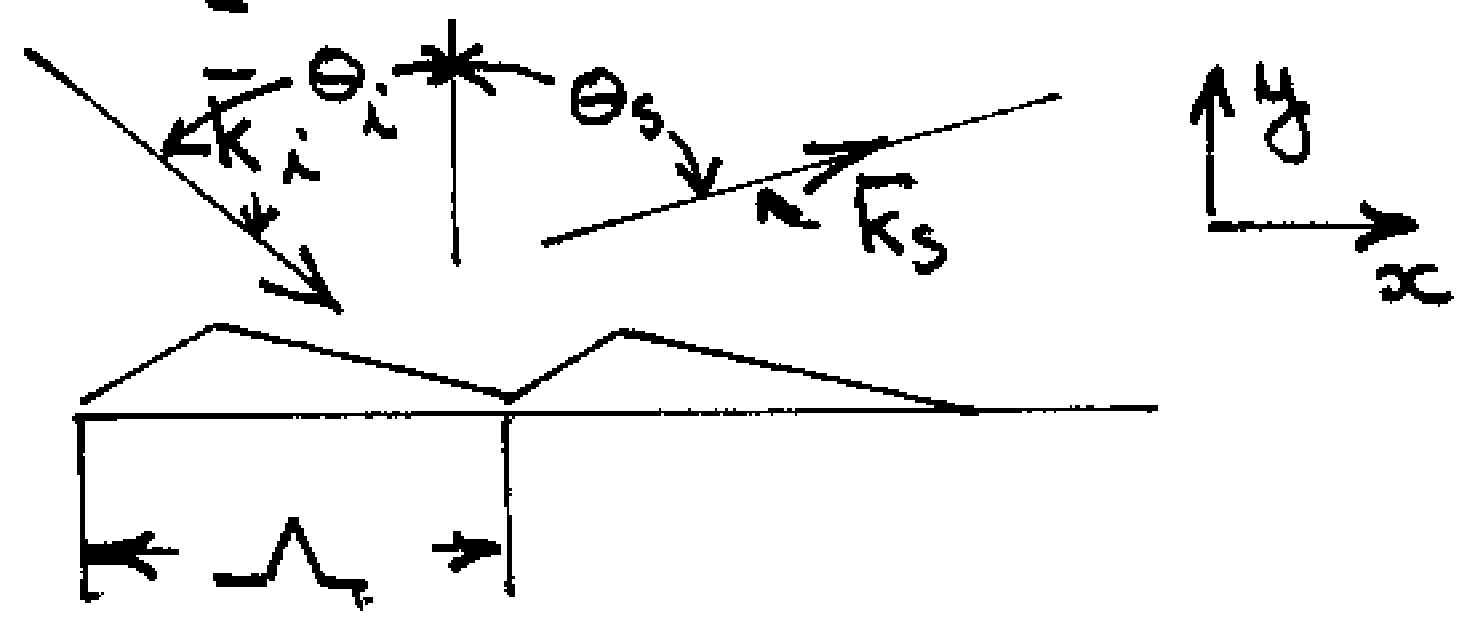
Grating Eq.ation And Re.olution



① conservation of momentm $\hbar \bar{k}_s = \hbar \bar{k}_i + \hbar \bar{k}_G$

② conservation of energy $\hbar \omega_s = \hbar \omega_i + 0$

Only if the grating is travelling (i.e. of the form $f(\omega t - \bar{k}_G \cdot \bar{r})$ for some f, can it have energy $\hbar\omega$

Resolve ① into components

$k_{sx} = k_{ix} + k_G$

$k_{sy} = k_{iy} + k_{Gy}$ with $k_{sx}^2 + k_{sy}^2 = k_s^2$ and $k_{ix}^2 + k_{iy}^2 = k_i^2$

③ $k_G = (m \frac{2\pi}{\Lambda})$, m an integer, since

$y(x) = \sum_m y_m e^{i \frac{2\pi}{\Lambda} m x}$ is the profile of the grating which multiplies the incident wave

$$y(x)(E_i) = (E_{io} e^{i k_{ix} x})(\sum y_m e^{i \frac{2\pi}{\Lambda} m x}) \quad (\text{at } y = 0)$$

$$= E_{io} \sum \underbrace{y_m e^{i(k_{ix} + \frac{2\pi}{\Lambda} m)x}}_{\text{generates the scattered field}}$$

