Dispersion of a Gaussian Pulse Propagating on an Optical Fiber

$$E(t) = e^{-(t/T_p)^2} e^{i\omega_0 t} \qquad \leftarrow \text{phasor of pulse at } z=0$$

Fourier Transform of $E(t)$

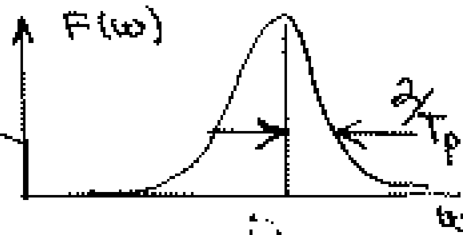
$$E(\omega) = \int E(t)\, e^{-i\omega t} = \int e^{-(t/T_p)^2 - i(\omega-\omega_0)t}\, dt$$

$$= \int e^{-\frac{1}{T_p^2}\left(t + i\frac{(\omega-\omega_0)}{2}T_p^2\right)^2} e^{-\frac{(\omega-\omega_0)^2 T_p^2}{4}}\, d\omega$$

$$= e^{-\frac{(\omega-\omega_0)^2}{4}T_p^2} \int e^{-\eta^2}\, d\eta\, T_p$$

with $\eta = \left(t + \frac{i(\omega-\omega_0)}{2}T_p^2\right) \cdot \frac{1}{T_p}$

$$= e^{-\frac{(\omega-\omega_0)^2}{4}T_p^2} \sqrt{\pi}\, T_p$$



After propagating a distance z a phase change given by e^{-ikz} has occurred where

$$k(\omega) \simeq k(\omega_0) + \frac{\partial k}{\partial \omega}(\omega-\omega_0) + \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}(\omega-\omega_0)^2$$

Thus the output spectrum of the pulse is:

$$E(\omega, z) = e^{-\frac{(\omega-\omega_0)^2}{4}T_p^2} \sqrt{\pi}\, T_p\, e^{-i k(\omega_0) z}\, e^{-i\frac{\partial k}{\partial \omega}(\omega-\omega_0)} \times$$

$$\times\, e^{-i\frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}(\omega-\omega_0)^2} \cdot T_p \cdot \sqrt{\pi}$$

The output pulse is the inverse transform

$$E(t, z) = \frac{1}{2\pi} T_p \sqrt{\pi} \int e^{i(\omega-\omega_0)t}\, e^{i(\omega_0 t - k(\omega_0) z)}$$

$$\times\, e^{-(\omega-\omega_0)^2 \frac{T_p^2}{4}}\, e^{-i\frac{\partial k}{\partial \omega}(\omega-\omega_0) z}\, e^{-i\frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}(\omega-\omega_0)^2 z}\, d\omega$$

Using the same technique of completing the square; this is (letting $\Delta\omega = \omega - \omega_0$)

$$\frac{1}{2\pi} T_p \sqrt{\pi} \left[\int e^{-\Delta\omega^2 T_p^2/4 + i\Delta\omega\left(t - \frac{\partial k}{\partial\omega} z\right) + -i\frac{1}{2}\frac{\partial^2 k}{\partial\omega^2}(\omega-\omega_0)^2 z} \, d\omega \right]$$

$$\times \underbrace{e^{i(\omega_0 t - k(\omega_0) z)}}_{\text{phase velocity term}}$$

$$= \frac{1}{2\pi} T_p \sqrt{\pi} \left[\int e^{-\left(T_p^2/4 + \frac{i}{2}\frac{\partial^2 k}{\partial\omega^2} z\right)\Delta\omega^2} e^{+i\Delta\omega\left(t - \frac{\partial k}{\partial\omega} z\right)} \, d\omega \right] \times$$

$$\times e^{i(\omega_0 t - k(\omega_0) z)}$$

Write this as (letting $A = \left(T_p^2/4 + \frac{i}{2}\frac{\partial^2 k}{\partial\omega^2} z\right)$

$$= \frac{1}{2\pi} T_p \sqrt{\pi} \left[\int e^{-A\left(\Delta\omega - i\left(t - \frac{\partial k}{\partial\omega} z\right)/A\right)^2} \frac{d\,\Delta\omega\, A}{A} \right.$$

$$\times \left[e^{i(\omega_0 t - k(\omega_0) z)} \times e^{-\left(t - \frac{\partial k}{\partial\omega} z\right)^2/A} \right]$$

Letting $\eta = A\left(\Delta\omega - \overbrace{i\left(t - \frac{\partial k}{\partial\omega} z\right)/A}^{\text{constant for the integration}}\right)$ this is

$$= \frac{1}{2\pi} T_p \sqrt{\pi}\, e^{i(\omega_0 t - k(\omega_0) z)} e^{-\left(t - \frac{\partial k}{\partial\omega} z\right)^2/A} \frac{1}{A} \underbrace{\int e^{-\eta^2} d\eta}_{\sqrt{\pi}}$$

$$E(t,z) = \frac{T_p/2}{A} e^{i(\omega_0 t - k(\omega_0) z)} e^{-\left(t - \frac{\partial k}{\partial\omega} z\right)^2/A}$$

