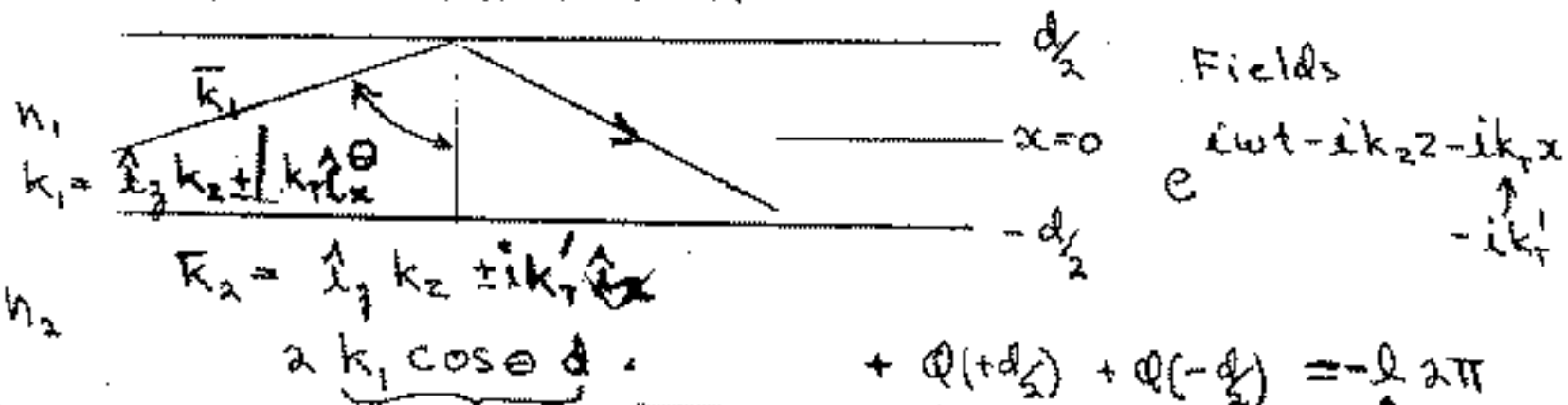


Basics of Modes In An Optical Fiber - a) Slab Waveguide ①

Consider a Slab First of All



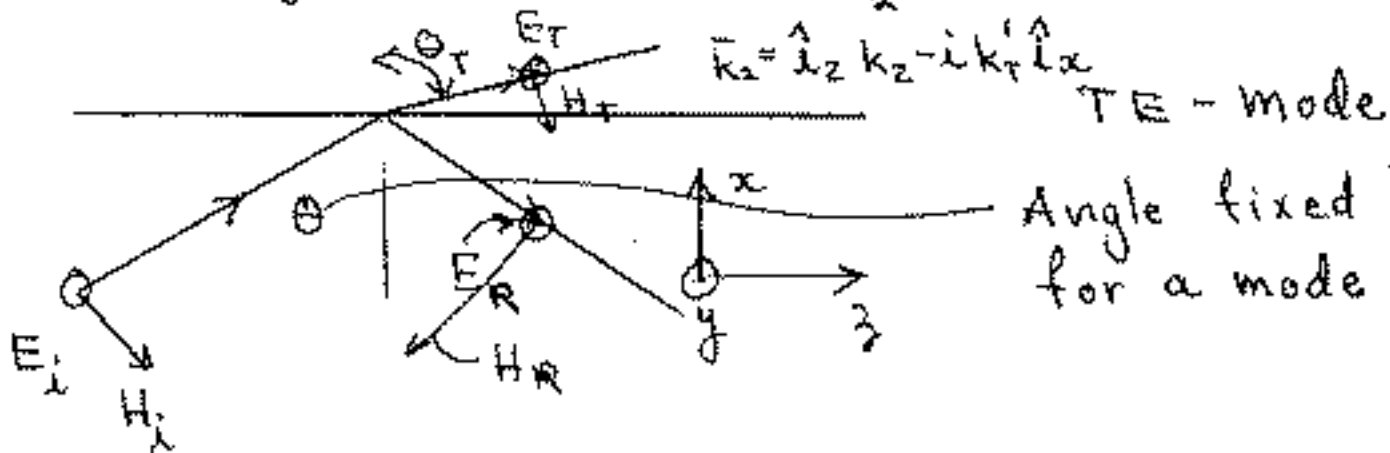
thus $k_{\perp} d \cos \theta = -l(\pi) + \tan^{-1}\left(\frac{k_T}{k_T}\right)$ phase changes upon reflection at the two boundaries $(= -2 \tan^{-1}\frac{k_T}{k_T})$

k_T is imaginary for $|x| > d$

k_x is imaginary for $|x| > d/2$

Calculation

of d



$$\textcircled{1} E_L + E_R = E_T$$

$$H_L \cos \theta - H_R \cos \theta = H_T \cos \theta_T$$

$$c \sin \theta_1 = \frac{c}{n_1} \sin \theta_1 = \frac{c}{n_2} \sin \theta_2 = \frac{c}{n_1} \cos \theta_1 = \frac{c}{n_2} \cos \theta_2 = \frac{c}{n_1} \sin \theta_1$$

$$\textcircled{2} (E_i k_T - E_R k_T) \text{ ~~also~~ } = E_T k'_T$$

$$E_L + E_R = E_T \quad (3)$$

$$E_i - E_H = -E_T \frac{ik'_T}{k_T} \quad (4)$$

$$E_i = \frac{1}{2} (K_T + i k_T') E_T$$

$$E = \frac{1}{2} (K_+ + K_-) E$$

$$\frac{E_R}{E_i} = \frac{(k_T + i k_T')}{(k_T - i k_T')} = e^{+2i \tan^{-1} \frac{k_T'}{k_T}}$$

