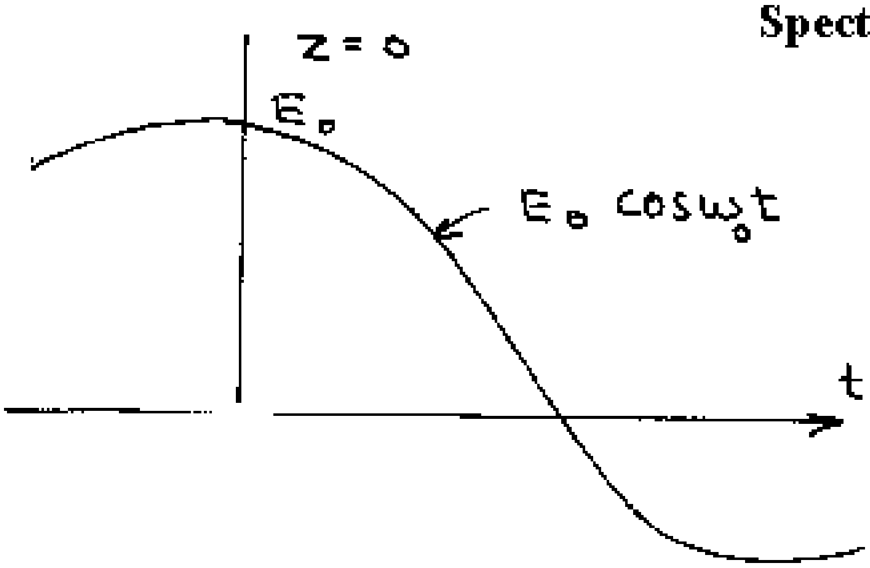
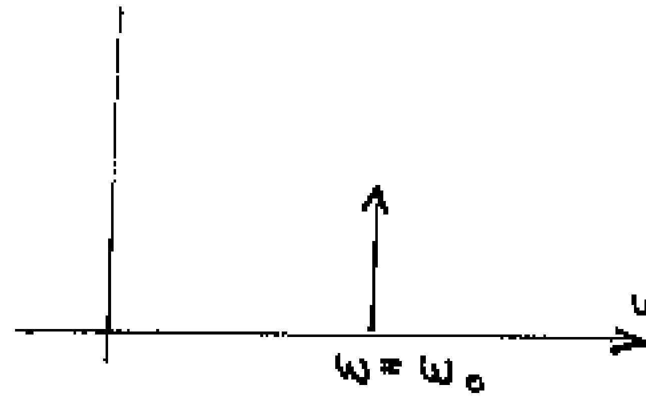
Spectral Content



Spectrum $\longrightarrow$

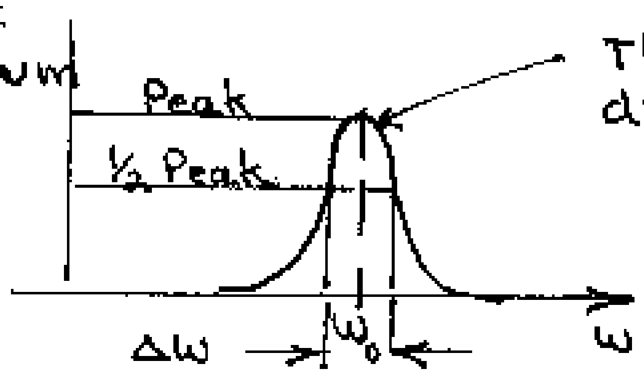


a) Ideal Continuous Cosine at $z = 0$

b) Laser – Non-ideal and Thus Has

1) ω_0 varying in time (stability) (drift).
2) rapid variation due to "phase fluctuations" and amplitude fluctuations.
3) When modulated with a signal the electric field spectrum is further broadened.

Resultant Spectrum



The whole pattern can drift in time (stability)

$\Delta\omega = 2\pi \, \Delta f$ (Spectral Width) or (Linewidth)

Example question!

