# Lecture 4: Bode Plots

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## Context

- In the last lecture:
  - We discussed analyzing circuits with a sinusoidal input, (in the frequency domain, a single frequency at a time)
  - How to simplify our notation with Phasors
  - and introduced Bode plots
- In this lecture, we will:
  - Review how get a transfer function for a circuit
  - How to put the transfer function into a standard form
  - Find why magnitude and phase plots are a useful form.
  - How to create an approximate Bode plot for a circuit.

# Bode plots

Since the majority of this lecture is on how to create approximate Bode plots by hand, it is fair to ask why we should do so when it can be done quickly on a computer.

The answer is that a few features of transfer functions that we will exploit for our graphs will appear often in different contexts, and design of a circuit for a particular purpose will often entail putting together several of these features, and the language of the circuit designer will use these constructs: poles, zeros, resonances, etc.







# Find the Transfer Function

- Excite a system with an input voltage $v_{in}$
- Define the output voltage $v_{any}$ to be any node voltage (branch current)
- For a complex exponential input, the "transfer function" from input to output( or any voltage or current) can then be written:

$$H(\omega) = \frac{n_1 + n_2 j\omega + n_3 (j\omega)^2 + \cdots}{d_1 + d_2 j\omega + d_3 (j\omega)^2 + \cdots}$$

This is found by using phasor notation to change circuits into networks of complex resistors, then applying Kirchoff's laws repeatedly

## Bode Plot Overview

- Then put the transfer function into standard form:

$$H(\omega) = G_0 (j\omega)^K \frac{(1 + j\frac{\omega}{\omega_{z1}})(1 + j\frac{\omega}{\omega_{z2}})\cdots(1 + j\frac{\omega}{\omega_{z3}})}{(1 + j\frac{\omega}{\omega_{p1}})(1 + j\frac{\omega}{\omega_{p2}})\cdots(1 + j\frac{\omega}{\omega_{p3}})}$$

- Each of the frequencies: $\omega_i = \frac{1}{\tau_i}$ correspond to time constants which are features of the circuit, and are called break frequencies.
- Those that appear in the numerator are called zeros, and those that appear in the denominator are called poles.





## Breakpoints

- Since the transfer function will always result in a real voltage, the following features can appear:
  - Real zeros
  - Real poles
  - Conjugate pairs of zeros
  - Conjugate pairs of holes

Each of these can appear in multiple orders (two poles at the same frequency, for example)

Additional features of the function are the constant $G_0$, and the order of the overall term →K

# Adding them up

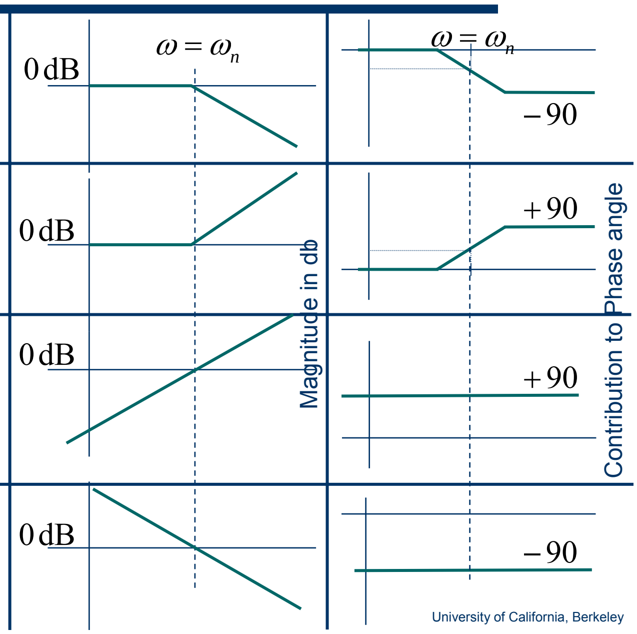
- If we take the log magnitude of each factor (as in db), they add to find the total magnitude.
- The phase angle adds as well, each factor contributes to the overall phase change.
- So we just catalog each of the features, and then add their magnitudes (in db) or contributions to the phase angles