Solution for k_z the propagation coefficient in the direction of propagation ②

$$k'_T = k_T \tan(k_T \frac{d}{2} + l \frac{\pi}{2}) \quad ①$$

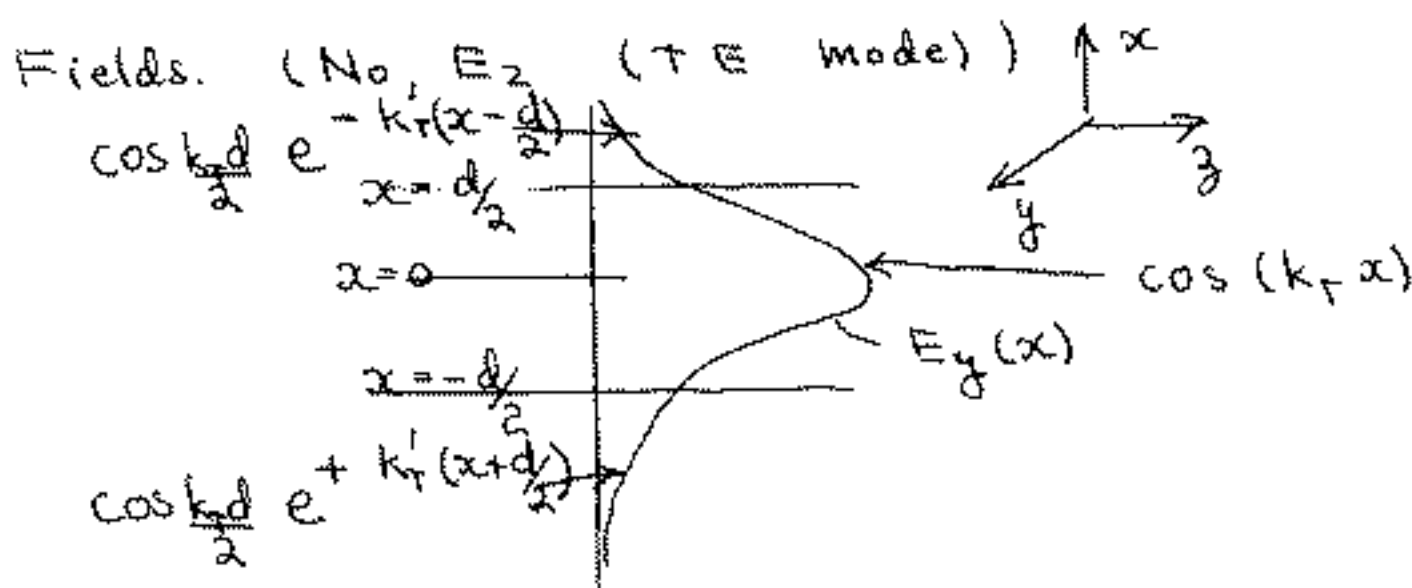
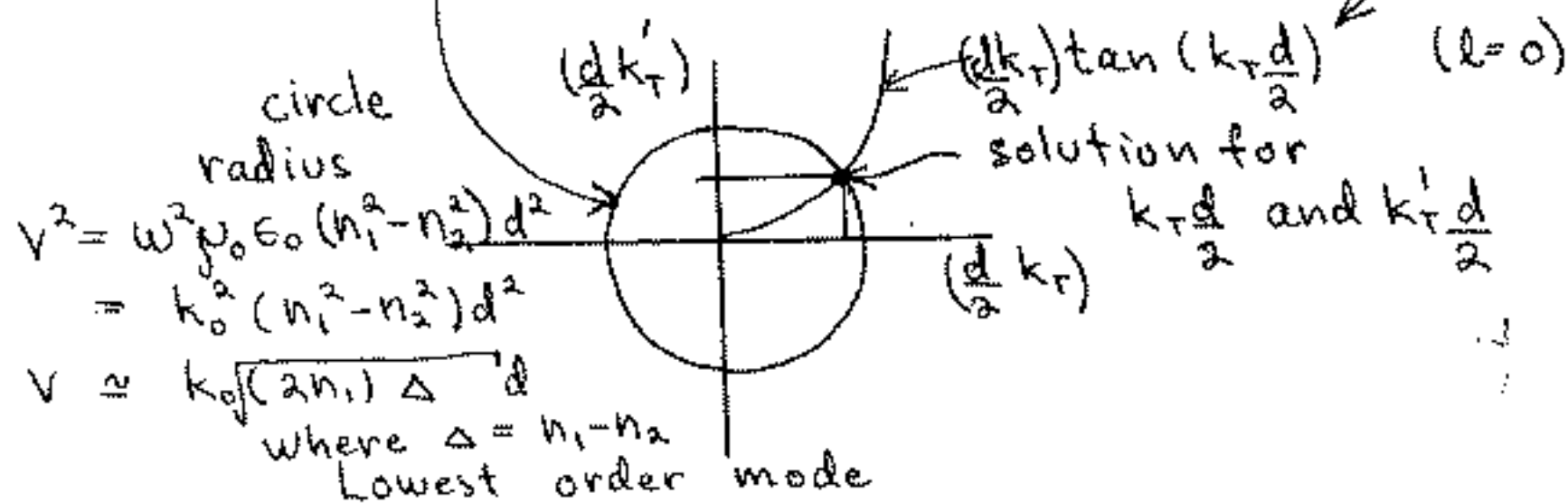
Also have

$$k_T^2 + k_z^2 = \omega^2 \mu_0 \epsilon_0 n_1^2 \quad ②$$

$$-(k'_T)^2 + k_z^2 = \omega^2 \mu_0 \epsilon_0 n_2^2 \quad ③$$

② - ③ yields

$$k_T^2 + (k'_T)^2 = \omega^2 \mu_0 \epsilon_0 (n_1^2 - n_2^2)$$



k_z - the mode propagation constant can now be determined from either Eq(2) or Eq(3) above