Convert the first A to polar form and rationalize the $\frac{1}{A}$ in the exponential. Thus

$$E(t,z) = \left[\frac{T_p/2 \quad e^{-i \tan^{-1}\left(\frac{2 \frac{\partial^2 k}{\partial \omega^2} z}{T_p^2} \right)}}{\underbrace{\left(\left(\frac{T_p}{2}\right)^4 + \left(\frac{\partial^2 k}{\partial \omega^2} \frac{z}{2}\right)^2 \right)^{1/2}}_{\text{Decrease in amplitude as the pulse spreads}}} \right] \times$$

phase change due to dispersion

$$\times \; e^{-\left(t - \frac{\partial k}{\partial \omega} z\right)^2 \left(\frac{\frac{T_p^2}{4} - \frac{i}{2} \frac{\partial^2 k}{\partial \omega^2} z}{\left(\frac{T_p^2}{4}\right)^2 + \left(\frac{1}{2} \frac{\partial^2 k}{\partial \omega^2} z\right)^2} \right)}$$

pulse width term

frequency sweep generated across the pulse

$$\times \; \underbrace{e^{i(\omega_0 t - k(\omega_0) z)}}_{\text{carrier wave propagating at the phase velocity}}$$

Observations.

1) Pulse width $= \left[\left(\frac{T_p^2}{4}\right)^2 + \left(\frac{1}{2} \frac{\partial^2 k}{\partial \omega^2} z\right)^2 \right]^{1/2} \Big/ \left(\frac{T_p}{2}\right)$

$$= \left[T_p^2 + \left(\frac{2}{T_p} \frac{\partial^2 k}{\partial \omega^2} z \right)^2 \right]^{1/2}$$

$$= \left[T_p^2 + \Delta T_p^2 \right]$$

where $(\Delta T_p) = \frac{2}{T_p} \frac{\partial^2 k}{\partial \omega^2} z = \Delta \omega \frac{\partial^2 k}{\partial \omega^2} z$

This is the same result obtained from

$$\Delta T_p = \left(\frac{z}{v_g(\omega)} - \frac{z}{v_g(\omega + \Delta\omega)} \right) \quad \text{with } v_g = \frac{\partial \omega}{\partial k}$$

$$\cong \frac{z}{v_g^2} \frac{\partial v_g}{\partial \omega} \Delta \omega = z \frac{\partial^2 k}{\partial \omega^2} \Delta \omega$$

Furthermore

Pulse-width depends only on $\frac{\partial^2 n}{\partial \lambda^2}$

$k = \frac{\omega}{c} n$ where n is the refractive index

$$\frac{\partial k}{\partial \omega} = \frac{n}{c} + \frac{\omega}{c}\frac{\partial n}{\partial \omega}$$

$$\left[\frac{\partial^2 k}{\partial \omega^2} = \frac{2}{c}\frac{\partial n}{\partial \omega} + \frac{\omega}{c}\frac{\partial^2 n}{\partial \omega^2}\right]$$

If $\frac{\partial^2 n}{\partial \omega^2}$ is changed to $\frac{\partial^2 n}{\partial \lambda^2}$ where λ is the free-space wavelength ; the first term cancels.

$$\frac{\partial^2 n}{\partial \omega^2} = \frac{\partial}{\partial \omega}\left(\frac{\partial n}{\partial \lambda}\frac{\partial \lambda}{\partial \omega}\right) = \frac{\partial^2 n}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial \omega}\right)^2 + \frac{\partial n}{\partial \lambda}\left(\frac{\partial^2 \lambda}{\partial \omega^2}\right)$$

$$\lambda f = c = \lambda \frac{\omega}{2\pi} \quad \text{or} \quad \lambda = \frac{c}{\omega} 2\pi$$

Thus $\frac{\partial \lambda}{\partial \omega} = -\frac{c}{\omega^2} 2\pi = -\frac{\lambda}{\omega}$ and $\frac{\partial^2 \lambda}{\partial \omega^2} = 2\frac{c}{\omega^3} 2\pi = -\frac{2}{\omega}\left(-\frac{c}{\omega^2} 2\pi\right) = -\frac{2}{\omega}\frac{\partial \lambda}{\partial \omega}$

Thus $\frac{\omega}{c}\frac{\partial^2 n}{\partial \omega^2} = \frac{\omega}{c}\frac{\partial^2 n}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial \omega}\right)^2 - \frac{2}{c}\frac{\partial \lambda}{\partial \omega}$

Thus $\frac{\partial^2 k}{\partial \omega^2} = \frac{\omega}{c}\frac{\partial^2 n}{\partial \lambda^2}\left(\frac{\partial \lambda}{\partial \omega}\right)^2 = \frac{\lambda^2}{\omega c}\frac{\partial^2 n}{\partial \lambda^2}$

Finally $\Delta T_p = Z\frac{\partial^2 k}{\partial \omega^2}\Delta\omega = Z\Delta\omega\frac{\lambda^2}{\omega c}\frac{\partial^2 n}{\partial \lambda^2}$

$$= -Z\Delta\lambda \left(\frac{\lambda}{c}\frac{\partial^2 n}{\partial \lambda^2}\right)$$

Important Dispersion ~~Parameter~~
(ps/nm/km)