# Summary of Individual Factors

- Simple Pole: $\left(1 + j\frac{\omega}{\omega_{pn}}\right)^{-1}$
- Simple Zero: $1 + j\frac{\omega}{\omega_{zn}}$
- DC Zero: $j\frac{\omega}{\omega_0}$
- DC Pole: $\frac{1}{j\frac{\omega}{\omega_0}}$



# Example

- Consider the following transfer function

$$H(j\omega) = \frac{10^{-5} j\omega(1 + j\omega\tau_2)}{(1 + j\omega\tau_1)(1 + j\omega\tau_3)} \qquad \begin{aligned} \tau_1 &= 100\,\text{ns} \\ \tau_2 &= 10\,\text{ns} \\ \tau_3 &= 100\,\text{ps} \end{aligned}$$

- Break frequencies: invert time constants

$$\omega_1 = 10\,\text{Mrad/s} \qquad \omega_2 = 100\,\text{Mrad/s} \qquad \omega_3 = 10\,\text{Grad/s}$$

$$H(j\omega) = \frac{\dfrac{j\omega}{10^5}(1 + j\dfrac{\omega}{\omega_2})}{(1 + j\dfrac{\omega}{\omega_1})(1 + j\dfrac{\omega}{\omega_3})}$$







# Magnitude

- Recall log of products is sum of logs

$$|H(j\omega)|_{\text{dB}} = 20\log\left|\frac{\dfrac{j\omega}{10^5}(1 + j\dfrac{\omega}{\omega_2})}{(1 + j\dfrac{\omega}{\omega_1})(1 + j\dfrac{\omega}{\omega_3})}\right|$$

$$= 20\log\left|\frac{j\omega}{10^5}\right| + 20\log\left|1 + j\frac{\omega}{\omega_2}\right|$$

$$- 20\log\left|1 + j\frac{\omega}{\omega_1}\right| - 20\log\left|1 + j\frac{\omega}{\omega_3}\right|$$

- Plot each factor separately and add them graphically

# Phase

- Since $\angle a \cdot b = \angle a + \angle b$

$$\angle H(j\omega) = \angle \frac{10^{-5} j\omega(1 + j\omega\tau_2)}{(1 + j\omega\tau_1)(1 + j\omega\tau_3)}$$

$$\angle H(j\omega) = \angle\{\frac{j\omega}{10^5}\} + \angle\{1 + j\frac{\omega}{\omega_2}\}$$

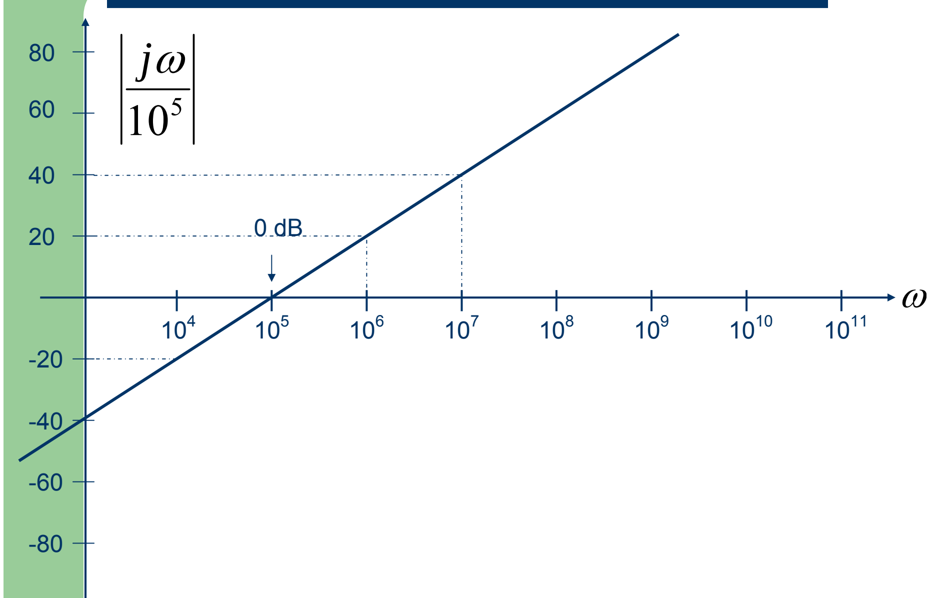
$$-\angle\{1 + j\frac{\omega}{\omega_1}\} - \angle\{1 + j\frac{\omega}{\omega_3}\}$$