Fields - Maxwell's Equations

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} = - \mu_0 i \omega \vec{H}$$

$$H_x = \frac{1}{-\mu_0 i \omega} \frac{\partial E_y}{\partial z} = \frac{i k_z}{-\mu_0 i \omega} E_y$$

$$H_z = \frac{1}{-\mu_0 i \omega} \frac{\partial E_x}{\partial x}$$

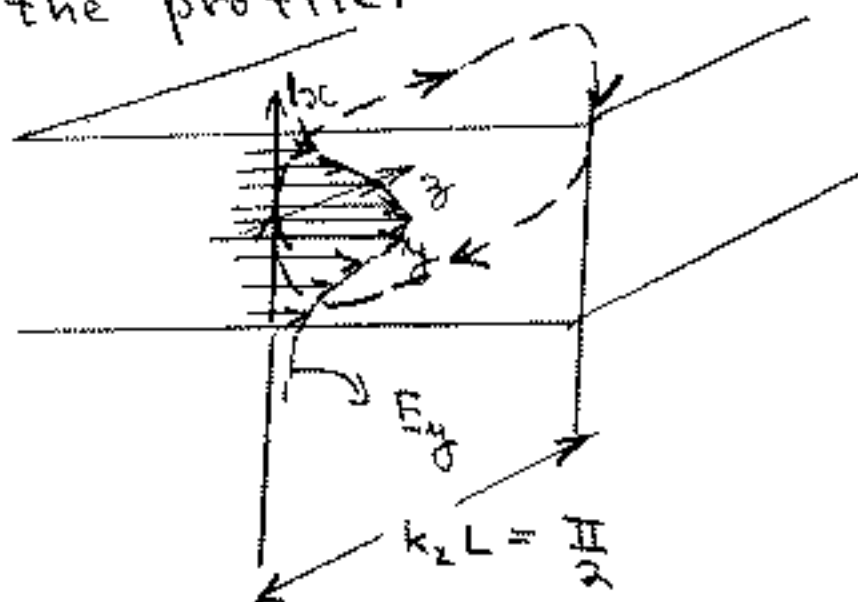
$$= \frac{k_T \sin k_T x}{\mu_0 i \omega} \quad x < \frac{d}{2}$$

$$= \frac{k_T'}{\mu_0 i \omega} e^{-k_T'(x - \frac{d}{2})} \cos \frac{k_T d}{2} \quad x > \frac{d}{2}$$

Note at $x = \frac{d}{2}$

$$k_T' = k_T \tan(k_T \frac{d}{2}) \quad \text{as required for continuity!}$$

H_x is also continuous in k_z is a constant across the profile.



$$\text{TM mode} \quad \phi = 2 \tan^{-1} \frac{\epsilon_1 k_T'}{\epsilon_2 k_T}$$

Boundary Conditions For ④ T.M. Waves

$$H_i = E_i \sqrt{\frac{\epsilon_0}{\mu_0}} n_1$$

$$H_i + H_R = H_T$$

$$E_i \cos \theta - E_R \cos \theta = E_T \cos \theta_T$$

$$E_i n_1 + E_R n_1 = E_T n_2$$

$$H_T = E_T \sqrt{\frac{\epsilon_0}{\mu_0}} n_2$$

$$E_i + E_R = E_T \frac{n_2}{n_1}$$

$$E_i - E_R = -E_T \frac{k_T}{k_i} \frac{n_1}{n_2}$$

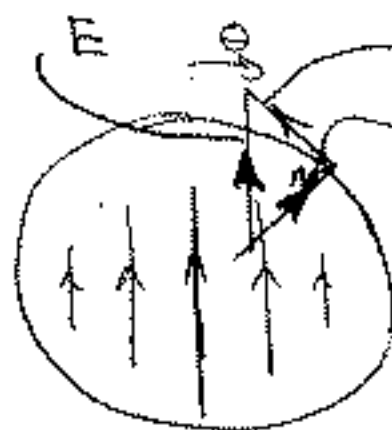
$$\therefore E_i = \frac{1}{2} \left(\frac{n_2}{n_1} - i \frac{n_1}{n_2} \frac{k_T}{k_i} \right)$$

$$E_R = \frac{1}{2} \left(\frac{n_2}{n_1} + i \frac{n_1}{n_2} \frac{k_T}{k_i} \right)$$

$$\therefore \frac{E_R}{E_i} = e^{2i \tan^{-1} \left(\frac{n_1^2}{n_2^2} \frac{k_T}{k_i} \right)}$$

The Modes cut-off in optical fibers. (TM & TE)

Lowest order mode is polarized



$$E_r = E \cos \theta$$

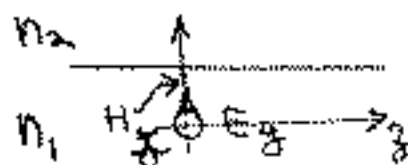
$$E_\theta = -E \sin \theta$$

Fields are of the form

oscillating $\rightarrow J_1(kr) e^{i\phi}$ inside

decaying $\rightarrow K_1(kr) e^{i\phi}$ outside

Reason T.E mode is cut-off



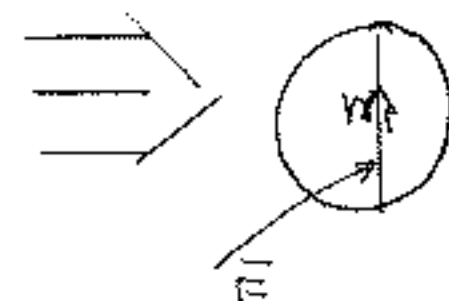
$$n_1 > n_2$$

Map?