If $\Delta f = 100$ MHz at $\lambda_0 = 1\ \mu m$, what is $\Delta\lambda$ in nanometers.

Ans. $\frac{1}{3000}$ nm

Sol$\underline{n}$ $\lambda f = c \Rightarrow \Delta\lambda f + \lambda \Delta f = 0 \Rightarrow \Delta\lambda = -\frac{\lambda}{f} \Delta f$

Thus $\Delta\lambda = -(\lambda^2/c)\Delta f = -(1\times10^{-9})^2 \times \frac{1}{3\times10^{10}} \times (10^8) \times \frac{1}{100} \times 10^9$ nm.

Representation of a time varying signal by a sum of sinusoidal signals, leading to a "broadened spectrum"

Simplest case: pure tone at $z=0$

$$f(t) = A\cos(\omega t+\varphi) = \frac{Ae^{i\varphi}}{2}e^{i\omega t} + \frac{Ae^{-i\varphi}}{2}e^{-i\omega t}$$

Amplitude) phase — All real; complex amplitude

$z \neq 0$ (single dimension)

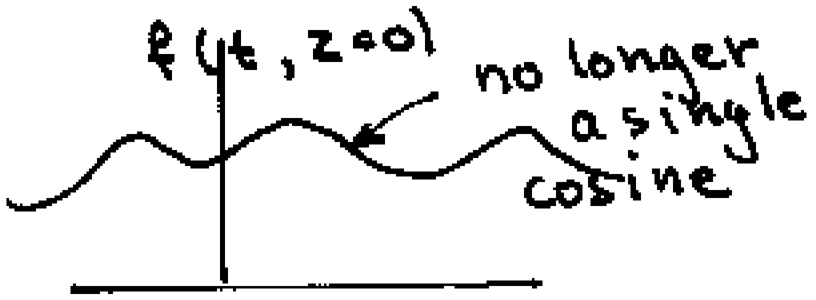
$$f(t,z) = A\cos(\omega t - kz + \varphi) = \frac{Ae^{i\varphi}}{2}e^{i\omega t - kz} + \underbrace{c.c.}_{\text{complex conjugate}}$$

$$\omega/k = f\lambda = c$$

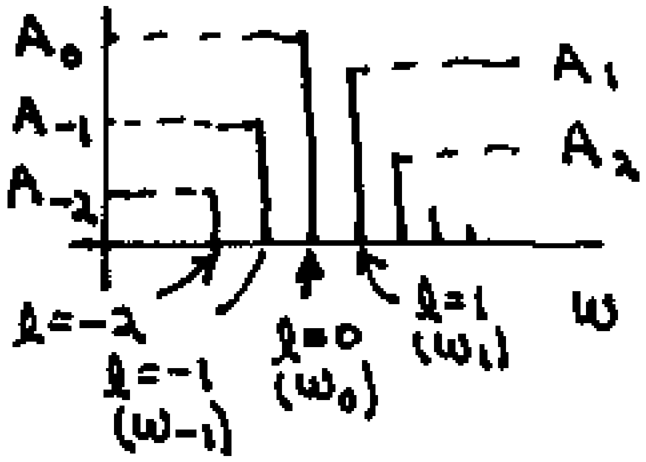
This is a single frequency wave. The exponential form is used because of its convenience and ease in analysis

For more complex signals, simply add other frequency components to the wave. Then $f(t,z)$ becomes

$$f(t,z) = \underset{\text{over index}}{\text{Sum}}\ A_k\cos(\omega_\ell t - k_\ell z + \varphi_\ell) = \sum_\ell A_\ell e^{i(\omega_\ell t - k_\ell z + \varphi_\ell)} + c.c.$$