Thus $k_{sx} = k_{ix} + \frac{2\pi}{\Lambda} m$: But $k_{sx} = k_s \sin\theta_s$

$k_{ix} = k_i \sin\theta_i$

Consequently $k_s \sin\theta_s = k_i \sin\theta_i + m \frac{2\pi}{\Lambda}$

Also $k_s = \omega/c$ $k_i = \omega/c = k_s$

Finally

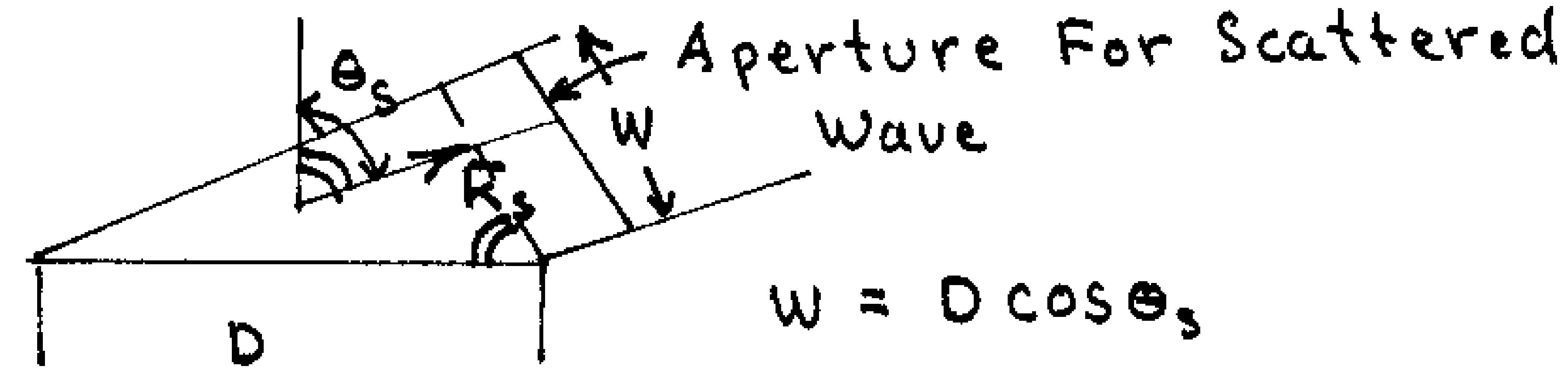
$$\boxed{\sin\theta_s = \sin\theta_i + \frac{cm}{\omega} \frac{2\pi}{\Lambda}}$$

order of the sat

Grating Equation

Important Parameters

Resolution



$$W = D\cos\theta_s$$

θ_D = Diffraction Angle $\simeq \dfrac{\lambda}{W} = \dfrac{\lambda}{D\cos\theta_s}$

θ_s given by $\sin\theta_s = \sin\theta_i + \dfrac{c}{\omega} m \dfrac{2\pi}{\Lambda}$

$\theta_s + \Delta\theta_s$ given by $\sin(\theta_s + \Delta\theta_s) = \sin\theta_i + (\lambda + \Delta\lambda)\dfrac{m}{\Lambda}$

$$\sin(\theta_s+\Delta\theta_s) = \sin\theta_s \underbrace{\cos\Delta\theta_s}_{\simeq 1} - \cos\theta_s \underbrace{\sin\Delta\theta_s}_{\simeq \Delta\theta_s}$$

$$\sin\theta_s - \sin(\theta_s + \Delta\theta_s) \simeq -\cos\theta_s\,\Delta\theta_s$$

$$= \Delta\lambda \frac{m}{\Lambda} \quad ; \quad \text{Thus } \boxed{\frac{\Delta\theta_s}{\Delta\lambda} = -\frac{m}{\Lambda\cos\theta_s}} \text{ Angular Dispersion}$$

Letting $\Delta\theta_s = \theta_D = \dfrac{\lambda}{D\cos\theta_s}$ this is

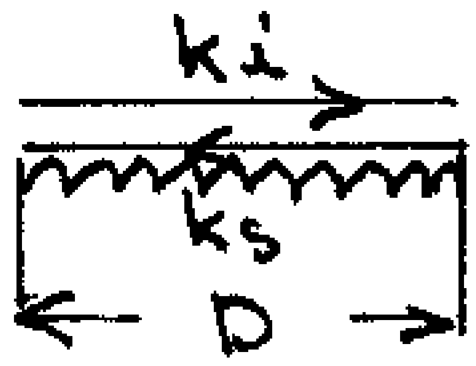
Resolution

$$-\frac{\lambda}{D} = \Delta\lambda \frac{m}{\Lambda} \quad \text{or} \quad \boxed{\frac{\Delta\lambda}{\lambda} = -\frac{\Lambda}{D}\cdot\frac{1}{m} = -\frac{1}{mN}}$$

where N is the number of periods.

