t) = v(0)e^{-t/RC}$$

Solving for the Current ($t > 0$)



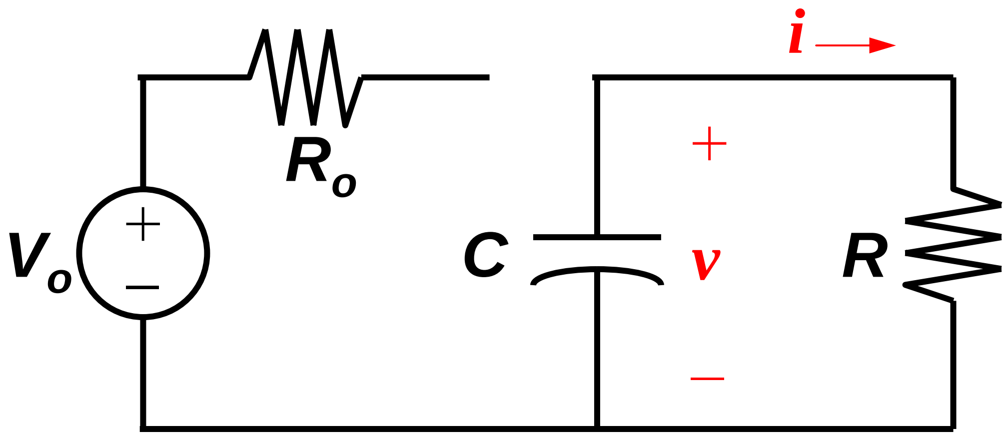
$$v(t) = V_o e^{-t/RC}$$

- Note that the current changes abruptly:
  $i(0^-) = 0$

$$\text{for } \ t > 0, \ i(t) = \frac{v}{R} = \frac{V_o}{R} e^{-t/RC}$$

$$\Rightarrow \ i(0^+) = \frac{V_o}{R}$$

Solving for Power and Energy Delivered ($t > 0$)



$$v(t) = V_o e^{-t/RC}$$

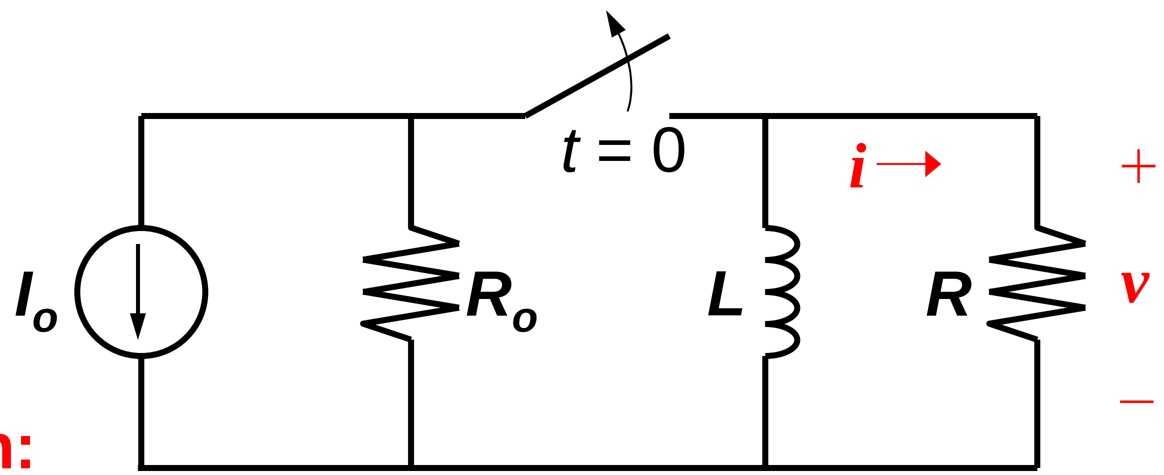
$$p = \frac{v^2}{R} = \frac{V_o^2}{R} e^{-2t/RC}$$