TE mode needs inner core to not cut-off

HE₁₁ linearly polarized



Common Applications of Bessel Functions

Bessel equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y(x) = 0$
 integer

Let $y(x) = \frac{z(x)}{\sqrt{x}}$; this becomes

$$\frac{d^2 z}{dx^2} + \left(1 - \frac{n^2 - 1/4}{x^2}\right) z = 0$$

showing the likeness to sines and cosines.

Solutions: $y = J_n(x) = \frac{x^n}{2^n n!} \left[1 - \frac{x^2}{2^{2(n+1)}} + \dots\right]$

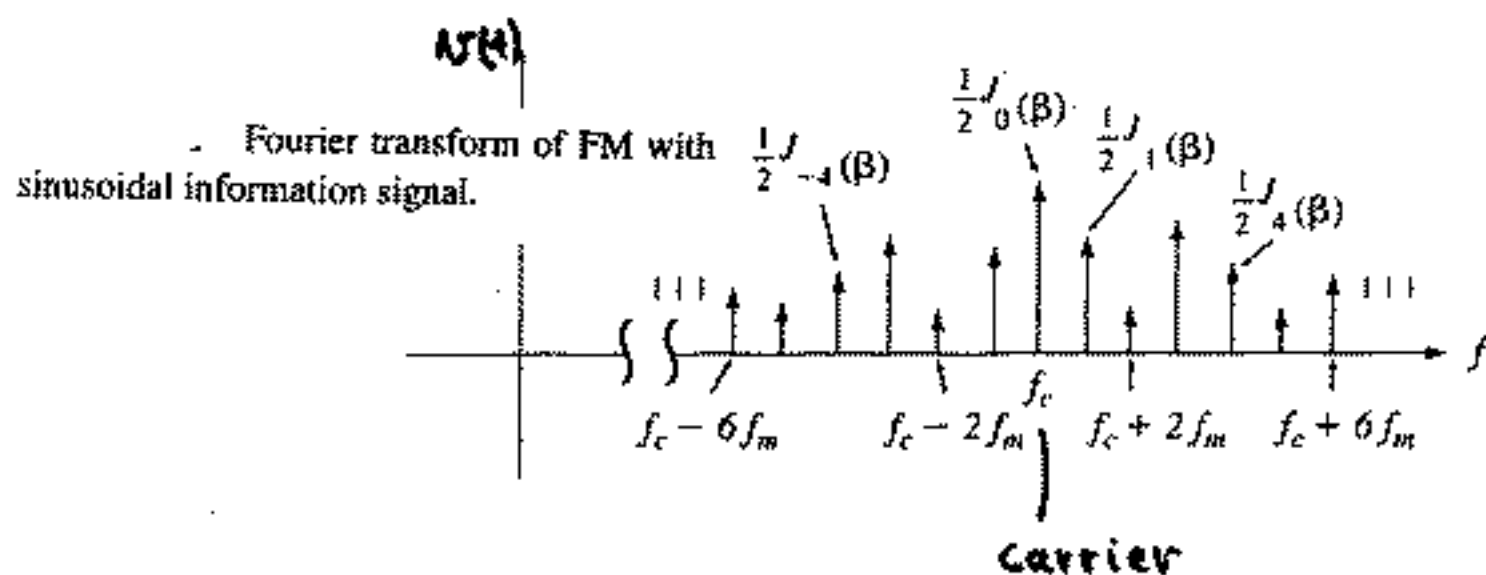
second independent solution is the Neuman function $\frac{2^n n!}{2^{2(n+1)}}$

1) F. M. modulation

Signal of the form

$$v(t) = A \cos(\omega_c t + \beta \sin \omega_m t)$$

$$= A \sum J_n(\beta) \cos[(\omega_c + n \omega_m) t]$$



2) Solutions of fields in cylindrical coordinates (i.e., the optical fiber) and more generally vibrations and waves in cylindrical coordinates

3) Diffraction from a uniformly illuminated circular aperture

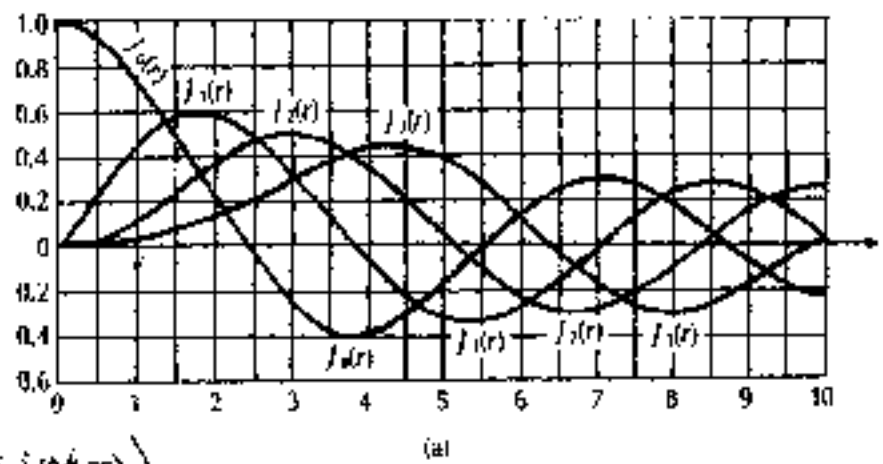
4) A transmission line on which one capacitor or inductor is initially excited and this excitation subsequently travels in both the + and - directions.