"Spectrum" $\longrightarrow$ simply the A_ℓ plotted versus ω

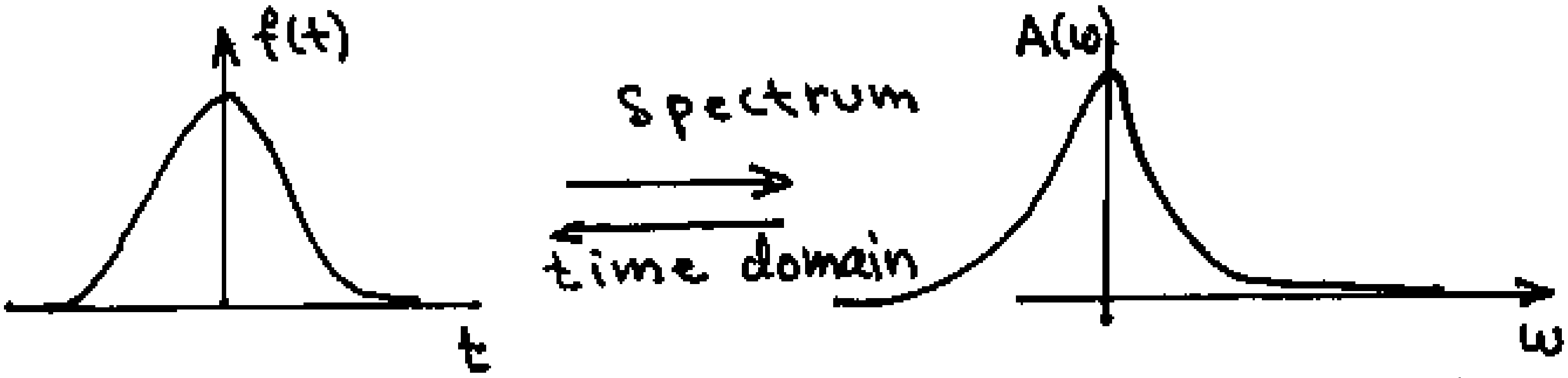


One can also write this as

$$f(t,z) = \int A(\omega)\, e^{i(\omega t - kz + \varphi_\ell)} \frac{d\omega}{2\pi} + c.c.$$

where $\frac{A(\omega)}{2\pi} = \sum_\ell A_\ell\, \delta(\omega - \omega_\ell)$ since the integral just eliminates the delta "distributions" leaving the sum. We often plot $A(\omega)$ indicating the delta "distributions" with arrows

The form $f(t) = \int A(\omega)\, e^{i(\omega t - kz + \varphi)} \frac{d\omega}{2\pi} + c.c.$ is then true also when $A(\omega)$ is not a sequence (sum) of δ-distributions but is a "continuous" amplitude (Note: k and φ are functions of ω)



To calculate $A(\omega_0)$ simply multiply $f(t)$ by $e^{-i\omega_0 t}$ (where ω_0 is the frequency one wishes to find $A(\omega)$) and integrate over all time. Thus

$$\int_{-\infty}^{+\infty} f(t)\, e^{-i\omega_0 t}\, dt = \int_{-\infty}^{+\infty} e^{-i\omega_0 t} \int_{-\infty}^{+\infty} d\omega\, A(\omega)\, e^{i(\omega t - kz + \varphi)} \frac{dt}{(2\pi)}$$

Note: t only occurs in the exponent so it can be integrated. Thus one only needs $\int_{-\infty}^{+\infty} e^{-i\omega_0 (t)}\, e^{i\omega t}\, dt = \int_{-\infty}^{+\infty} e^{i(\omega - \omega_0)t}\, dt$

To "get" this to converge write it as

$$\int_{-\infty}^{+\infty} e^{i(\omega-\omega_0)t}\, dt = \int_{-\infty}^{0} e^{i(\omega-\omega_0)t}\, dt + \int_{0}^{\infty} e^{i(\omega-\omega_0)t}\, dt$$

Add small $\pm\epsilon t$ terms so that the $-\infty$ and $+\infty$ limits give 0. For example

$$\int_{-\infty}^{0} e^{i(\omega-\omega_0)t + \epsilon t}\, dt = \frac{1}{i(\omega-\omega_0)+\epsilon}$$; the $-\infty$ limit $= 0$ since it has the factor $e^{\epsilon t} \to 0$ as $t \to -\infty$

For the other term use $-\epsilon t$ in the exponent. Now sum the two terms: and take the limit

$$\lim_{\epsilon \to 0} \left(\frac{1}{i(\omega-\omega_0)+\epsilon} - \frac{1}{i(\omega-\omega_0)-\epsilon} \right) = \lim_{\epsilon\to 0} \left(\frac{-2\epsilon}{-\epsilon^2 - (\omega-\omega_0)^2} \right)$$

Next.