$$w = \int_0^t p(x)dx = \int_0^t \frac{V_o^2}{R} e^{-2x/RC} dx$$

$$= \frac{1}{2} C V_o^2 \left(1 - e^{-2t/RC}\right)$$

Natural Response of an RL Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

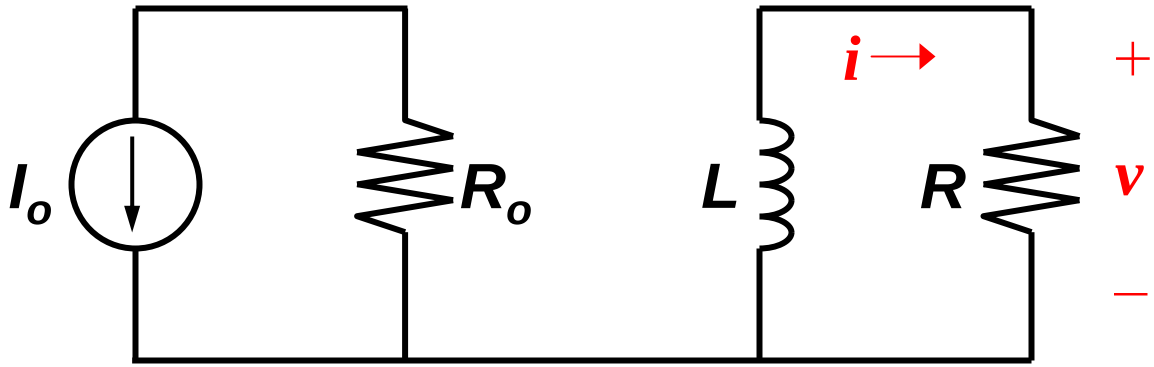
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- t<0 the entire system is at steady-state; and the inductor is → like short circuit
- The current flowing in the inductor at $t = 0^-$ is I_o and V across is 0.

Solving for the Current ($t \geq 0$)

- For t > 0, the circuit reduces to

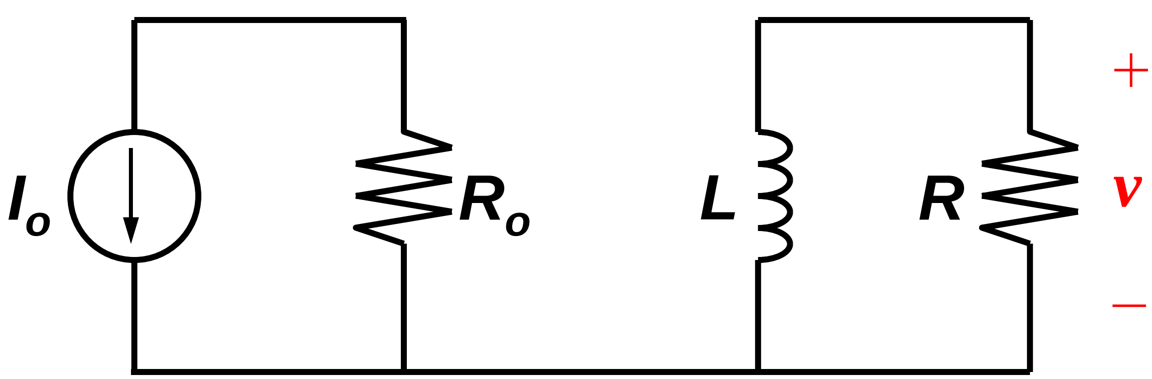


- Applying KVL to the LR circuit:
- $v(t)=i(t)R$
- At $t=0^+$, $i=I_{0,}$
- At arbitrary t>0, $i=i(t)$ and $v(t) = -L\dfrac{di(t)}{dt}$

- Solution: $i(t) = i(0)e^{-(R/L)t} = I_0 e^{-(R/L)t}$

Solving for the Voltage (t > 0)

$$i(t) = I_o e^{-(R/L)t}$$



- Note that the voltage changes abruptly:

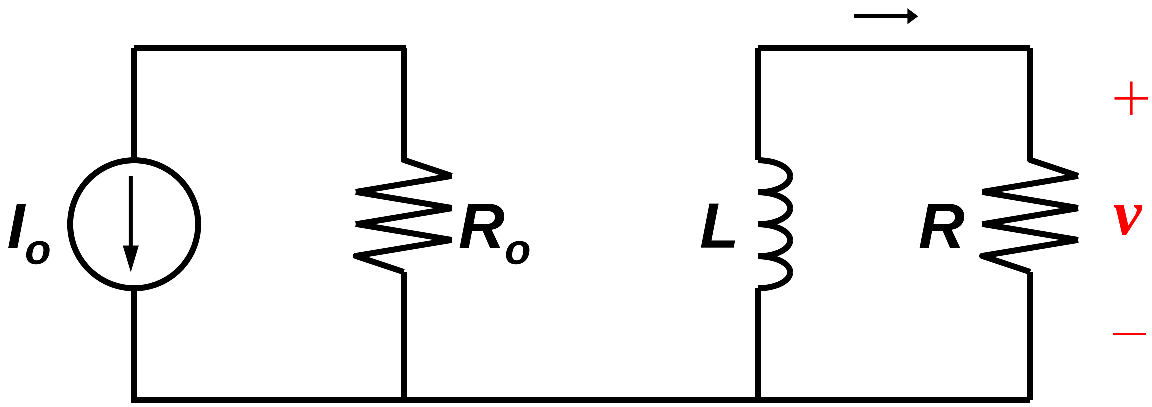
$$v(0^-) = 0$$

$$\text{for } t > 0,\ v(t) = iR = I_o R e^{-(R/L)t}$$

$$\Rightarrow\ v(0^+) = I_0 R$$

Solving for Power and Energy Delivered (t > 0)

