EE100Su08 Lecture #11 (July 21st 2008)

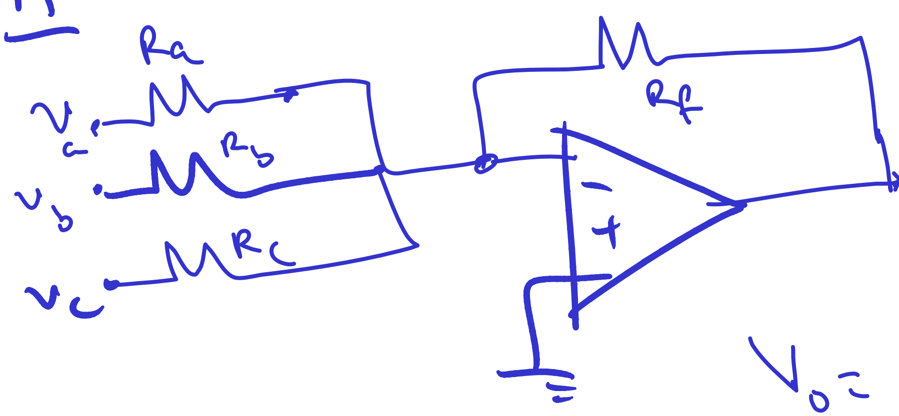
- Bureaucratic Stuff
  - Lecture videos should be up by tonight
  - HW #2: Pick up from office hours today, will leave them in lab.
    REGRADE DEADLINE: Monday, July 28th 2008, 5:00 pm PST, Bart's office hours.
  - HW #1: Pick up from lab.
  - Midterm #1: Pick up from me in OH
    REGRADE DEADLINE: Wednesday, July 23rd 2008, 5:00 pm PST. Midterm: drop off in hw box with a note attached on the first page explaining your request.
- OUTLINE
  - QUESTIONS?
  - Op-amp MultiSim example
  - Introduction and Motivation
  - Arithmetic with Complex Numbers (Appendix B in your book)
  - Phasors as notation for Sinusoids
  - Complex impedances
  - Circuit analysis using complex impedances
  - Derivative/Integration as multiplication/division
  - Phasor Relationship for Circuit Elements
  - Frequency Response and Bode plots
- Reading
  - Chapter 9 from your book (skip 9.10, 9.11 (duh)), Appendix E* (skip second-order resonance bode plots)
  - Chapter 1 from your reader (skip second-order resonance bode plots)

Op-amps: Conclusion

- Questions?

Ex 5.2: What do you mean by a "weighting factor"?

A: Related to the summing amplifier!



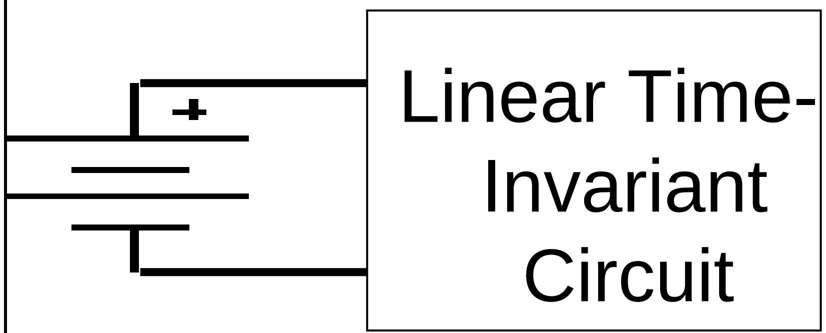
weighting factor

p. 163 of your book

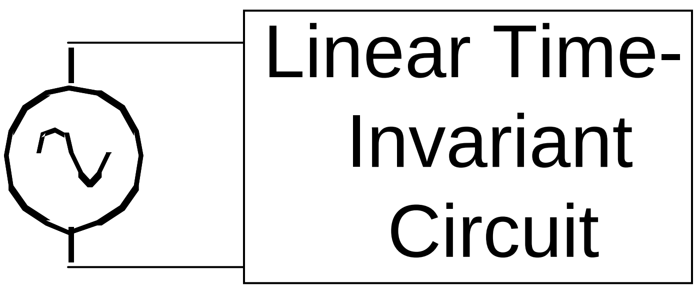
$$V_o = -\frac{R_f}{R_a} v_a - \frac{R_f}{R_b} v_b - \frac{R_f}{R_c} v_c$$

- MultiSim Example

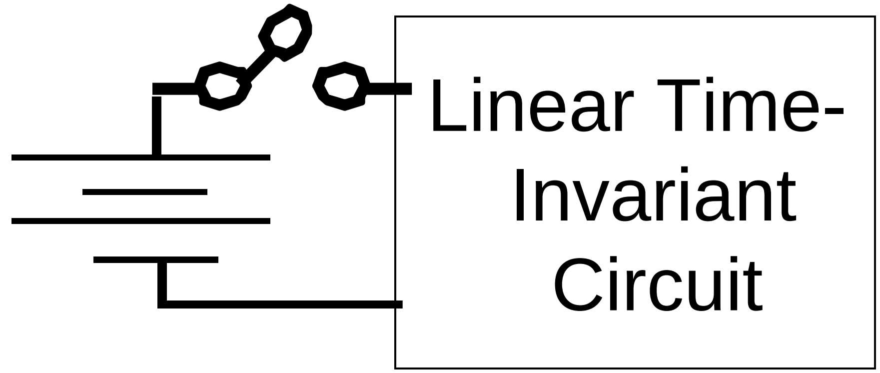
Types of Circuit Excitation



Steady-State Excitation (DC Steady-State)