Thus the resolution is determined by the number of periods and the order, m.

Note: Proof of DFB Grating Resolution Using The Uncertainty Relation



Grating Period $\Lambda = \dfrac{\lambda}{2}$ $(m = 1)$

$\Delta k \Delta x \simeq \dfrac{1}{2}$; $\Delta k = 2\pi \dfrac{\Delta f}{c} = -2\pi \dfrac{\Delta\lambda}{\lambda^2}$

$\Delta x = D$

$$\therefore -2\pi \frac{\Delta\lambda}{\lambda^2} D \simeq \frac{1}{2} \Rightarrow \frac{\Delta\lambda}{\lambda} \simeq -\frac{1}{4\pi}\frac{\lambda}{D} = -\frac{\Lambda}{2\pi D}$$

Grating as a mirror

$E_+ e^{i(\omega t - k_+)z}$ $\qquad$ $E_- e^{i(\omega t + k_-)z}$

$k_+ = \frac{2\pi}{\lambda} n$

$k_- = \frac{2\pi}{\lambda} n$

The fields are coupled through the Fourier expansion of the periodic dielectric coefficient

$$\epsilon(z) = \sum_m \epsilon_m e^{i(m\frac{2\pi}{\Lambda})z}$$

For tuned coupler

$$k_- = k_+ + m\frac{2\pi}{\Lambda}$$

For $m=-1$ $\quad \Lambda = \frac{\lambda}{2n}$

If κ is the coupling of the − and + going waves

+ for + going wave

$$\frac{dE_+}{dz} = +i\kappa E_-$$

for − going wave

$$\frac{dE_-}{dz} = -i\kappa E_+$$

Conservation of Energy

$$(E_+)^*\frac{dE_+}{dz} + E_+\frac{dE_+^*}{dz}$$

$$-(E_-)^*\frac{dE_-}{dz} - E_-\frac{dE_-^*}{dz}$$

$$= i\kappa E_- E_+^* - i\kappa E_-^* E_+$$

$$+ i\kappa E_+ E_-^* - i\kappa E_+^* E_-$$

$$= \frac{d}{dz}(E_+E_+^* - E_-E_-^*) = 0$$

Thus Power in the + wave − Power in the − wave is a constant.