$$i(t) = I_o e^{-(R/L)t}$$



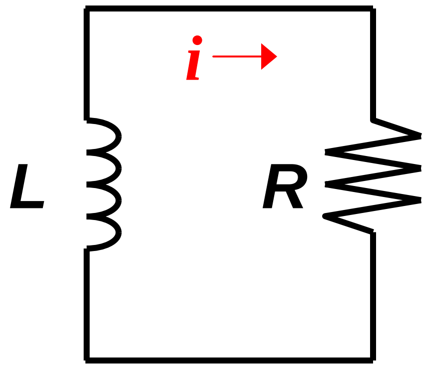
$$p = i^2 R = I_o^2 R e^{-2(R/L)t}$$

$$w = \int_0^t p(x)dx = \int_0^t I_o^2 R e^{-2(R/L)x} dx$$

$$= \frac{1}{2} L I_o^2 \left(1 - e^{-2(R/L)t}\right)$$

Natural Response Summary

RL Circuit



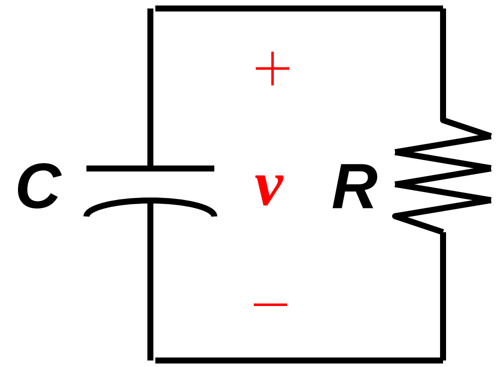
- Inductor current cannot change instantaneously

$$i(0^-) = i(0^+)$$

$$i(t) = i(0)e^{-t/\tau}$$

- time constant $\tau = \dfrac{L}{R}$

RC Circuit



- Capacitor voltage cannot change instantaneously

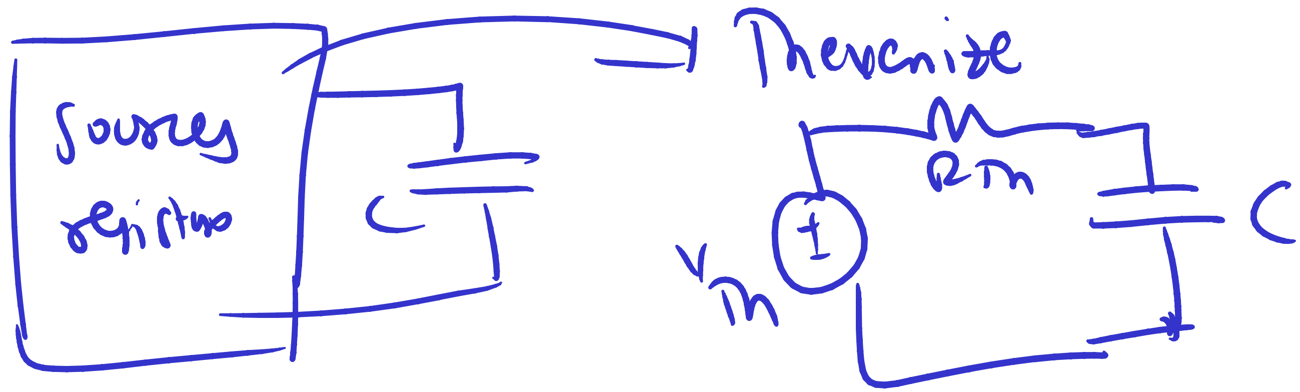
$$v(0^-) = v(0^+)$$

$$v(t) = v(0)e^{-t/\tau}$$

- time constant $\tau = RC$

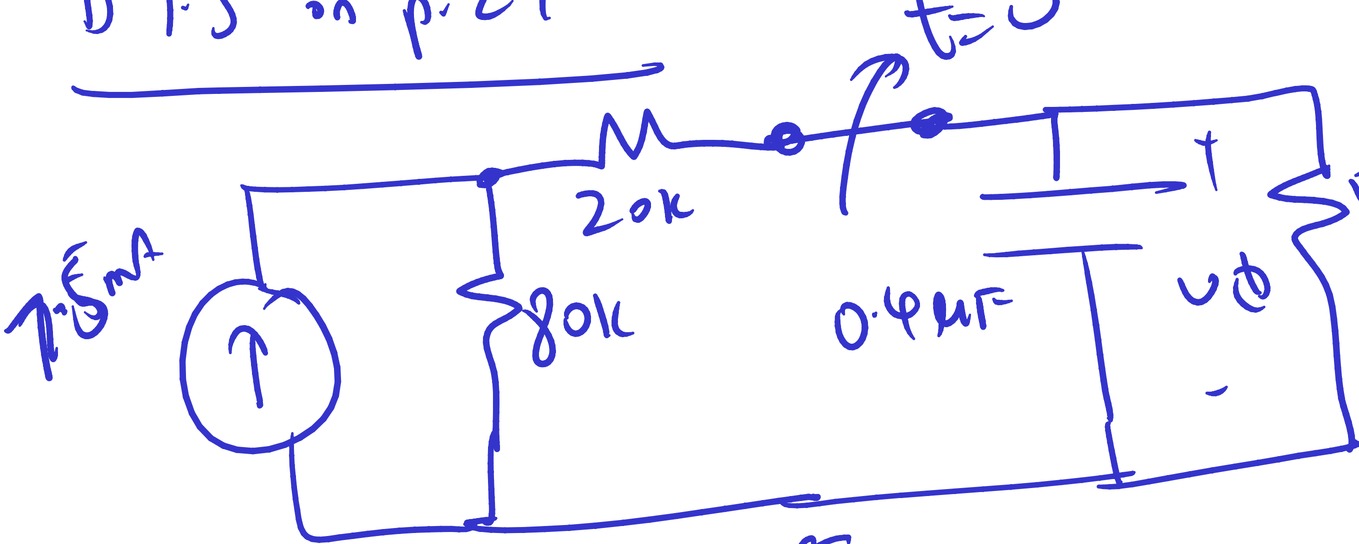
Application of Thevenin's Theorem to RC/RL Circuits