Bessel Functions + Cylindrical Functions

(5)

Bessel
Function
of
Real
Argument

(Analogous to $\sin(x)$)



Bessel
Function
of
Imaginary
Argument

(Analogous to e^{-x})

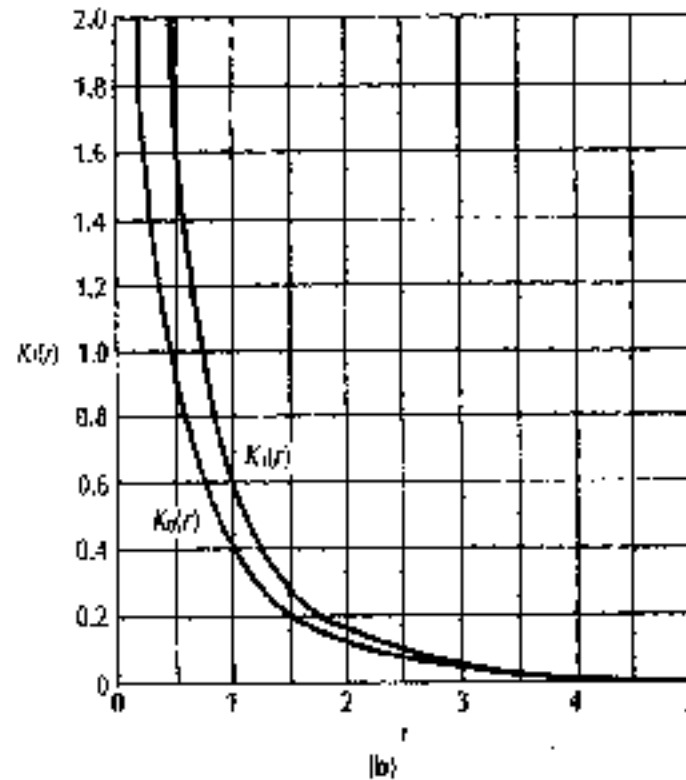


Table 4.2.1 Zeros of Bessel Functions

$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
2.405					
	3.832				
5.520		5.136			
	7.016		6.380		
8.634		8.417		7.580	
	10.173		9.761		8.772
11.792		11.620		11.065	
	13.324		13.015		12.338
14.931		14.796		14.372	
	16.471		16.223		15.700
18.071		17.960		17.616	
	19.616		19.409		18.980
21.212		21.117		20.827	
	22.760		22.583		22.218

2.8.1) The Dispersion Relationship

For a cylindrical guide with $n(r) = n_1$ for $r < a$ and n_2 for $r > a$, in general one cannot obtain the solutions in terms of only E_z or only H_z . Then both E_z and H_z must be taken into account when writing the solution and applying the boundary conditions. The solutions are of the form:

$$E_z(r_T) = A J_s(k_T r) e^{j(s\phi)} \quad (2.8.1.1a)$$

$$H_z(r_T) = B J_s(k_T r) e^{j(s\phi)} \quad (2.8.1.1b)$$

for $r < a$ and

$$E_z(r_T) = A' K_s(k'_T r) e^{j(s\phi)} \quad (2.8.1.2a)$$

$$H_z(r_T) = B' K_s(k'_T r) e^{j(s\phi)} \quad (2.8.1.2b)$$

for $r > a$. Thus there are four constants to solve. The appropriate equations come from the four boundary conditions on the continuity of H_z , E_z , E_ϕ and H_ϕ (Problem 1). The final determinantal dispersion relation is equal to:

$$\left[\frac{1}{k_{Ta}} \frac{J'_s(k_{Ta})}{J_s(k_{Ta})} + \frac{1}{k'_{Ta}} \frac{K'_s(k'_{Ta})}{K_s(k'_{Ta})} \right] \left[\frac{(k_0)^2 n_1^2}{k_{Ta}} \frac{J'_s(k_{Ta})}{J_s(k_{Ta})} + \frac{(k_0)^2 n_2^2}{k'_{Ta}} \frac{K'_s(k'_{Ta})}{K_s(k'_{Ta})} \right] \quad (2.8.1.3)$$

$$= s^2 \left[k_T^2 \left[\frac{1}{(k_{Ta})^2} + \frac{1}{(k'_{Ta})^2} \right]^2 \right]$$

When $s = 0$ the equation immediately factors, the first being the TE_{0p} and the second, the TM_{0p} mode dispersion relationships, respectively, where $0(s)$ stands for the order of the Bessel function and p indicates the successive zero of the Bessel function. If $s \neq 0$ although this factorization does not occur, there is an approximation which is useful for weakly guiding structures ($n_1 \approx n_2$) and which makes the expression still relatively easy to analyze. Then $k_2 \approx k_0 n_1 = k_0 n_2$ so that these factors cancel out of Eq. (2.8.1.3). Now we factor the above and use the appropriate recursion relationships for the Bessel functions. To do this let $X = \frac{J'_s}{k_{Ta} J_s}$ and $Y = \frac{K'_s}{k'_{Ta} K_s}$ where k_{Ta} and $(k')_{Ta}$ arguments for J_s and K_s respectively, are implied. Then the above can be written as:

$$\left[X+Y \right]^2 \dots s^2 \left[\frac{1}{(k_{Ta})^2} + \frac{1}{(k'_{Ta})^2} \right]^2 = 0 \quad (2.8.1.4)$$