- Plot each factor separately and add them graphically

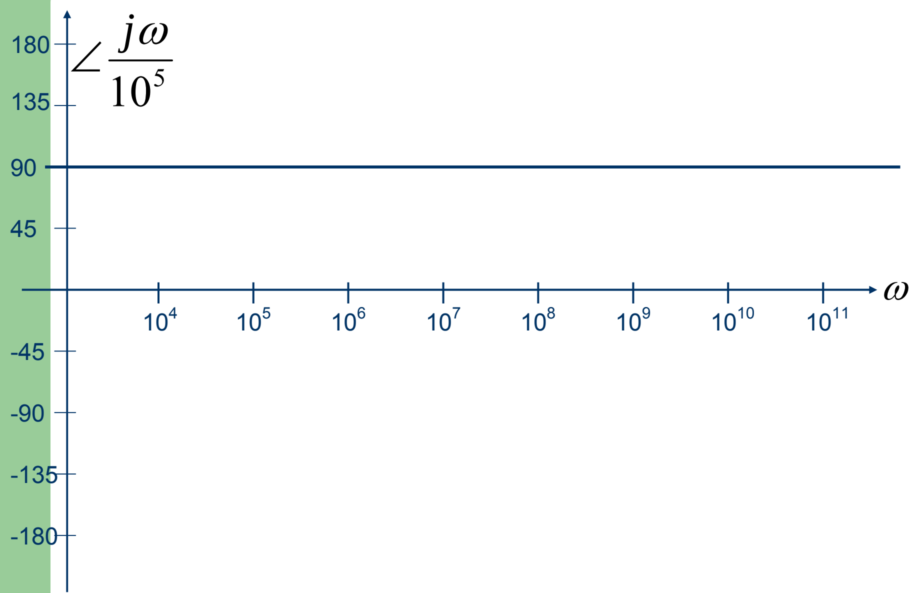




# Magnitude Bode Plot: DC Zero



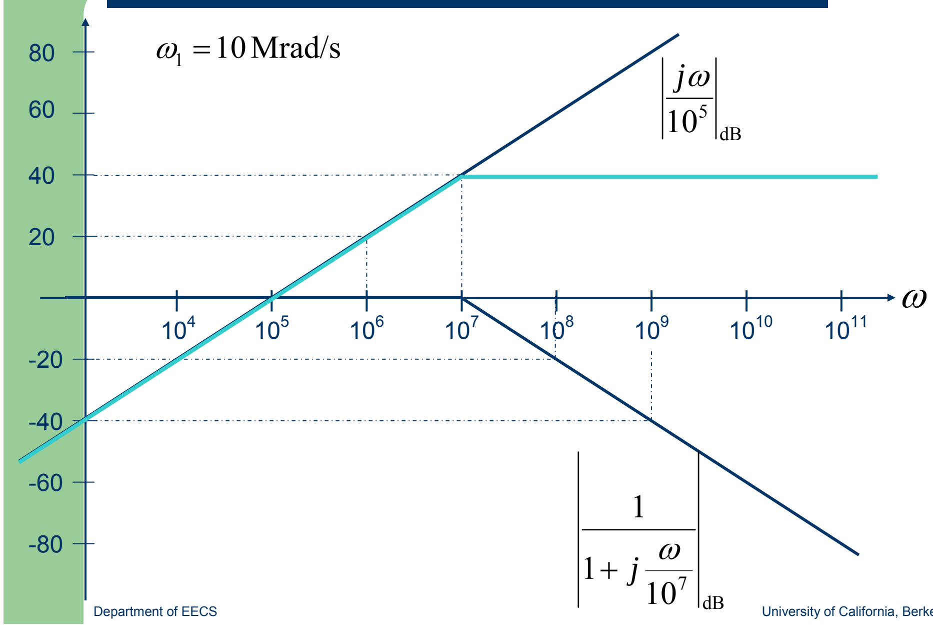
# Phase Bode Plot: DC Zero



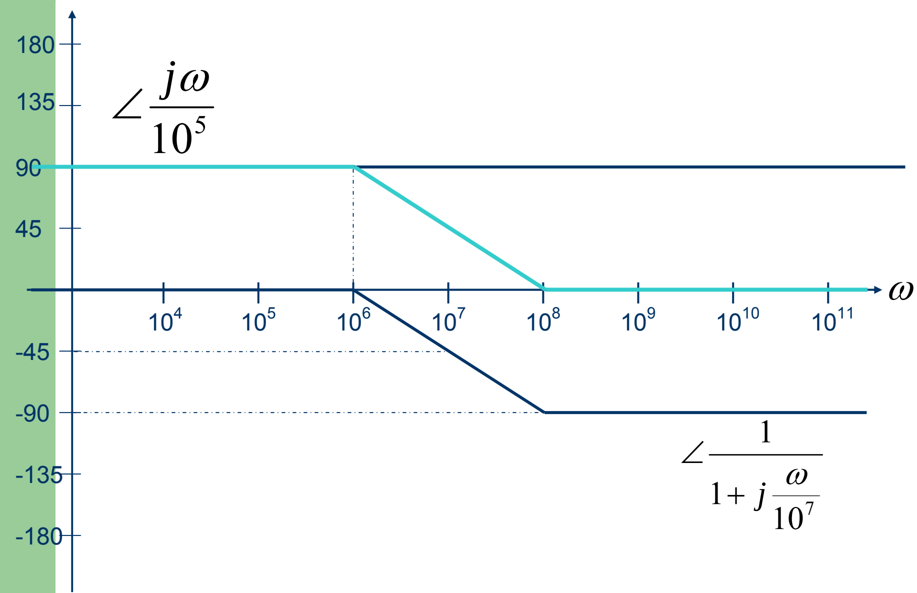





# Magnitude Bode Plot: Add First Pole



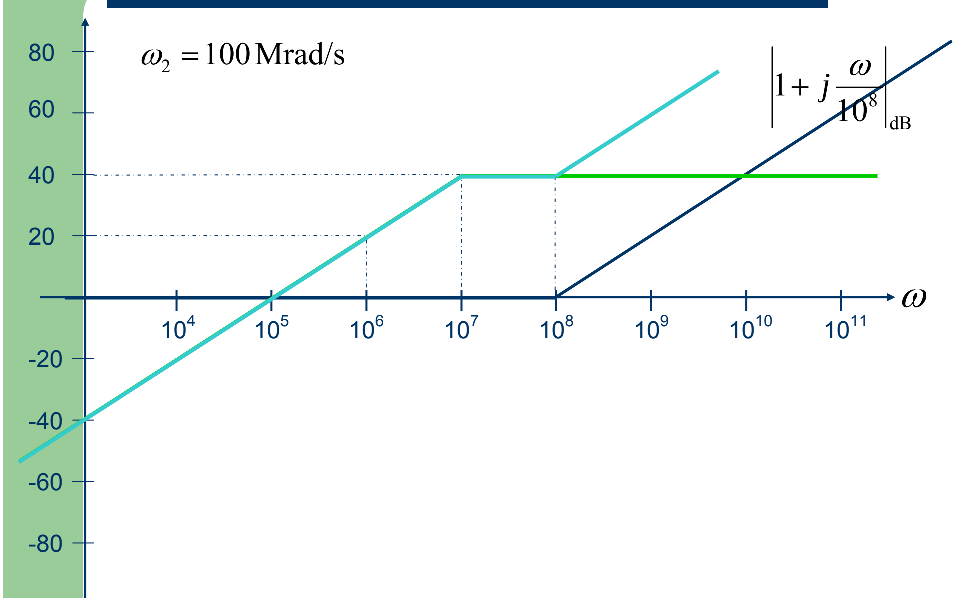
# Phase Bode Plot: Add First Pole



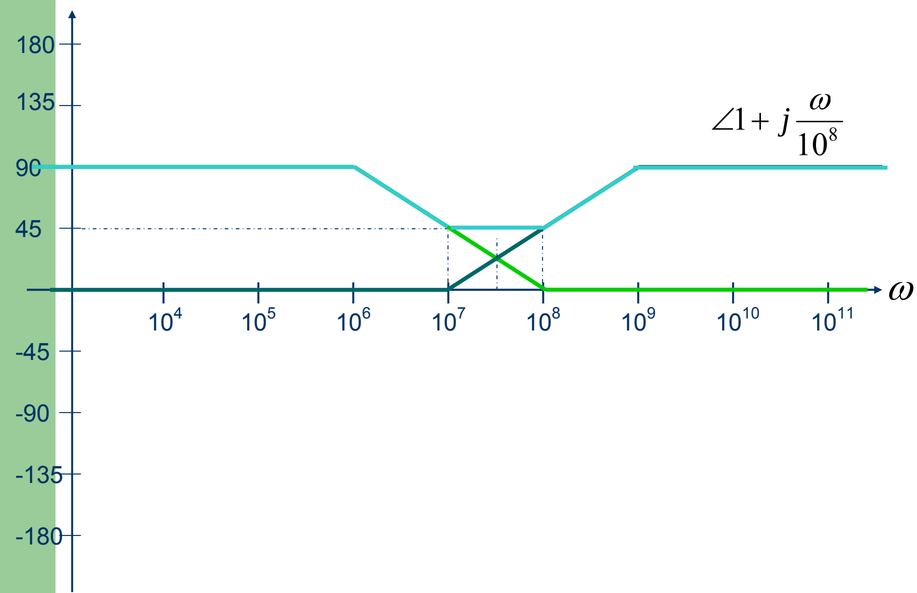





# Magnitude Bode Plot: Add 2nd Zero



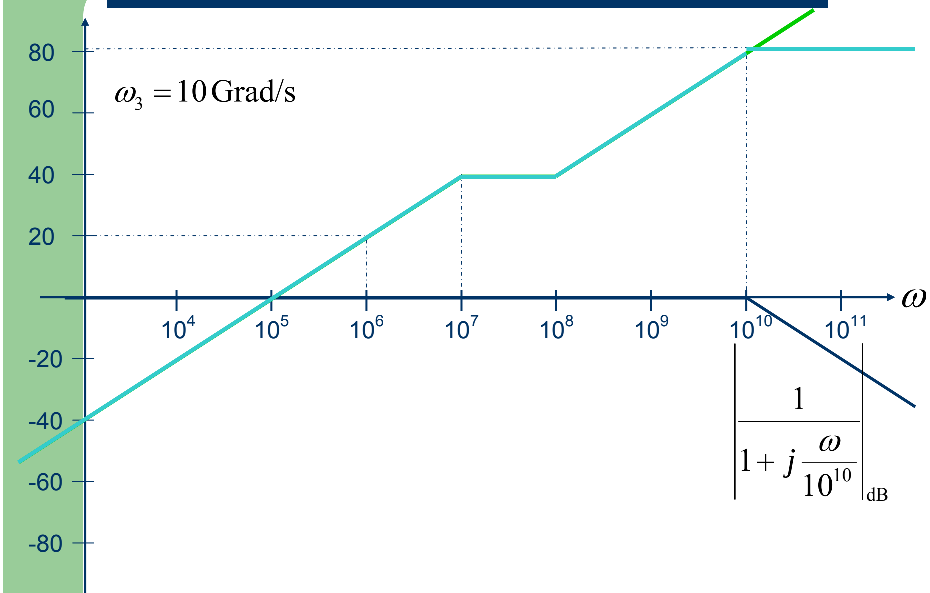