③

Note: $\dfrac{2\epsilon}{\epsilon^2+(\omega-\omega_0)^2}$

① - if $(\omega-\omega_0)\neq 0$ this $\to 0$ as $\epsilon\to 0$

② - if $(\omega-\omega_0)=0$ this $\to \dfrac{2}{\epsilon}\to\infty$ as $\epsilon\to 0$

③ $$\int_{-\infty}^{+\infty}\frac{2\epsilon}{\epsilon^2+(\omega-\omega_0)^2}\,d\omega = \int_{-\infty}^{+\infty}\frac{2\epsilon^2}{\epsilon^2+(\omega-\omega_0)^2}\,d\frac{(\omega-\omega_0)}{\epsilon}$$

constant

$$= \int_{-\infty}^{+\infty}\frac{2}{1+\left(\frac{\omega-\omega_0}{\epsilon}\right)^2}\,d\left(\frac{\omega-\omega_0}{\epsilon}\right) = \int_{-\infty}^{+\infty}\frac{2}{1+x^2}\,dx \;:\; x=\frac{\omega-\omega_0}{\epsilon}$$

$$= 2\tan^{-1}\Big|_{-\infty}^{\infty} = 2\left(\pi/2-(-\pi/2)\right) = 2\pi$$

Thus we have shown that $\displaystyle\int_{-\infty}^{+\infty} dt\, e^{i(\omega-\omega_0)t} = 2\pi\,\delta(\omega-\omega_0)$

Thus from the previous page (the 2π cancels)

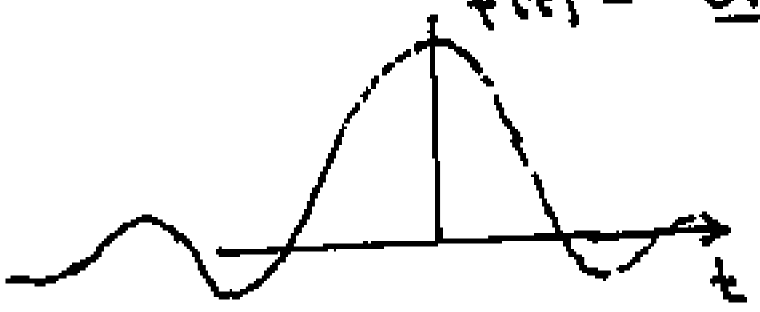
$$\int_{-\infty}^{+\infty} f(t)\,e^{-i\omega_0 t}\,dt = \int d\omega\; A(\omega)\,e^{-ikz+i\varphi}\,\delta(\omega-\omega_0)$$

$$= A(\omega)\,e^{-ikz+i\varphi}$$

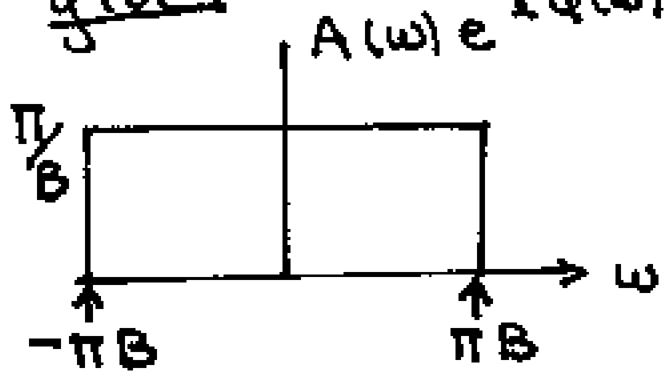
Thus one can obtain $A(\omega)e^{i\varphi}$ by knowing $f(t)$ (and e^{-ikz} of course).