Recall:



Examples

D 7.3 on p. 240



Switch closed for a long time and opens at $t=0$.

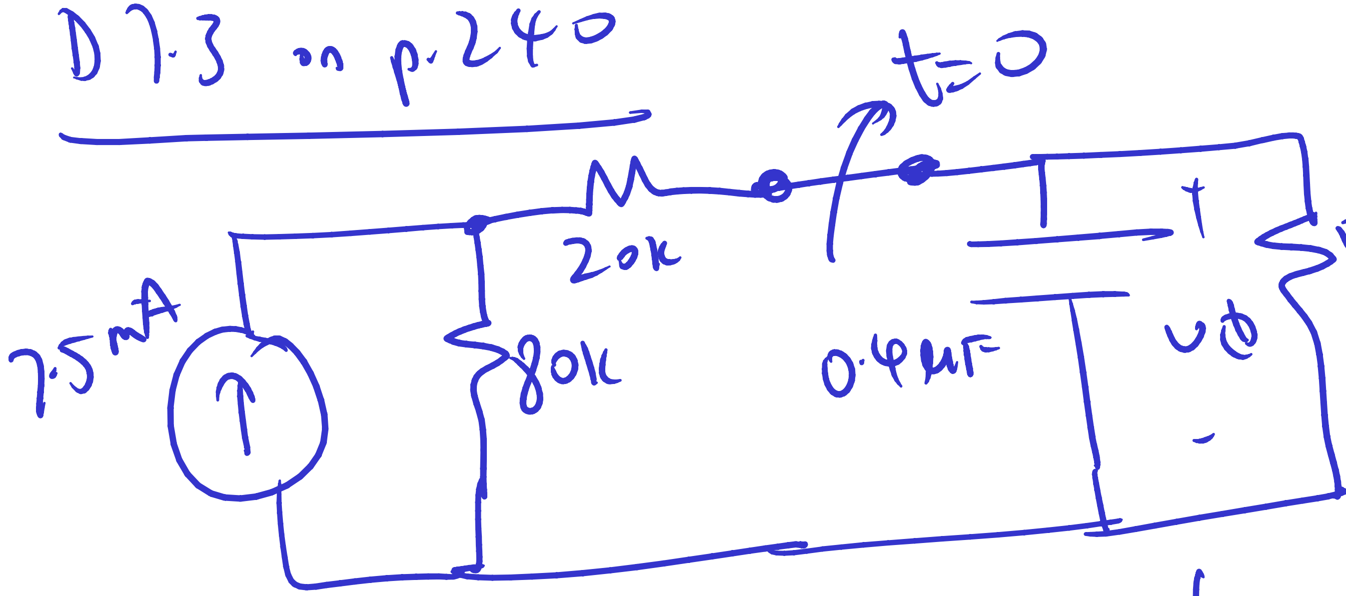
(1) Find $v(t=0^-)$
$= v(t=0^+)$

[Note: 0^- means at the infinitisemally small time instant before discontinuity]

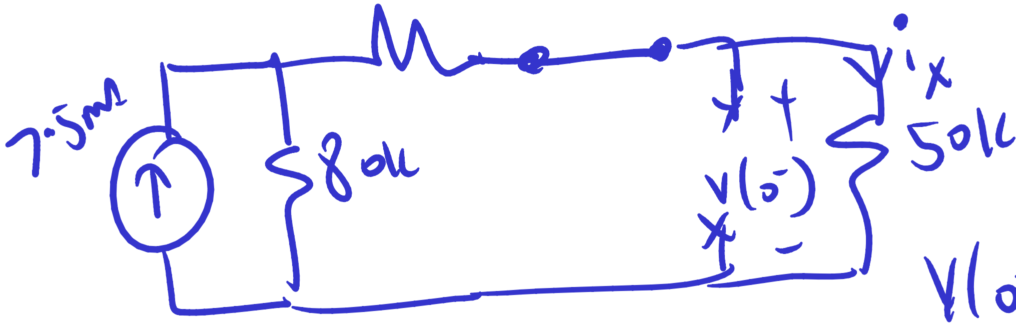
Examples

D 7.3 on p. 240

Switch closed for a long time and opens at $t = 0$.