# Phase Bode Plot: Add 2nd Zero



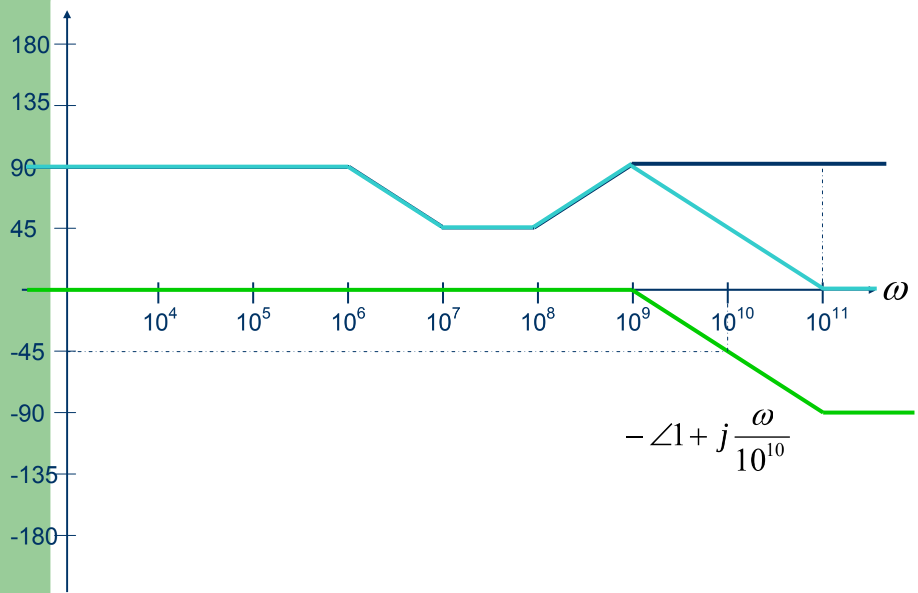





# Magnitude Bode Plot: Add 2nd Pole



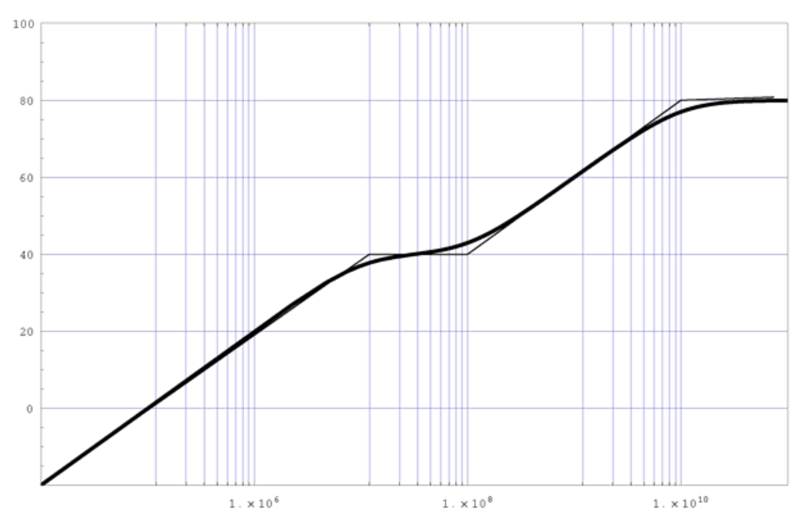
# Phase Bode Plot: Add 2nd Pole



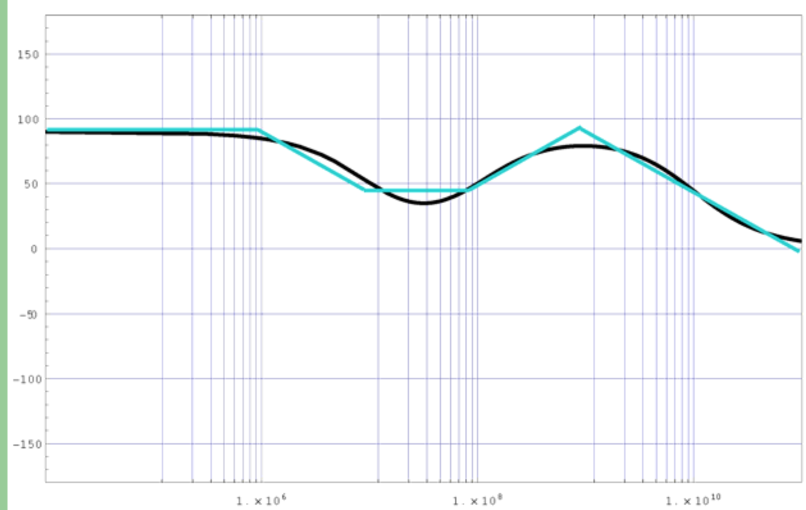





# Comparison to "Actual" Mag Plot



# Comparison to "Actual" Phase Plot

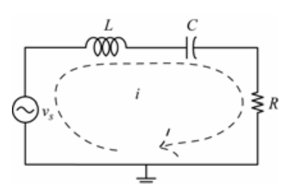







# Second Order Transfer Function

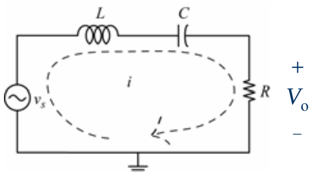