Factoring;

$$\left[X+Y-s \left[\frac{1}{(k_{Ta})^2} + \frac{1}{(k'_{Ta})^2} \right] \right] \left[X+Y+s \left[\frac{1}{(k_{Ta})^2} + \frac{1}{(k'_{Ta})^2} \right] \right] = 0 \quad (2.8.1.5)$$

The positive factor gives the lowest order fiber mode (HE₁₁), in particular, and the HE_{sp} modes in general.

$$\left[X + Y + s \left[\frac{1}{(k_T a)^2} + \frac{1}{(k_T' a)^2} \right] \right] = 0 \quad (2.8.1.6)$$

Multiplying through by $J_s K_s$ gives:

$$\frac{1}{k_T a} J_s' K_s + \frac{1}{k_T' a} K_s' J_s + s J_s K_s \left[\frac{1}{(k_T a)^2} + \frac{1}{(k_T' a)^2} \right] = 0 \quad (2.8.1.7)$$

Using the recursion relationship for the Bessel functions

$$J_s' = [J_{s-1} - J_{s+1}]/2 = J_{s-1} - \frac{s}{k_T a} J_s \quad (2.8.1.8a)$$

$$K_s' = -[K_{s-1} + K_{s+1}]/2 = -K_{s-1} - \frac{s}{k_T' a} J_s \quad (2.8.1.8b)$$

Substituting for J_s' and K_s' and subsequently for $J_s = -J_{s-2} + 2 \frac{s-1}{k_T a} J_{s-1}$ and $K_s = K_{s-2} + 2 \frac{s-1}{k_T' a} K_{s-1}$ in the resulting $J_s K_{s-1}$ and $K_s J_{s-1}$ terms respectively, this can be written as:

$$\begin{aligned} \frac{1}{k_T a} \left[J_{s-1} \frac{K_{s-2}}{2} + \frac{s-1}{k_T} K_{s-1} J_{s-1} - \frac{s}{k_T a} K_s J_s \right] + \frac{1}{k_T' a} \left[K_{s-1} \frac{J_{s-2}}{2} - \frac{s-1}{k_T'} J_{s-1} K_{s-1} - \frac{s}{k_T' a} J_s K_s \right] \\ + s K_s J_s \left[\frac{1}{(k_T a)^2} + \frac{1}{(k_T' a)^2} \right] = 0 \end{aligned} \quad (2.8.1.9)$$

One observes a cancellation of both the $J_{s-1} K_{s-1}$ and the $J_s K_s$ terms resulting in:

$$k_T' a = -k_T a \frac{K_{s-1} J_{s-2}}{K_{s-2} J_{s-1}}$$

Note HE_{2p} ^{s=2} same as TE_{0p} ≈ TM_{0p} (when n₁ ≈ n₂)

These are the so-called HE_{sp} modes. In particular if $s = 1$ these are the HE_{1p} modes, in which case this can be written as:

$$k_T' a = k_T a \frac{K_0 J_1}{K_1 J_0} \quad (2.8.1.11)$$

where the relationship between the -1 and +1 order Bessel functions have been used. k_T and k_T' are determined by plotting Eq. (2.8.1.11) on the $k_T' a$, $k_T a$ plane and determining the intersection with the circle $k_T^2 + k_T'^2 = V^2 = \omega^2 \mu_0 \epsilon_0 (n_1^2 - n_2^2)$, in total analogy to the slab wave-guide solutions. The behavior is dominated by $\frac{J_1}{J_0}$. This is zero for $k_T a = 0$ and infinite at the first root of J_0 which is 2.405, as is shown in Fig.

2.8.1.1. Below this the fiber is a single mode guide with no cut-off. (Note that only for $s = 1$ does the step from Eq. (2.8.1.10) to Eq. (2.8.1.11) occur since for $s > 1$ the sub-indicies in 10 are all positive)

The HE_{sp} , s greater than 1 modes are given by the intersection of Eq. (2.8.1.10) as it stands with circle of radius V . Their behavior is seen to be dominated by $-\frac{J_{s-2}}{J_{s-1}}$. This goes aperiodically from $-\infty$ to $+\infty$ between the zeros of J_{s-1} with the crossovers on the k_{Ta} axis being at the zeros of J_{s-2} . The p 'th zero determines the cut-off of the HE_{sp} , $s \geq 1$ modes. This is illustrated in Fig. (2.8.1.2) for $s = 2$. Since the cutoffs of the higher order HE_{sp} , $p > 1$ modes are determined by the $p-1$ 'th zero of J_1 , H_{3p} is seen to be degenerate with $H_{1,p-1}$.

To obtain the EH modes, one takes the other factor in Eq. (2.8.1.5):

$$X+Y-s \left[\frac{1}{(k_{Ta})^2} + \frac{1}{(k'_{Ta})^2} \right] = 0 \quad (2.8.1.12)$$