Sinusoidal (Single-Frequency) Excitation →AC Steady-State



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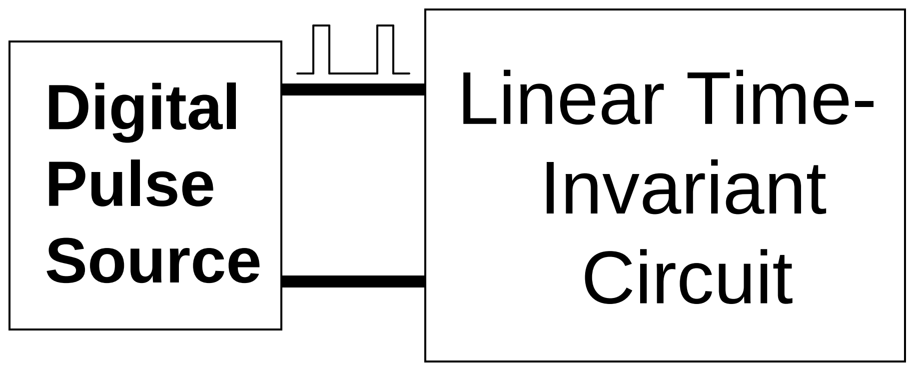
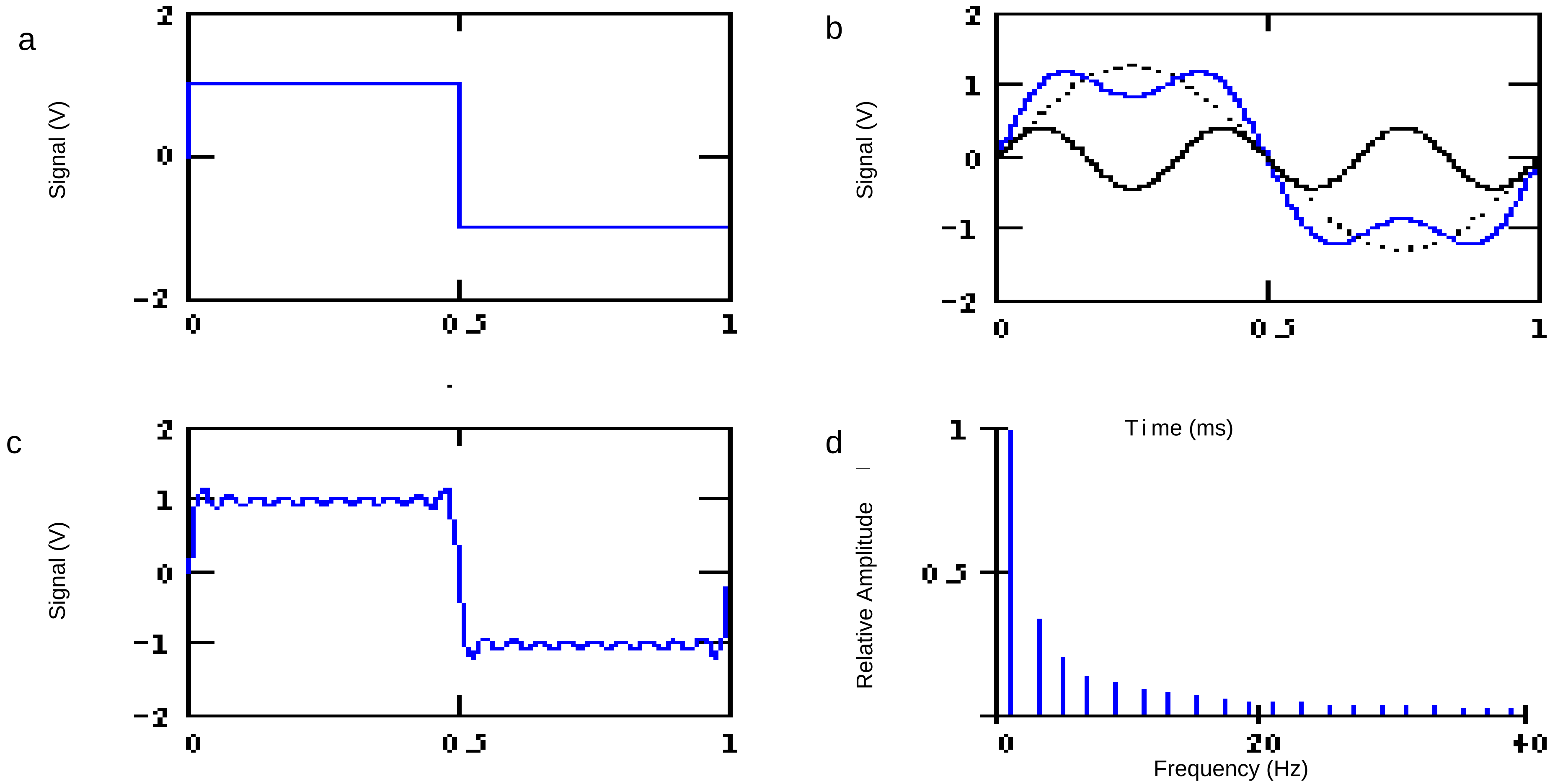


Transient Excitation

Why is Single-Frequency Excitation Important?

- Some circuits are driven by a single-frequency sinusoidal source.
- Some circuits are driven by sinusoidal sources whose frequency changes slowly over time.
- You can express any periodic electrical signal as a sum of single-frequency sinusoids – so you can analyze the response of the (linear, time-invariant) circuit to each individual frequency component and then sum the responses to get the total response.
- This is known as Fourier Transform and is tremendously important to all kinds of engineering disciplines!

Representing a Square Wave as a Sum of Sinusoids

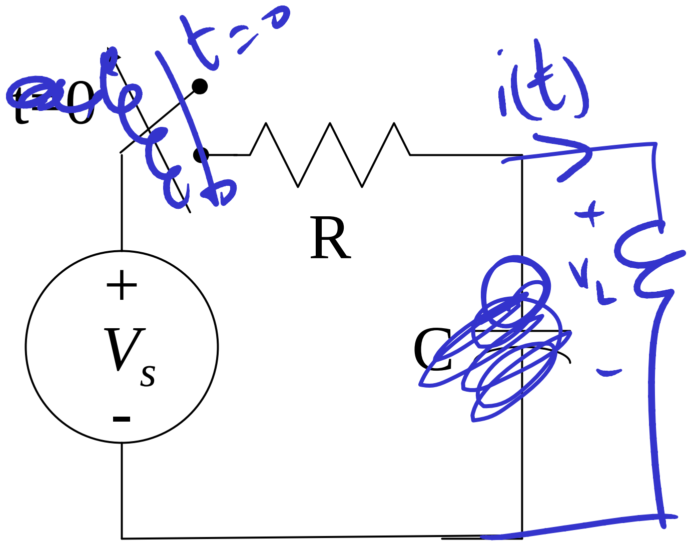


(a)Square wave with 1-second period. (b) Fundamental component (dotted) with 1-second period, third-harmonic (solid black) with1/3-second period, and their sum (blue). (c) Sum of first ten components. (d) Spectrum with 20 terms.