Solutions

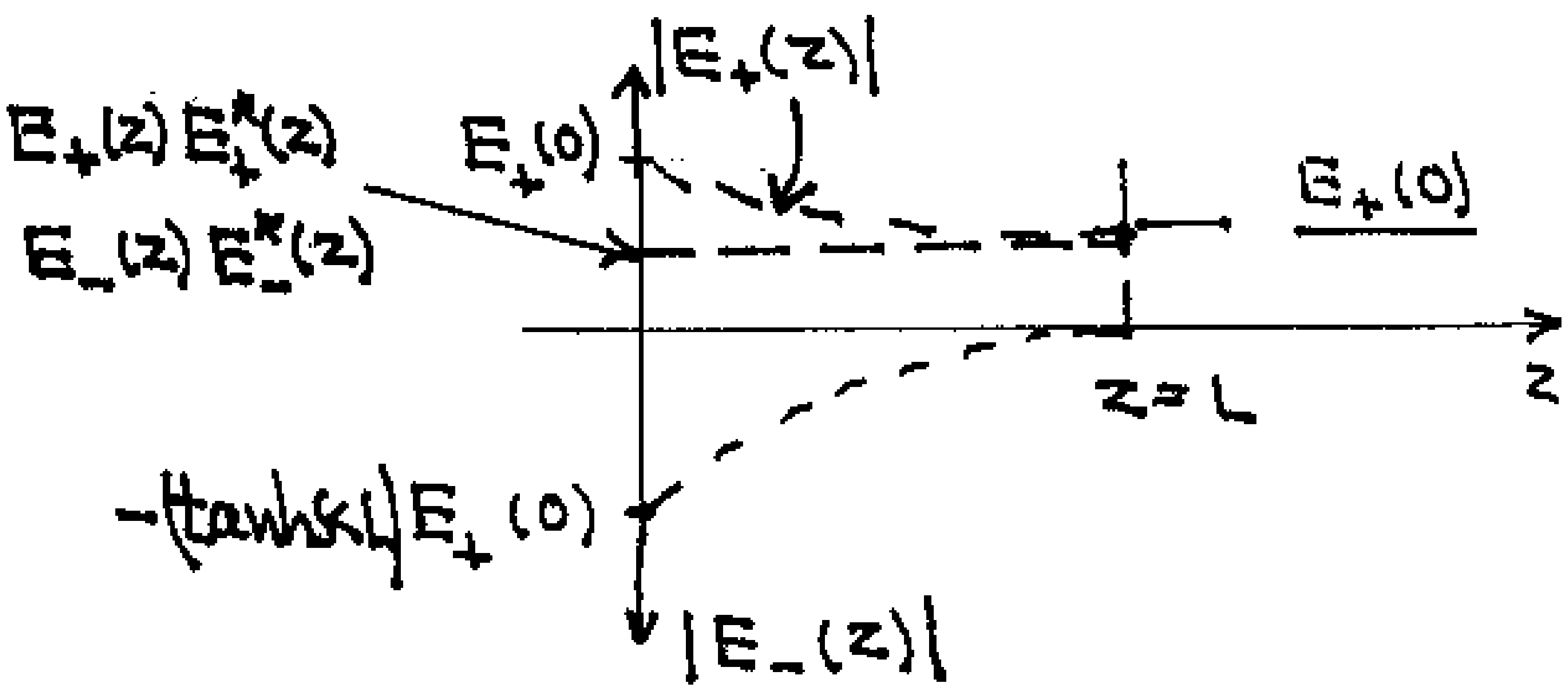
With Boundary Condition

$$E_+\big|_{z=0} = E_+(0)$$

$$E_-\big|_{z=L} = 0$$

$$E_+(z) = \frac{\cosh\kappa(z-L)}{\cosh\kappa L}E_+(0)$$

$$E_-(z) = +\left(\frac{i\sinh\kappa(z-L)}{\cosh\kappa L}\right)E_+(0)$$



Reflectivity at $z=0$

$$R = \frac{E_-(0)}{E_+(0)} = -i\tanh(\kappa L)$$

$\frac{\pi}{2}$ phase shift.

What if the grating is not "tuned" to $\Lambda = \lambda/2n$ (2)

$$\underbrace{\cancel{e^{i\omega t}}\ \underline{e}^{-ikz}}_{+\text{ going wave form}} \frac{dE_+}{dz} = i\kappa\, \underbrace{e^{+ip\frac{2\pi}{\Lambda}z}}_{\text{p'th Fourier component of Grating}}\ \underbrace{e^{+ikz}\ \cancel{e^{i\omega t}}}_{-\text{ going wave-form}} E_-$$

or $\frac{dE_+}{dz} = i\kappa\, e^{i2\Delta\beta z}$ $\underbrace{\Delta\beta = k - \frac{p\pi}{\Lambda}}_{\text{detuning}}$

Also $\frac{dE_+}{dz} = -i\kappa\, e^{-i2\Delta\beta z}$

Solutions For $E_+(z)$ and $E_-(z)$ with $E_+(0)$ given and $E_+(L)=0$

$$E_+(z) = \frac{-\Delta\beta \sinh S(z-L) + iS\cosh S(z-L)}{\Delta\beta \sinh SL + iS\cosh SL}\left(E_+(0)\right)$$

$$E_-(z) = \frac{K^*\sinh S(z-L)}{\Delta\beta \sinh SL + iS\cosh SL}\cdot\left(E_+(0)\right)$$

where $S = \left(|K|^2 - |\Delta\beta|^2\right)^{1/2}$

Mirror Reflectivity $R = \dfrac{-K^*\sinh SL}{\Delta\beta\sinh SL + iS\cosh SL}$

At $\Delta\beta = 0$, $S = |K|$

$$|r(0)| = \tanh|KL|$$

which approaches 1 if $|KL|$ is large enough. Also, when $|\Delta\beta| > |K|$, S becomes purely imaginary and $|r(0)|^2$ vanishes when

$$\sin^2\sqrt{(\Delta\beta L)^2 - |KL|^2} = 0, \quad \text{or} \quad |\Delta\beta/K| = \sqrt{1 + (n\pi/|KL|)^2}\,.$$