- The series resonant circuit is one of the most important elementary circuits:




- The physics describes not only physical LCR circuits, but also approximates mechanical resonance (mass-spring, pendulum, molecular resonance, microwave cavities, transmission lines, buildings, bridges, …)

# Series LCR Analysis

- With phasor analysis, this circuit is readily analyzed




$$V_s = I\, j\omega L + I \frac{1}{j\omega C} + I\,R$$

$$V_s = I\left( j\omega L + \frac{1}{j\omega C} + R \right)$$

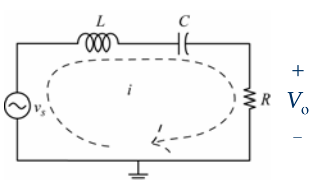
$$V_0 = I\,R = \frac{V_s}{j\omega L + \frac{1}{j\omega C} + R} R$$





# Second Order Transfer Function

- So we have:




$$H(j\omega) = \frac{V_0}{V_s} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}$$

- To find the poles/zeros, let's put the *H* in canonical form:

$$H(j\omega) = \frac{V_0}{V_s} = \frac{j\omega CR}{1 - \omega^2 LC + j\omega RC}$$

- One zero at DC frequency → can't conduct DC due to capacitor

# Poles of 2nd Order Transfer Function

- Denominator is a quadratic polynomial:

$$H(j\omega)=\frac{V_0}{V_s}=\frac{j\omega CR}{1-\omega^2 LC+j\omega RC}=\frac{j\omega\frac{R}{L}}{\frac{1}{LC}+(j\omega)^2+j\omega\frac{R}{L}}$$

$$H(j\omega)=\frac{j\omega\frac{R}{L}}{\omega_0^2+(j\omega)^2+j\omega\frac{R}{L}} \qquad \omega_0^2\equiv\frac{1}{LC}$$

$$H(j\omega)=\frac{j\frac{\omega\omega_0}{Q}}{\omega_0^2+(j\omega)^2+j\frac{\omega\omega_0}{Q}} \qquad Q\equiv\frac{\omega_0 L}{R}$$