$$J'_s = -J_{s+1} + \frac{s}{k_{Ta}} J_s \quad (2.8.1.13a)$$

$$K'_s = -K_{s+1} + \frac{s}{k'_{Ta}} K_s \quad (2.8.1.13b)$$

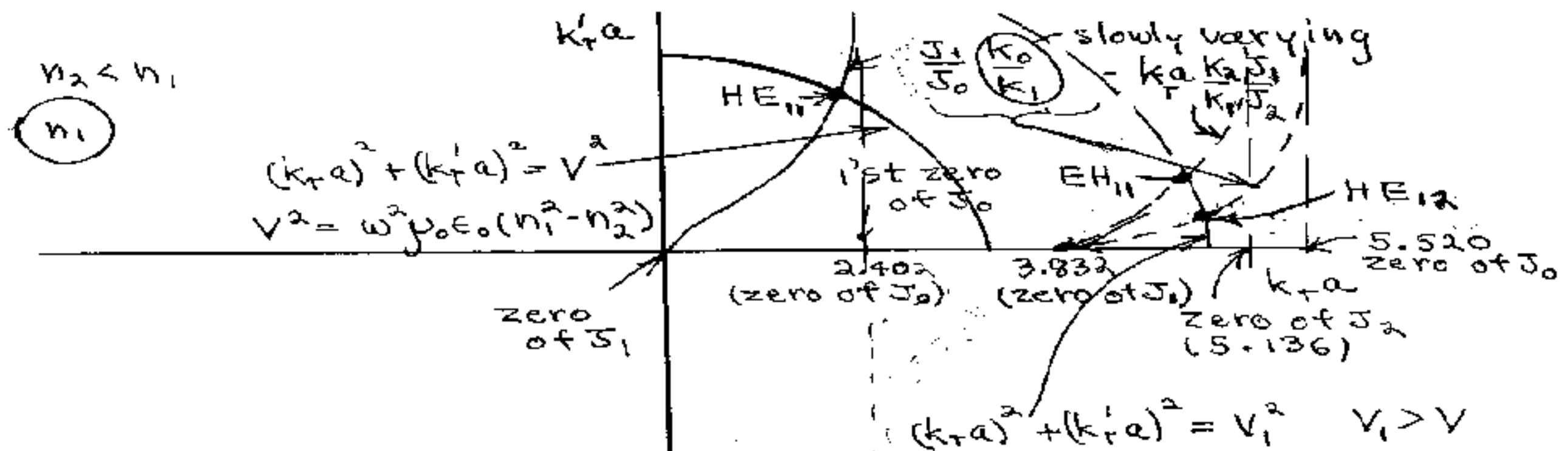
The above are obtained by using $J_{(s-1)} + J_{(s+1)} = \frac{2s}{k_{Ta}} J_s$ and $K_{(s-1)} + K_{(s+1)} = -\frac{2s}{k'_{Ta}} K_s$ in the derivative formulas (Eq. (2.8.1.8a) and Eq. (2.8.1.8b)) to eliminate $J_{(s+1)}$ and $K_{(s+1)}$ Eq. (2.8.1.13a) and Eq. (2.8.1.13b) are used to cancel the s terms in Eq. (2.8.1.12), leaving;

$$k'_{Ta} = -k_{Ta} \frac{K_{s+1}}{K_s} \frac{J_s}{J_{s+1}}$$

Note same as (2.8.1.0) when s replaced by $s-2$ (2.8.1.14)

Thus EH_{sp} same as $HE_{s-2,p}$ for $s \geq 2$

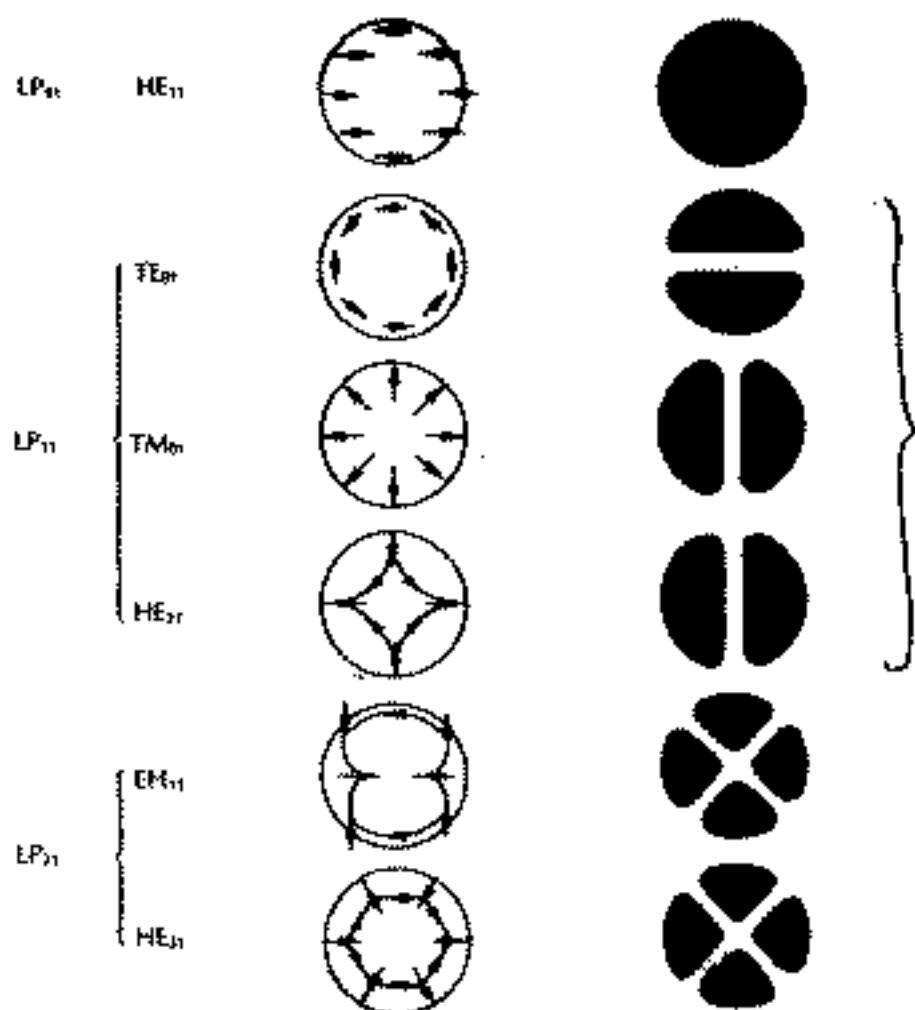
The intersection of this relationship with the circle of radius V in the k_{Ta} , k'_{Ta} plane gives the solutions for the EH modes of the optical fiber. In this approximation (weakly guiding), these are seen to be degenerate with $HE_{s+2,p}$.