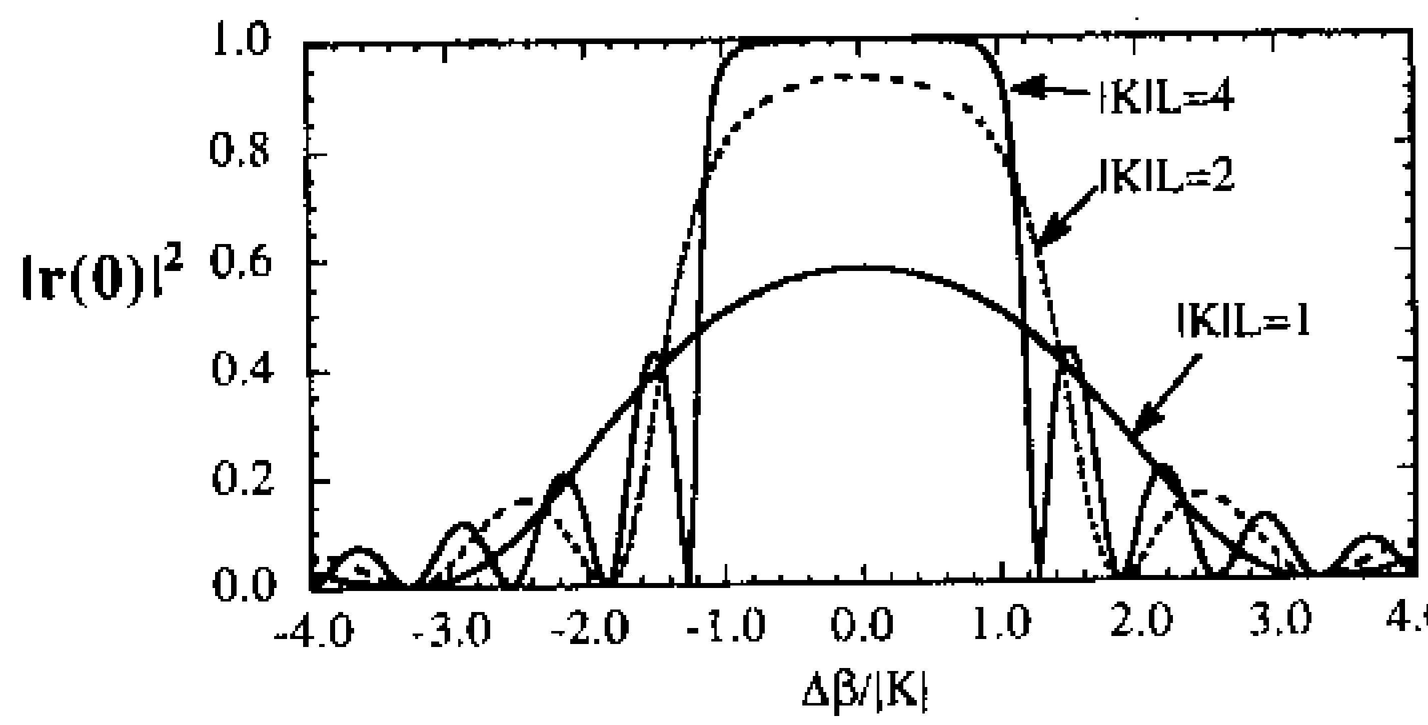


Figure 8.26. A plot of the reflectivity $|r(0)|^2$ vs. the ratio $\Delta\beta/|K|$ for three different values of $|KL|$. For a large value of $|KL|$, the bandwidth for $\Delta\beta$ is about $2|K|$.