Steady-State Sinusoidal Analysis

- Also known as AC **steady-state**
- Any steady state voltage or current in a linear circuit with a sinusoidal source is a sinusoid.
  - This is a consequence of the nature of particular solutions for sinusoidal forcing functions.
- All AC steady state voltages and currents have the same frequency as the source.
- In order to find a steady state voltage or current, all we need to know is its magnitude and its phase relative to the source
  - We already know its frequency.
- Usually, an AC steady state voltage or current is given by the particular solution to a differential equation.

Example: 1st order RC Circuit with sinusoidal excitation



p. 335-337: "sinusoid"

Suppose $v_s(t) = V_m \cos(\omega t + \phi)$

[Find expression for determining $i(t)$? $(t \geq 0)$]

KVL: $iR + v_L = v_s$

$$\Rightarrow \; iR + L\frac{di}{dt} = v_s(t)$$

Solution:

$$i(t) = \underbrace{\frac{-V_m}{\sqrt{R^2+\omega^2 L^2}}\cos(\phi-\theta)\,e^{-(R/L)t}}_{\text{Transient}} + \underbrace{\frac{V_m}{\sqrt{R^2+\omega^2 L^2}}\cos(\omega t+\phi-\theta)}_{\text{steady-state}}$$

Sinusoidal Sources Create Too Much Algebra

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

Guess a solution

Two terms to be general

$$x_P(t) = A\sin(wt) + B\cos(wt)$$

$$(A\sin(wt) + B\cos(wt)) + \tau \frac{d(A\sin(wt) + B\cos(wt))}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$(A - \tau B - F_A)\sin(wt) + (B + \tau A - F_B)\cos(wt) = 0$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

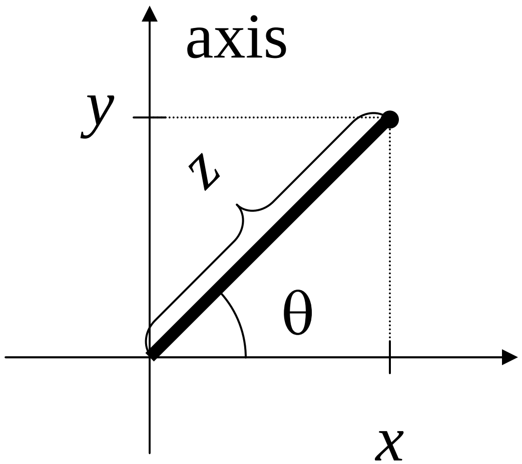
$$(A - \tau B - F_A) = 0$$

$$(B + \tau A - F_B) = 0$$

$$A = \frac{F_A + \tau F_B}{\tau^2 + 1} \qquad B = -\frac{\tau F_A - F_B}{\tau^2 + 1}$$

Phasors (vectors that rotate in the complex plane) are a clever alternative.

Complex Numbers (1)



$$j = \sqrt{(-1)}$$

- x is the real part
- y is the imaginary part
- z is the magnitude
- θ is the phase

$$x = z\cos\theta \qquad y = z\sin\theta$$

$$z = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

- Rectangular Coordinates

  $\mathbf{Z} = x + jy$

- Polar Coordinates:

  $\mathbf{Z} = z \angle \theta$

- Exponential Form:

$$\mathbf{Z} = |\mathbf{Z}|e^{j\theta} = ze^{j\theta}$$

$$\mathbf{Z} = z(\cos\theta + j\sin\theta)$$

$$1 = 1e^{j0} = 1\angle 0°$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90°$$

Complex Numbers (2)

Euler's Identities

Bart's fav. identity

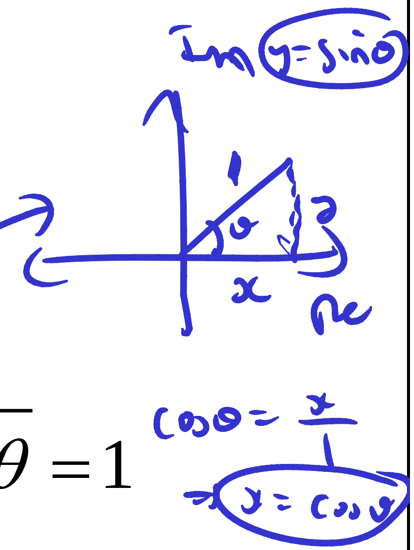
$$e^{i\pi}+1=0$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$$



Exponential Form of a complex number

$$\mathbf{Z} = \left|\mathbf{Z}\right| e^{j\theta} = z e^{j\theta} = z\angle\theta$$

Arithmetic With Complex Numbers

- To compute phasor voltages and currents, we need to be able to perform computation with complex numbers.
  - Addition
  - Subtraction
  - Multiplication
  - Division
- Later use multiplication by $j\omega$ to replace:
  - Differentiation
  - Integration