$t = 0^-$



$$v(0^-) = \left\{\left(\frac{80k}{150k}\right) 7.5\,mA\right\} \cdot (50k)$$

$$\Rightarrow \boxed{v(0^-) = 200\,V}$$

$$v(0^-) = (i_x)(50k)$$

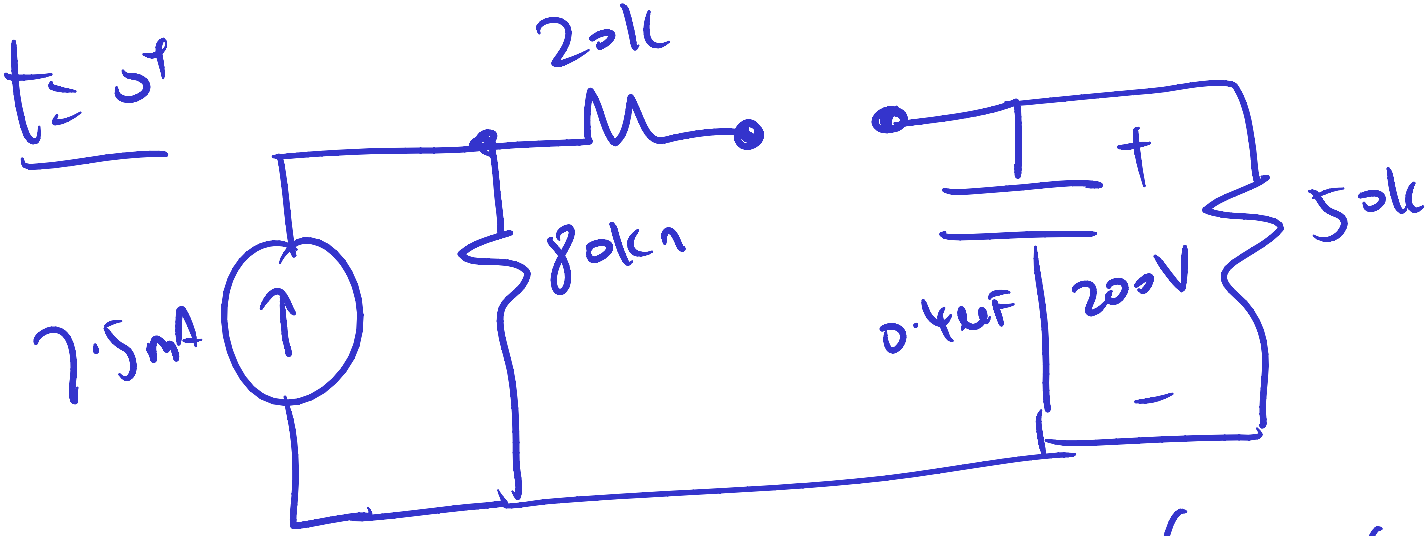
Example (contd.)

$\therefore\ v(0^+) = v(0^-) = 200\,V$

[Because of continuity of voltage across capacitor]