Example: Filter Response

$f(t) = \dfrac{\sin\pi Bt}{Bt} \;\rightarrow\; A(\omega)e^{i\varphi(\omega)} = \displaystyle\int_{-\infty}^{+\infty} e^{-i\omega t}\,\frac{\sin\pi Bt}{t}\,dt$



gives



Check

$$f(t) = \int_{-\pi B}^{+\pi B}\frac{d\omega}{2\pi}\,e^{i\omega t}\underbrace{\left(\frac{2\pi}{2B}\right)}_{A(\omega)e^{i\varphi(\omega)}} = \frac{e^{i\pi Bt}-e^{-i\pi Bt}}{2iBt} = \frac{\sin\pi Bt}{Bt}$$

The "Uncertainty Relation" (Relationship between a function and its Fourier Spectrum)

relates to "conjugate variables" (ω, t), (p_x, x) (k_y, y), (E_x, H_x) etc.

Consider the function $\omega - x t$

where ω and t can be complex and x is complex

$$(\omega - xt)(\omega^* - x^* t^*) \geq 0$$

Multiplying and averaging

$$\overline{|\omega|^2} - \overline{\omega t^*} x^* - \overline{\omega^* t} x + |x|^2 \overline{|t|^2} \geq 0$$

Pick $x = \dfrac{\overline{\omega t^*}}{\overline{|t|^2}}$ (Note: x is a constant and thus not averaged). Substituting for x gives

$$\overline{|\omega|^2} - 2\frac{\overline{\omega t^*}\,\overline{\omega^* t}}{\overline{|t^2|}} + \frac{\overline{\omega t^*}\,\overline{\omega^* t}}{\overline{|t^2|}} \geq 0 \Rightarrow \boxed{\overline{|\omega|^2}\,\overline{|t|^2} \geq \overline{\omega t^*}\,\overline{\omega^* t}}$$

Example: ω and t. Let the signal be of the form $f(t)$ (real) (normalized to 1). Let $\bar{t} = \bar{\omega} = 0$ (for simplicity)

and $F(\omega)$ = Fourier Transform of $f(t)$. Then

$$\overline{\omega t} = \int_{-\infty}^{+\infty} t\, \underset{(?)^2}{\omega(t)}\, f(t)\, dt \overset{?}{=} \int_{-\infty}^{+\infty} \omega\, t(\omega)\, F(\omega)\, d\omega$$

ω and t taken real

$$= \int_{-\infty}^{+\infty} t\, \omega(t) \int_{-\infty}^{+\infty} \underset{\text{Amplitude}}{F(\omega)}\, e^{i\omega t}\, d\omega\, dt$$

frequency

(to interpret $\omega(t)$ properly express it as a derivative with respect to (it))

$$= \int_{-\infty}^{+\infty} t\, \frac{\partial}{\partial it} \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega\, dt$$

$$= \underbrace{\int_{-\infty}^{+\infty} \frac{\partial}{\partial it}\left(t \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega\right) dt}_{=0 \text{ at } t=-\infty \text{ and } t=+\infty} - \frac{1}{i} \underbrace{\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega\right) dt}_{\int_{-\infty}^{+\infty} f(t)\, dt = 1}$$

Thus $\overline{\omega t} = i$. Thus $\overline{\omega^2}\,\overline{t^2} \geq 1$.

Example: Gaussian $A e^{-t^2/2T^2}$ has the Fourier transform $e^{-\omega^2 T^2/2}$. These are Gaussian distributions with $\bar{t} = \bar{\omega} = 0$ and $\overline{t^2} = T^2$, $\overline{\omega^2} = \frac{1}{T^2}$ so

$$\overline{\omega^2}\,\overline{t^2} = 1$$