Addition

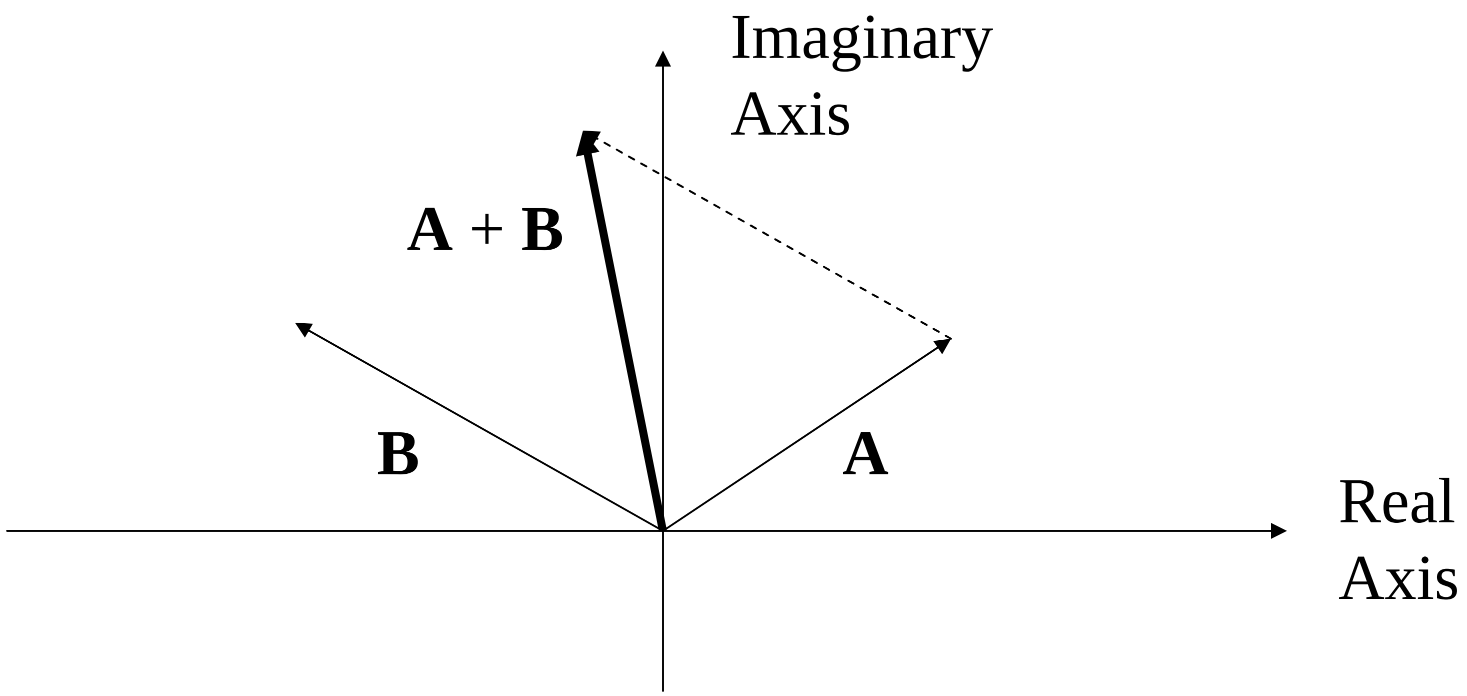
- Addition is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} + \mathbf{B} = (x + z) + j(y + w)$$

Addition



Subtraction

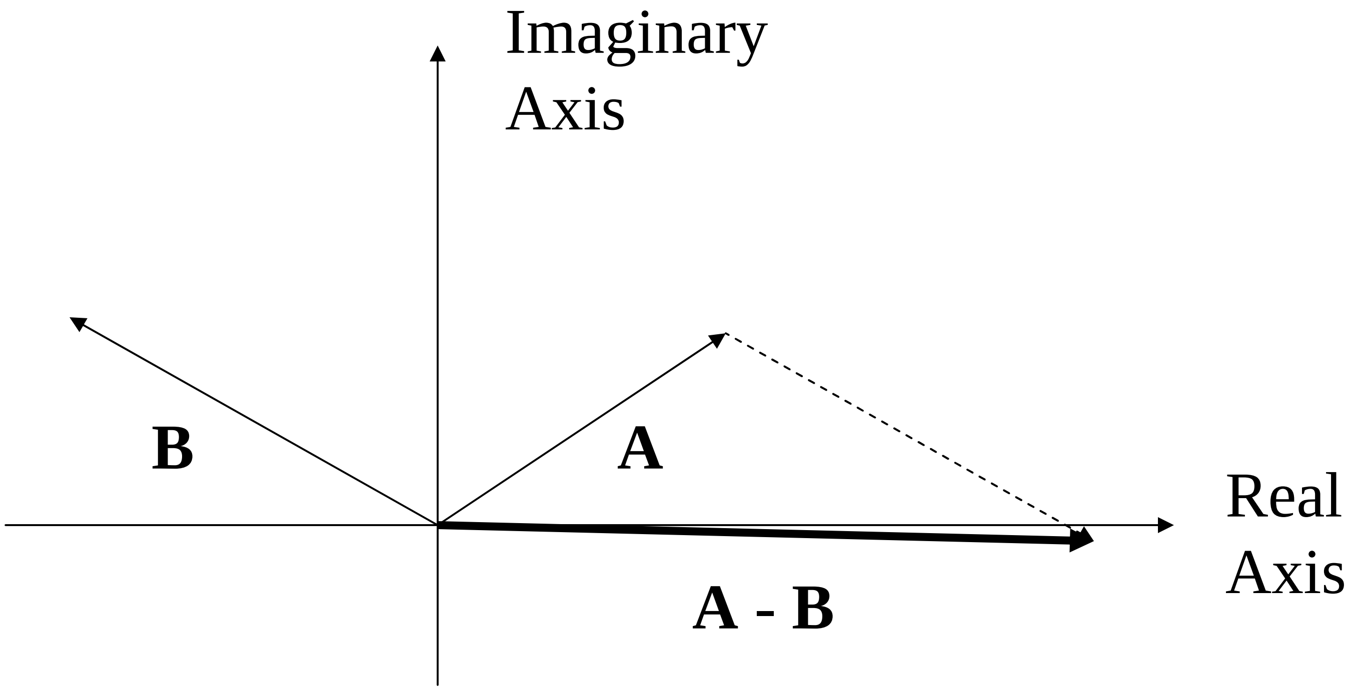
- Subtraction is most easily performed in rectangular coordinates:

$$\mathbf{A} = x + jy$$

$$\mathbf{B} = z + jw$$

$$\mathbf{A} - \mathbf{B} = (x - z) + j(y - w)$$

Subtraction



Multiplication

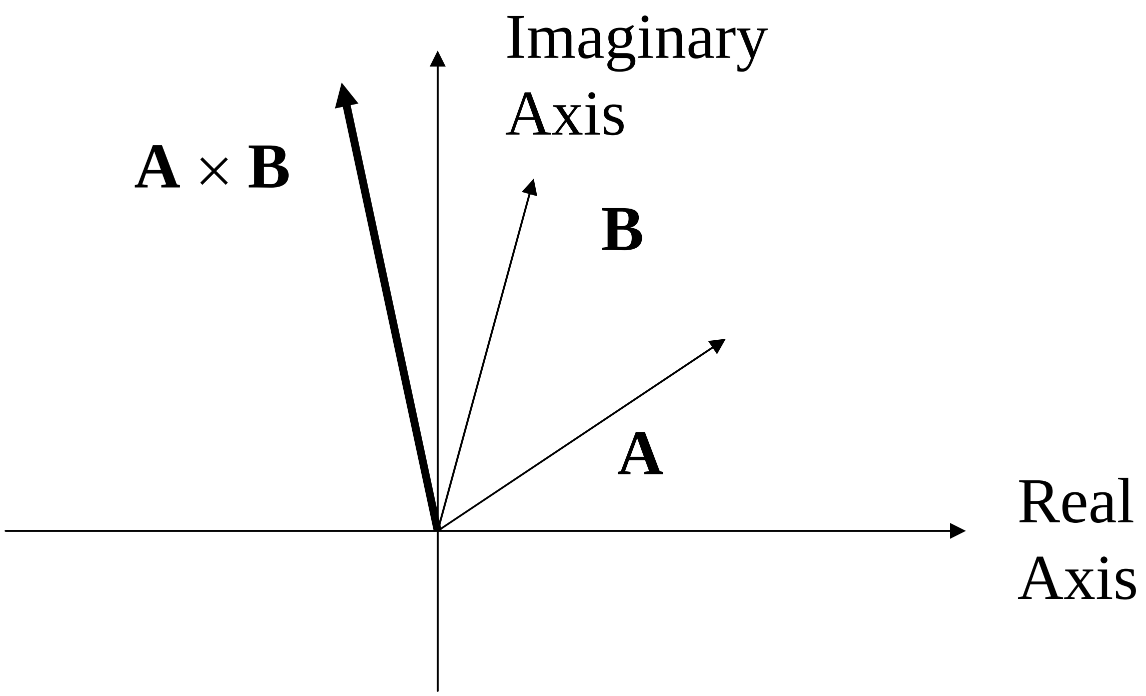
- Multiplication is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} \times \mathbf{B} = (A_M \times B_M) \angle (\theta + \phi)$$

Multiplication



Division

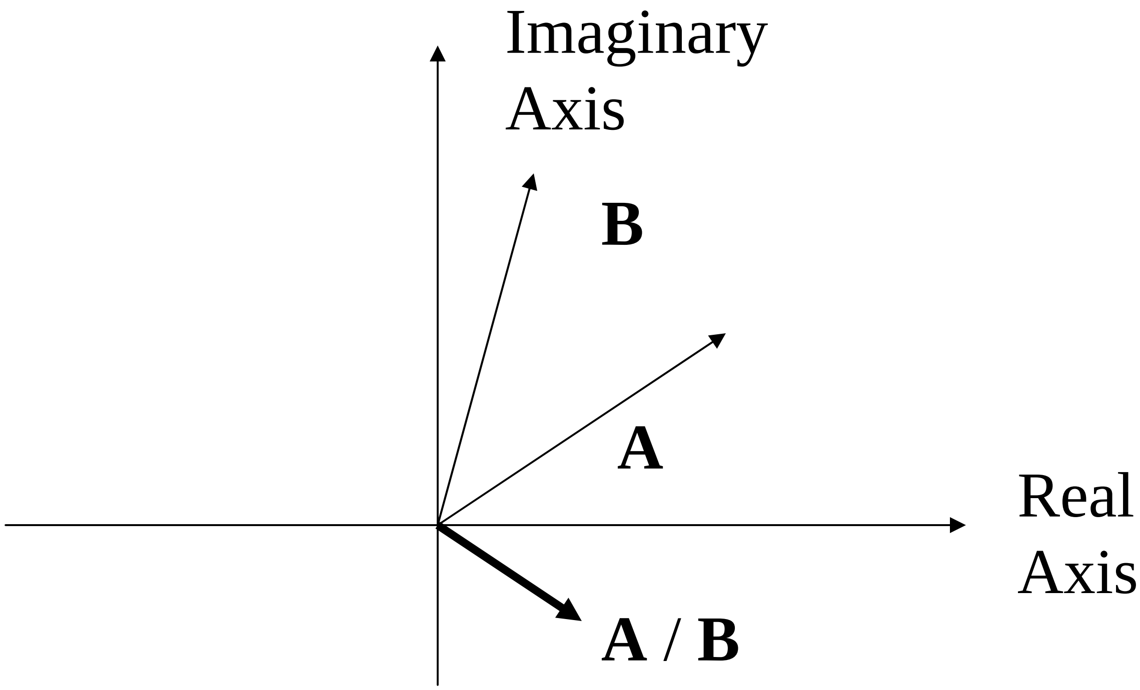
- Division is most easily performed in polar coordinates:

$$\mathbf{A} = A_M \angle \theta$$

$$\mathbf{B} = B_M \angle \phi$$

$$\mathbf{A} / \mathbf{B} = (A_M / B_M) \angle (\theta - \phi)$$

Division



Arithmetic Operations of Complex Numbers

- Add and Subtract: it is easiest to do this in rectangular format
  - Add/subtract the real and imaginary parts separately
- Multiply and Divide: it is easiest to do this in exponential/polar format
  - Multiply (divide) the magnitudes
  - Add (subtract) the phases

$$\mathbf{Z_1} = z_1 e^{j\theta_1} = z_1 \angle \theta_1 = z_1 \cos\theta_1 + jz_1 \sin\theta_1$$

$$\mathbf{Z}_2 = z_2 e^{j\theta_2} = z_2 \angle \theta_2 = z_2 \cos\theta_2 + jz_2 \sin\theta_2$$

$$\mathbf{Z_1} + \mathbf{Z}_2 = (z_1 \cos\theta_1 + z_2 \cos\theta_2) + j(z_1 \sin\theta_1 + z_2 \sin\theta_2)$$

$$\mathbf{Z_1} - \mathbf{Z}_2 = (z_1 \cos\theta_1 - z_2 \cos\theta_2) + j(z_1 \sin\theta_1 - z_2 \sin\theta_2)$$

$$\mathbf{Z_1} \times \mathbf{Z}_2 = (z_1 \times z_2) e^{j(\theta_1 + \theta_2)} = (z_1 \times z_2) \angle (\theta_1 + \theta_2)$$

$$\mathbf{Z_1} / \mathbf{Z}_2 = (z_1 / z_2) e^{j(\theta_1 - \theta_2)} = (z_1 / z_2) \angle (\theta_1 - \theta_2)$$

Phasors

- Assuming a source voltage is a sinusoid time-varying function

$$v(t) = V\cos(\omega t + \theta)$$

→ $V\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = V\,\mathrm{Re}\{\cos(\omega t+\theta) + j\sin(\omega t+\theta)\}$

- We can write:

$$v(t) = V\cos(\omega t + \theta) = V\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[Ve^{j(\omega t+\theta)}\right]$$

$$Define\ \ Phasor\ \ as\ \ Ve^{j\theta} = V\angle\theta$$

← Do NOT put a "time" anywhere in a phasor!