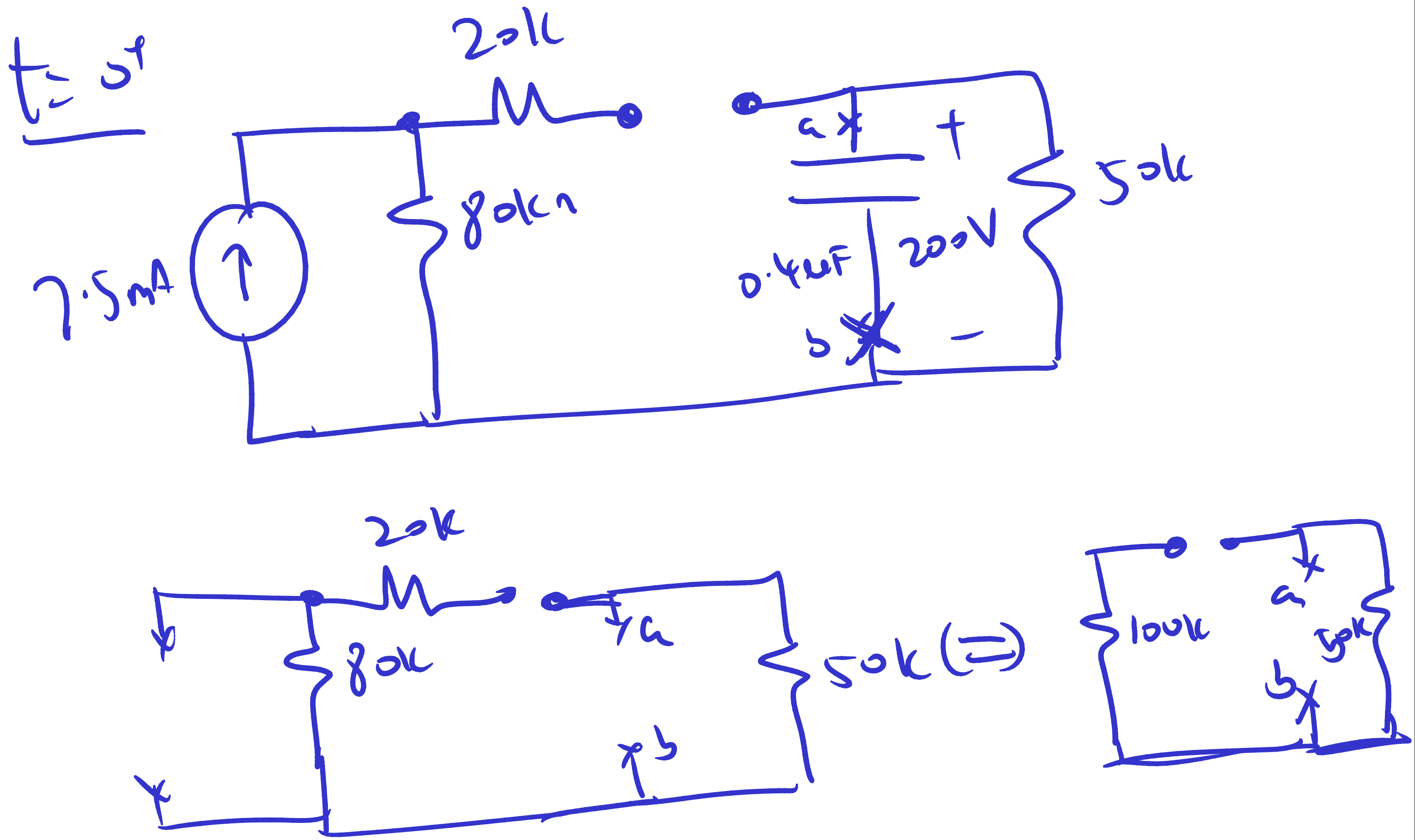
(b) Find τ, $t \geq 0$

$t = 0^+$

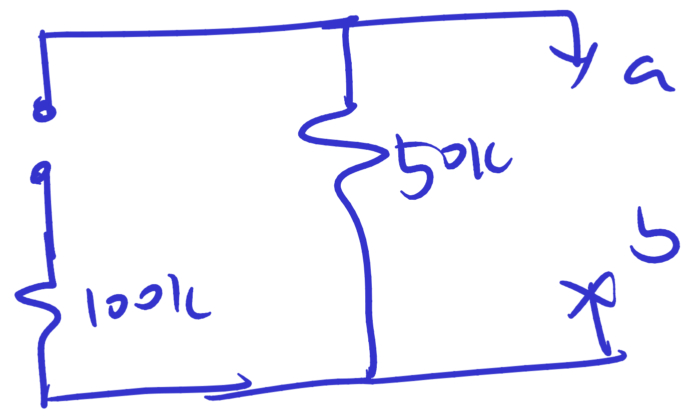


$\tau = R_{Th} \cdot C = (50k)(0.4\mu F) = \underline{20ms}$

Example (contd.)

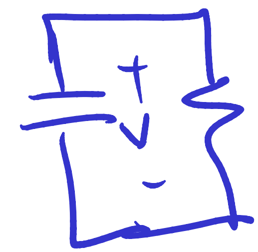


Example (contd.)



$\Rightarrow R_{ab} = R_{Th} = 50k$

$t \geq 0$

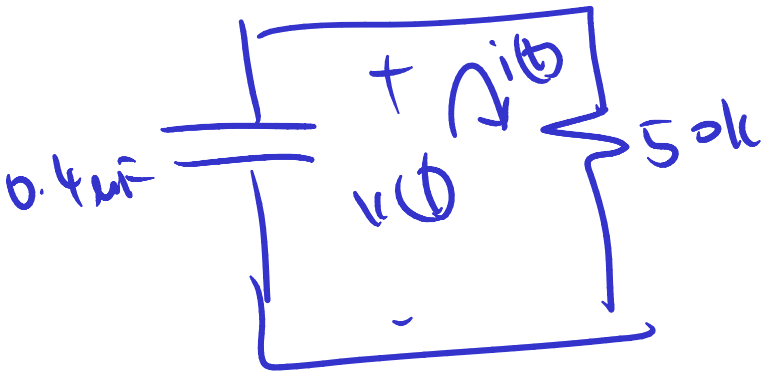


(C) Find $v(t)$ $t \geq 0$

$$v(t) = \cancel{V_{final}}^{\,0} + (V_{initial} - \cancel{V_{final}}^{\,0})\, e^{-t/\tau}$$

$$\boxed{v(t) = 200\, e^{-t/20\,ms} \text{ Volts}}$$

Example (contd.)