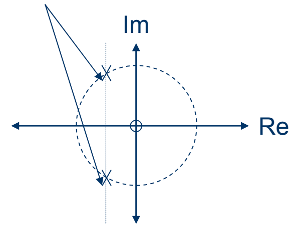
# Finding the poles…

- Let's factor the denominator:

$$(j\omega)^2+j\frac{\omega\omega_0}{Q}+\omega_0^2=0$$

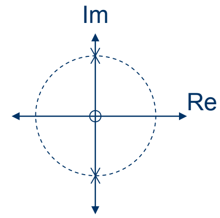
$$\omega=-\frac{\omega_0}{2Q}\pm\sqrt{\frac{\omega_0^2}{4Q^2}-\omega_0^2}=-\frac{\omega_0}{2Q}\pm j\omega_0\sqrt{1-\frac{1}{4Q^2}}$$

- Poles are complex conjugate frequencies
- The $Q$ parameter is called the "quality-factor" or Q-factor
- This parameters is an important parameter: $Q\xrightarrow{R\to 0}\infty$



# Resonance without Loss

- The transfer function can parameterized in terms of loss. First, take the lossless case, $R$=0:




$$\omega = \left( -\frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} - \omega_0^2} \right)_{Q\to\infty} = \pm j\omega_0$$

- When the circuit is lossless, the poles are at *real* frequencies, so the transfer function blows up!
- At this *resonance* frequency, the circuit has zero imaginary impedance
- Even if we set the source equal to zero, the circuit can have a steady-state response

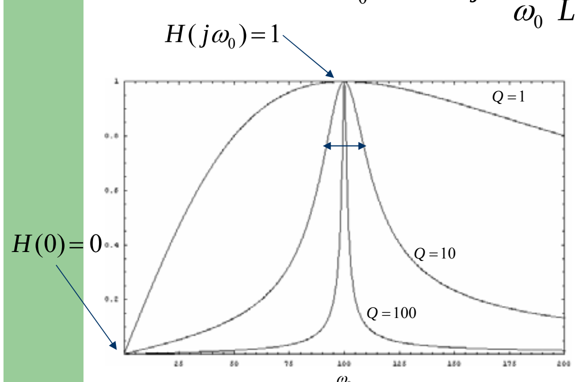




# Magnitude Response

- How strongly peaked the response is depends on $Q$

$$H(j\omega) = \frac{j\omega \frac{\omega_0}{\omega_0}\frac{R}{L}}{\omega_0^2 - \omega^2 + j\omega\frac{\omega_0}{\omega_0}\frac{R}{L}} = \frac{j\omega\frac{\omega_0}{Q}}{\omega_0^2 - \omega^2 + j\omega\frac{\omega_0}{Q}}$$




$$H(j\omega_0) = \frac{j\frac{\omega_0^2}{Q}}{\omega_0^2 - \omega_0^2 + j\omega_0\frac{\omega_0}{Q}} = 1$$

# Get to know your logs!

| dB | ratio | | | dB | ratio |
|---|---|---|---|---|---|
| -20 | 0.100 | | | 20 | 10.000 |
| -10 | 0.316 | | | 10 | 3.162 |
| -5 | 0.562 | | | 5 | 1.778 |
| -3 | 0.708 | | | 3 | 1.413 |
| -2 | 0.794 | | | 2 | 1.259 |
| -1 | 0.891 | | | 1 | 1.122 |

- Engineers are very conservative. A “margin” of 3dB is a factor of 2 (power)!
- Knowing a few logs by memory can help you calculate logs of different ratios by employing properties of log. For instance, knowing that the ratio of 2 is 3 dB, what’s the ratio of 4?