- Similarly, if the function is $v(t) = V\sin(\omega t + \theta)$

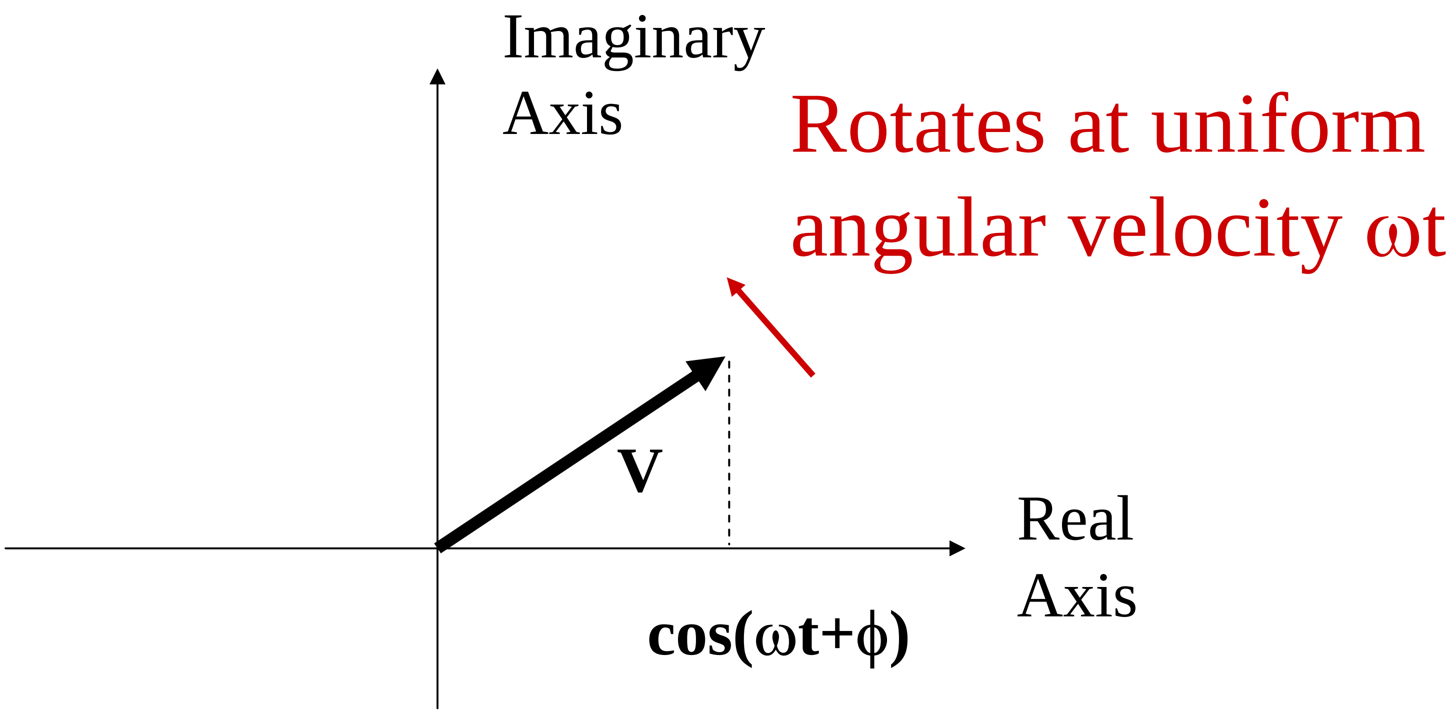
$$v(t) = V\sin(\omega t + \theta) = V\cos\left(\omega t + \theta - \frac{\pi}{2}\right) = \mathrm{Re}\left[Ve^{j(\omega t + \theta - \frac{\pi}{2})}\right]$$

$$Phasor\ = V\angle\left(\theta - \frac{\pi}{2}\right)$$

Note: $V\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = V\,\mathrm{Re}\left[e^{j\omega t}\cdot e^{j\theta}\right]$
$= \mathrm{Re}\left[(Ve^{j\theta})\cdot e^{j\omega t}\right]$

Phasor: Rotating Complex Vector

$$v(t) = V\cos(\omega t + \phi) = \text{Re}\left\{Ve^{j\phi}e^{jwt}\right\} = \text{Re}\left(\mathbf{V}e^{j\omega t}\right)$$

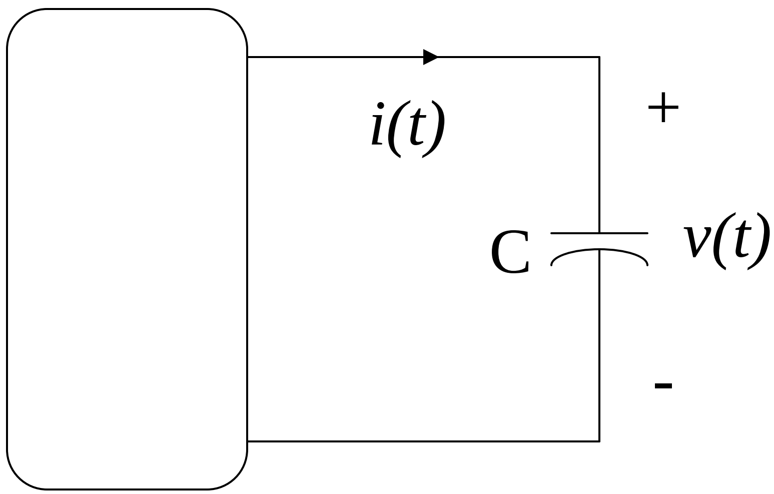


The head start angle is ϕ.

Complex Exponentials

- We represent a real-valued sinusoid as the real part of a complex exponential after multiplying by $e^{j\omega t}$.
- Complex exponentials
  - provide the link between time functions and phasors.
  - Allow derivatives and integrals to be replaced by multiplying or dividing by $j\omega$
  - make solving for AC steady state simple algebra with complex numbers.
- Phasors allow us to express current-voltage relationships for inductors and capacitors much like we express the current-voltage relationship for a resistor.