$v(0) = 200V$

$E(0) = \frac{1}{2} C [v(0)]^2$

$= \frac{1}{\cancel{2}} \cdot 0.4\mu F \cdot \overset{100}{\cancel{200}} \cdot 200$

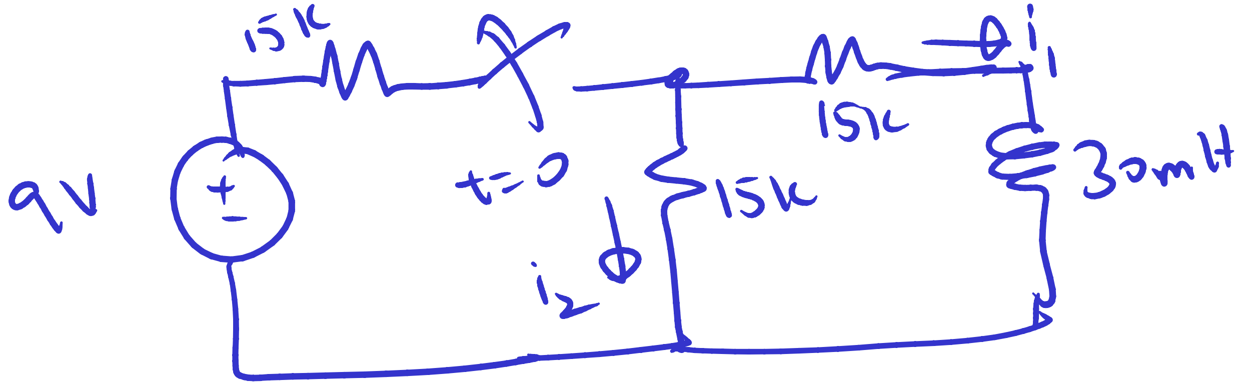
$= \underline{\underline{8\ mJ}}$

* Find $i(t) = \frac{v(t)}{50k}$ [Ohm's law]

Capacitor: $i = -C\frac{dv}{dt}$

$= -C\frac{d}{dt}\left(v_{initial}\, e^{-t/\tau}\right)$

Problem 7.2 on p. 265



Note:

$\Leftrightarrow v = L\frac{di}{dt}$

(a) Find $i_1(0^-)$, $i_2(0^-)$

(b) Find $i_1(0^+)$, $i_2(0^+)$

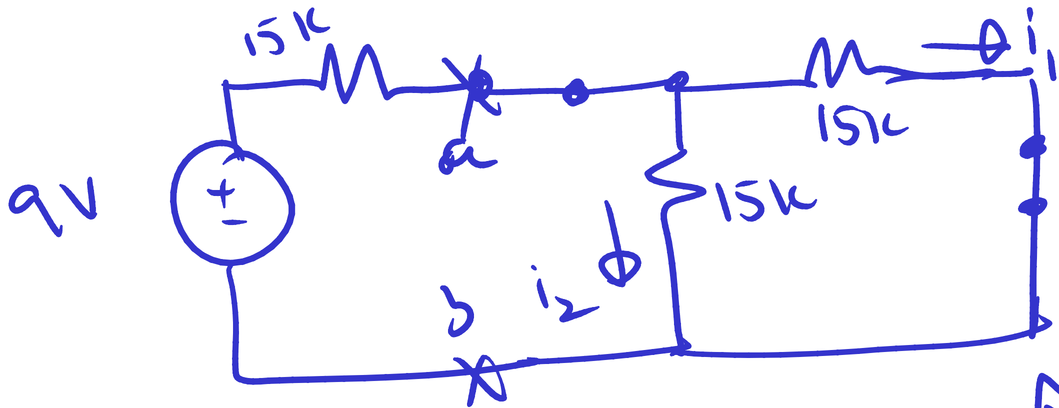
⋮

⇒ (1) Steady state, inductor is a short circuit

(2) Inductors maintain current across discontinuities

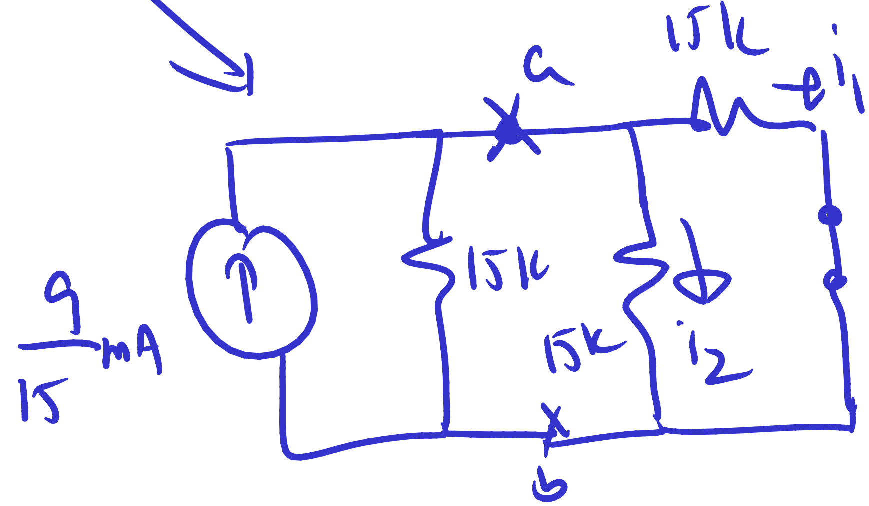
(3) $\tau = L/R$

Problem 7.2 on p. 265



$t = 0^-$

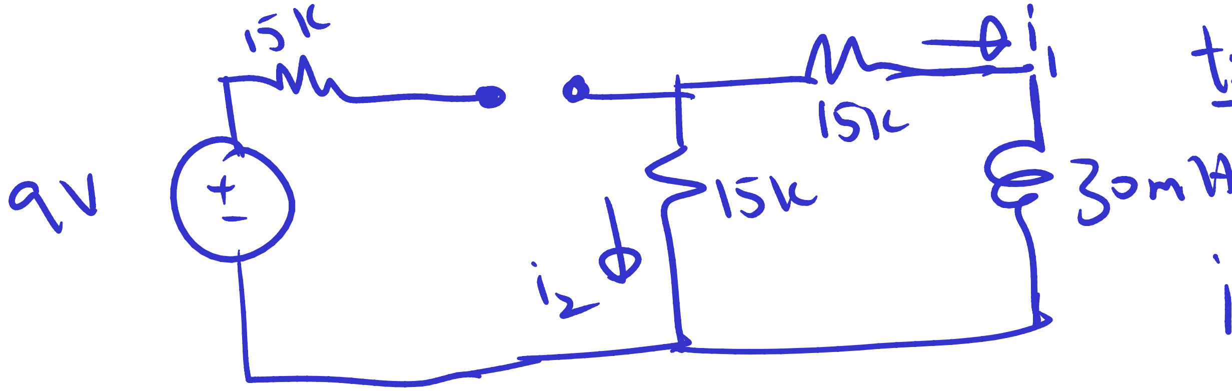
Norton at ab



(a) Find $i_1(0^-)$, $i_2(0^-)$

$$i_1(0^-) = i_2(0^-) = \frac{1}{5}\text{ mA}$$

Problem 7.2 on p. 265



$\underline{t = 0^+}$

$i_1(0^+) = i_1(0^-)$

$= \underline{0.2\ mA}$

(b) Find $i_1(0^+)$, $i_2(0^+)$

Notice: $i_2(0^+) = -i_1(0^+)$

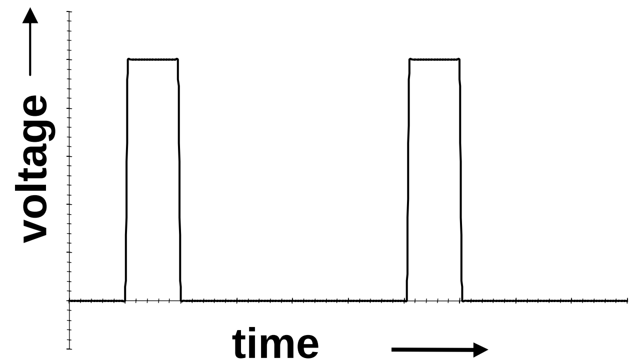
$= \underline{-0.2\ mA}$

(c) $i_1(t) = i_{1_{final}} + (i_{1_{initial}} - i_{1_{final}}) e^{-t/\tau}, \quad \tau = L/R$

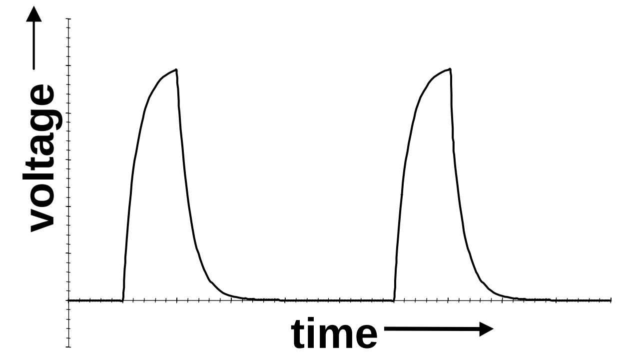
Digital Signals

We compute with pulses.

We send beautiful pulses in:



But we receive lousy-looking pulses at the output:



Capacitor charging effects are responsible!

- Every node in a real circuit has capacitance; it's the charging of these capacitances that limits circuit performance (speed)