Plot of $k'_T a - k_T a$ and its use in graphically determining the HE_{11} , the HE_{12} , and the EH_{11} Modes

Weakly Guided Modes in Fibers.

(6)



LP_{11} are linear combinations of TE_{01} , TM_{01} , and HE_{21} which are polarized (linearly)

For TM modes this factor is 0 (and $s=0$)

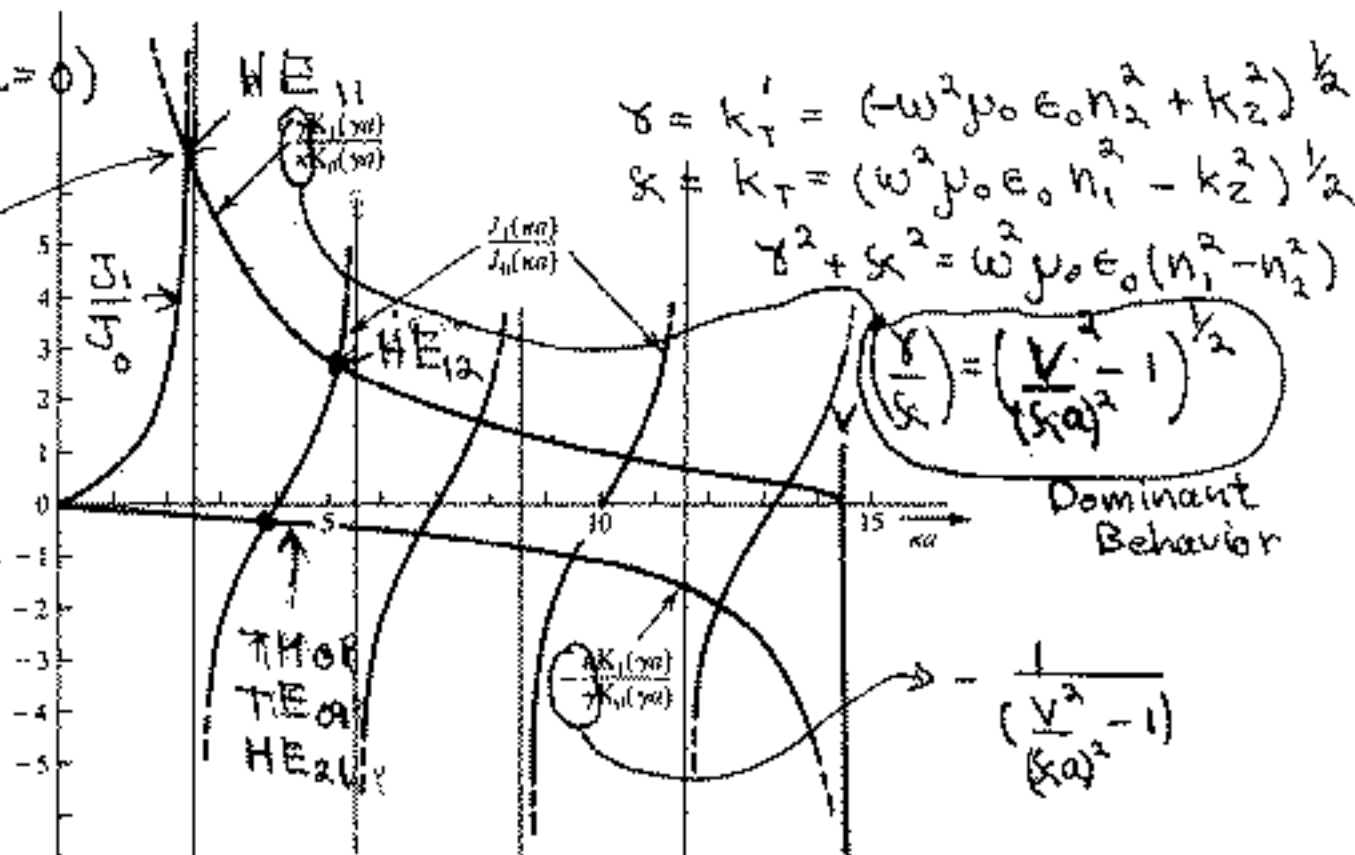
$$\left[\frac{1}{k_{TA}} \frac{J'_s(k_{TA})}{J_s(k_{TA})} + \frac{1}{k_{TA}} \frac{K'_s(k_{TA})}{K_s(k_{TA})} \right] \left[\frac{(k_0)^2 n_1^2 J'_s(k_{TA})}{k_{TA} J_s(k_{TA})} + \frac{(k_0)^2 n_2^2 K'_s(k_{TA})}{k_{TA} K_s(k_{TA})} \right] = s^2 \left[k_z^2 \left[\frac{1}{(k_{TA})^2} + \frac{1}{(k_{TA})^2} \right]^2 \right] \quad (s = l \text{ previously})$$

For TE modes This Factor is 0 (and $s=l=0$)

Only Mode That Does not cut-off

$$k_{TA} = k_{TA} \frac{K_0 J_1}{K_1 J_0}$$

or $\frac{\gamma}{K_0} \frac{K_1}{J_0} = \frac{J_1}{J_0}$



Alternative Way To View Solutions