I-V **Relationship for a Capacitor**



$$i(t) = C\frac{dv(t)}{dt}$$

Suppose that *v(t)* is a sinusoid:

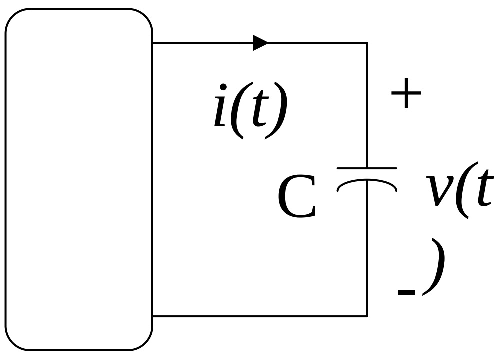
$$v(t) = \text{Re}\{Ve^{j(\omega t+\theta)}\}$$

$= \text{Re}[V \cdot \cos(\omega t + \theta) + jV \sin(\omega t + \theta)]$

$= V \cdot \cos(\omega t + \theta)$

Find *i(t)*.

Capacitor Impedance (1)



$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = V\cos(\omega t + \theta) = \frac{V}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right]$$

$$i(t) = C\frac{dv(t)}{dt} = \frac{CV}{2}\frac{d}{dt}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] = \frac{CV}{2}j\omega\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right]$$

$$= \frac{-\omega CV}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right] = -\omega CV\sin(\omega t + \theta) = \omega CV\cos(\omega t + \theta + \frac{\pi}{2})$$

$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\left(\theta + \frac{\pi}{2}\right)} = \frac{V}{\omega CV}\angle(\theta - \theta - \frac{\pi}{2}) = \frac{1}{\omega C}\angle(-\frac{\pi}{2}) = -j\frac{1}{\omega C} = \frac{1}{j\omega C}$$

$v(t) = V\cos(\omega t + \theta)$

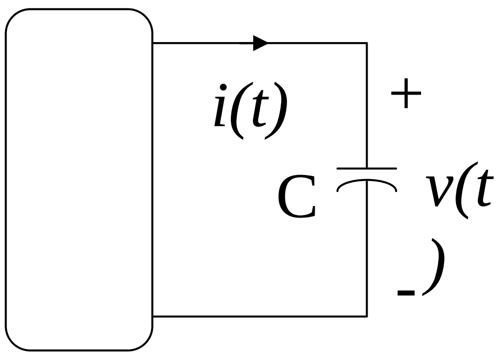
$i = C\frac{d}{dt}\left[V\cos(\omega t + \theta)\right]$

$= -CV\sin(\omega t + \theta) \cdot \omega$

Euler's formula

Impedance → $Z_c \triangleq \frac{\bar{V}}{\bar{I}} = \frac{V\angle\theta}{\omega CV\angle\theta + \pi/2}$

Capacitor Impedance (2)



$$i(t) = C\frac{dv(t)}{dt}$$

Phasor definition

$$v(t) = V\cos(\omega t + \theta) = \mathrm{Re}\left[Ve^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{V} = V\angle\theta$$

$$i(t) = C\frac{dv(t)}{dt} = \mathrm{Re}\left[CV\frac{de^{j(\omega t+\theta)}}{dt}\right] = \mathrm{Re}\left[j\omega CVe^{j(\omega t+\theta)}\right] \Rightarrow \mathbf{I} = I\angle\theta$$

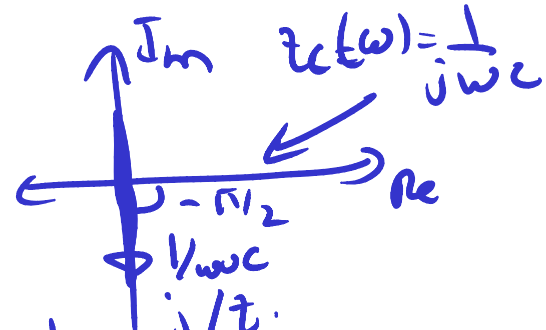
$$Z_c = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\theta}{I\angle\theta} = \frac{V}{j\omega CV}\angle(\theta-\theta) = \frac{1}{j\omega C}$$

Points to note:

(1) $Z_c(\omega) = \frac{1}{j\omega C} = \frac{-j}{\omega C}$

$|Z_c| \rightarrow \frac{1}{\omega C}$, $\angle \rightarrow -\pi/2$

Note: $\omega = 2\pi f$



$\alpha + j\beta$, $|Z|e^{j\angle Z}$.

Example

$v(t)$ = 120V cos(377t + 30°)

C = 2μF

- What is **V**?
- What is **I**?
- What is $i(t)$?

$\omega = 2\pi \cdot 60$ (wall outlet)

$\frac{\pi}{6}$

$\frac{120 \angle \pi/6}{\frac{1}{\omega C} \angle -\pi/2} = 120\omega C \angle \pi/6 + \pi/2 \Big|_{\omega = 377 \text{ rad/sec}}$

$\bar{V} \triangleq 120 \angle \frac{\pi}{6}$

$\bar{I} \longleftrightarrow i(t)$

$= \frac{\bar{V}}{Z_C} = \frac{120 \angle \pi/6}{\frac{-j}{\omega C}}$

$i(t) = C \frac{dv}{dt}$

$= (120V)\, 2\mu F \frac{d}{dt} \cos(377t + \frac{\pi}{6})$

$= -(2\mu F)(120V) \sin(377t + \pi/6) \cdot 377$

Computing the Current

Advantage of phasors!

[Capacitor: $i = C\frac{dv}{dt}$
$\Rightarrow \bar{I} = \bar{V}/\frac{1}{j\omega C}$]

[Inductor: READ THE BOOK]

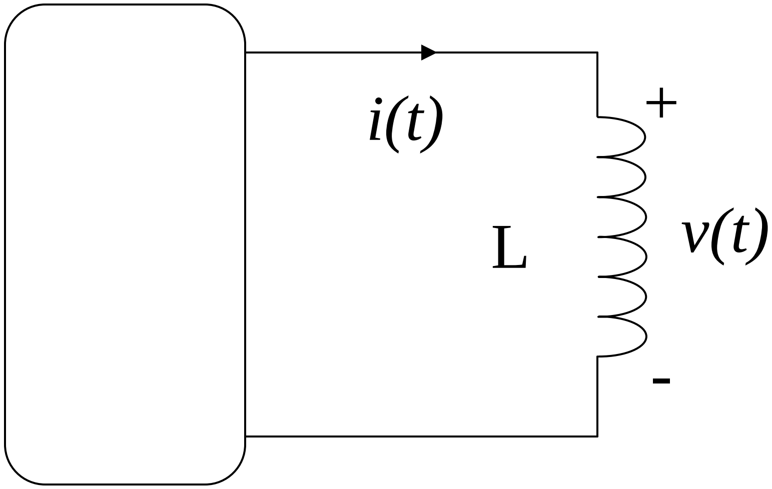
Note: The differentiation and integration operations become algebraic operations

$$\frac{d}{dt} \Rightarrow j\omega \qquad\qquad \int dt \Rightarrow \frac{1}{j\omega}$$

[Sections 9.1-9.4]

Inductor Impedance

[Section 9.4 in your book]



$$v(t) = L\frac{di(t)}{dt}$$

$$\mathbf{V} = j\omega L\,\mathbf{I}$$

Example

$$i(t) = 1\mu\text{A}\cos(2\pi\, 9.15\, 10^7 t + 30°)$$

$$L = 1\mu\text{H}$$

- What is **I**?
- What is **V**?
- What is *v(t)*?