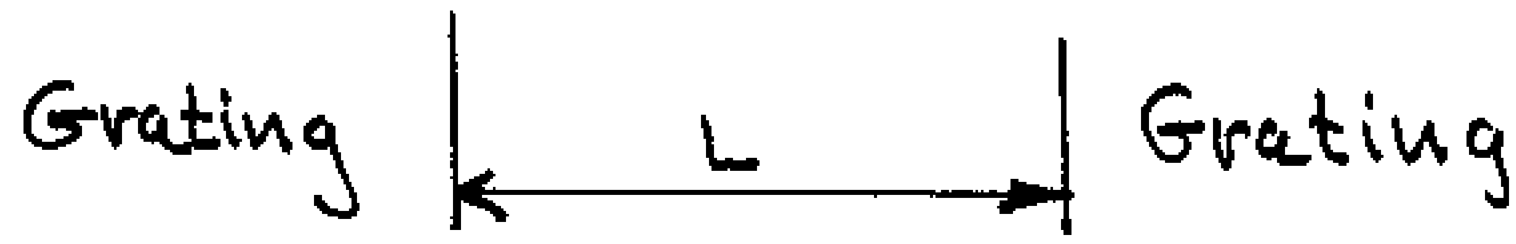
Note: Width of Reflectivity

$\Delta\beta/|K| = \frac{\Delta\beta L}{(|K|L)} \simeq \frac{\pi}{(|K|L)}$

But this is just Uncertainty Relation

$\Delta\beta L \simeq \pi$

Cavity with Grating Mirrors : The role of the $\pi/2$ phase shift.



Cavity Constraint $\quad r_1 r_2 e^{-2ikL} = 1$

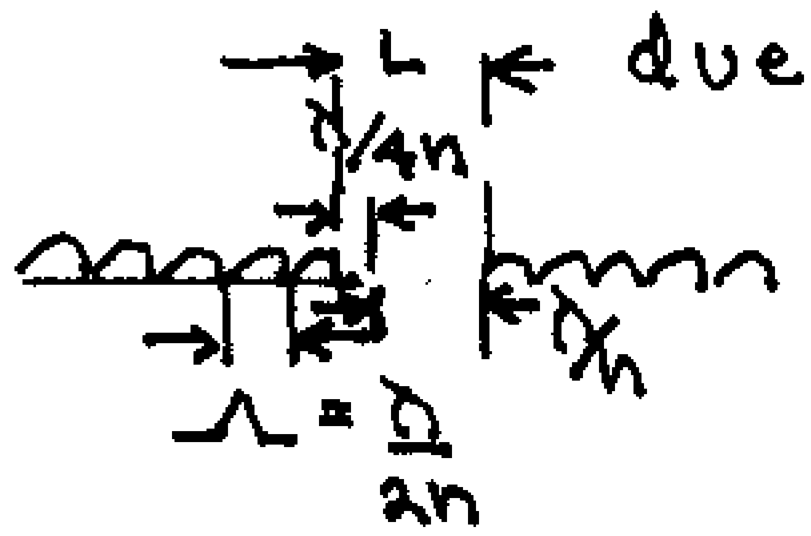
phase parameters $\quad r_1 = |r_1| e^{-i\pi/2}$

$$r_2 = |r_2| e^{-i\pi/2}$$

Thus $\quad -\frac{\pi}{2} - \frac{\pi}{2} - 2kL = -p(2\pi) \quad$ p - integer

For $p = 1 \quad L = \frac{\pi}{2k} = \frac{\lambda}{4n}$

In general $\quad L = p\frac{\lambda}{2n} - \frac{\lambda}{4n}$ — A ¼ wavelength shift due to the $\pi/2$ phase shift of the gratings



example $p = 3 \quad L = \frac{5}{4}\frac{\lambda}{n}$

$$= \frac{1}{4}\frac{\lambda}{n} + \frac{\lambda}{n}$$

Note DBR Laser (With a Quarter Wave length Sh...
The Cavity Mode is at the center of the Reflectivity Profile (at f_0 in the figure below)

However: If $L = l\Lambda$, the resonant wave lengths are

$$\frac{\lambda}{n} = \frac{l\Lambda}{(P/2 - 1/4)} = l\frac{2\Lambda}{(p - 1/2)}$$

In particular if $l = p$

$$\frac{\lambda}{n} = \frac{2\Lambda}{(1 - \frac{1}{2p})}$$

or $f = \frac{cn}{\lambda} = f = f_0\left(1 - \frac{1}{2p}\right)$

where $f_0 = \frac{c}{2\Lambda}$ is the tuned frequency