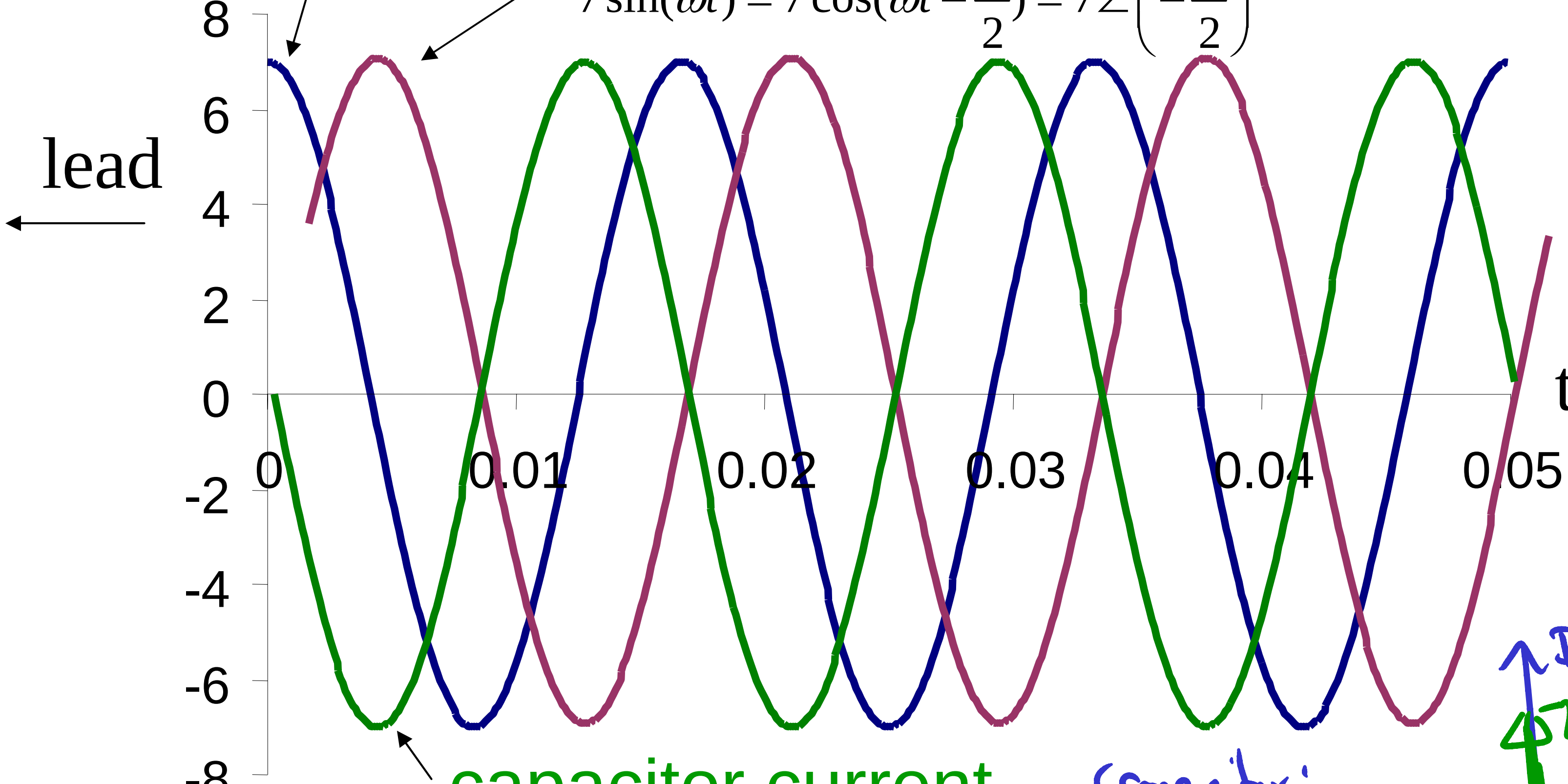
Phase

Voltage

$7\cos(\omega t) = 7\angle 0°$

inductor current

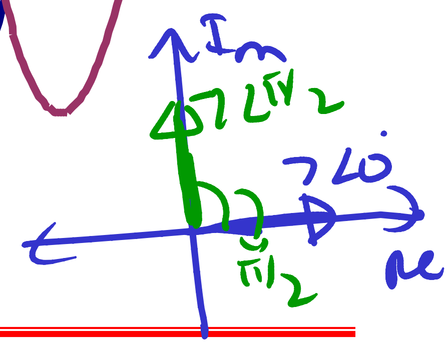
$7\sin(\omega t) = 7\cos(\omega t - \frac{\pi}{2}) = 7\angle\left(-\frac{\pi}{2}\right)$



capacitor current

$-7\sin(\omega t) = 7\cos(\omega t + \frac{\pi}{2}) = 7\angle\left(+\frac{\pi}{2}\right)$

capacitor:



Phasor Diagrams

- A phasor diagram is just a graph of several phasors on the complex plane (using real and imaginary axes).
- A phasor diagram helps to visualize the relationships between currents and voltages.
- Capacitor: I leads V by 90^o
- Inductor: V leads I by 90^o

Impedance

- AC steady-state analysis using phasors allows us to express the relationship between current and voltage using a formula that looks likes Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\mathbf{Z}$$

- **Z** is called impedance.

Some Thoughts on Impedance

- Impedance depends on the frequency ω.
- Impedance is (often) a complex number.
- Impedance allows us to use the same solution techniques for AC steady state as we use for DC steady state.