Example Binomial Distribution For

$m = 6 \quad p = \frac{1}{2}$ (Random walk $\longleftrightarrow$ Phase Fluctuation in a Laser)

$$P_m(n) = \binom{m}{n} p^n (1-p)^{n'}$$

As discussed in class n = number of steps to the right

n' = number of steps to the left.

Let $N = n - n'$ be the <u>net</u> no of steps to the right after m steps.

Also have $m = n + n'$

Thus eliminating n and n' above we obtain

$$P_m(N) = \binom{m}{\frac{N+m}{2}} (p)^{\frac{N+m}{2}} (1-p)^{\frac{m-N}{2}}$$

Numbers for

n	n'	$n-n' = N$
6	0	6
5	1	4
4	2	2
3	3	0
2	4	-2
1	5	-4
0	6	-6

$P_m(N)$

$P_6(0) = \dfrac{6!}{(3!)^2}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$

$= \dfrac{20}{64}$

$P_6(2) = \dfrac{6!}{(4!)(2!)}\left(\frac{1}{2}\right)^6$

$= \dfrac{15}{64}$

$P_6(4) = \dfrac{6}{64}$

$P_6(6) = \dfrac{1}{64}$

$P_m(n)$

$P_6(0) = \left(\frac{1}{2}\right)^6 = \dfrac{1}{64}$

$P_6(1) = \dfrac{6}{64}$

$P_6(2) = \dfrac{15}{64}$

$P_6(3) = \dfrac{20}{64}$

$P_6(4) = \dfrac{15}{64}$

$P_6(5) = \dfrac{6}{64}$

$P_6(6) = \dfrac{1}{64}$

Notes

$2\left(\sum_{2,4,6} P_m(N)\right) + P_m(0) = 1$

$P_m(-N) = P_m(N)$

$\sum_{n=0}^{6} P_m(n) = 1$

Review Questions!

a) What is $\bar{N} = \sum P(N)\, N$?

b) What is $\bar{n}$?

c) What are $\overline{(n-\bar{n})^2}$ and $\overline{(N-\bar{N})^2}$?

Proof that Bernouilli Distribution becomes a Gaussian Distribution as the number of trials "steps" goes to infinity -- An example of the Central Limit Theorem

$p(n) = \binom{m}{n}(p)^n (1-p)^{m-n}$ → Bernouilli Dist. for n steps to right

$$= \frac{m!}{n!\,(m-n)!}\, p^n (1-p)^{m-n}$$

$$= \frac{1}{n!}\frac{m \cdots (m-n+1)}{m^n}\,(pm)^n\left(1-\frac{pm}{m}\right)^{m-n}$$

$$= \frac{(\bar{n})^n e^{-\bar{n}}}{n!}$$ * note $\bar{n} = pm$

Let $n - n' = N$ → Net No. of steps to right

$n + n' = m$ → No of steps to left ($= m-n$)

$n = \frac{N+m}{2}$

$n' = \frac{m-N}{2}$

Thus

$$p(N) = \binom{m}{\frac{N+m}{2}}(p)^{\frac{N+m}{2}}(1-p)^{\frac{m-N}{2}}$$

↑ prob Dist^n for N

$\ln n! = (n+\frac{1}{2})\ln n - n + \frac{1}{2}\ln 2\pi$ ← Stirlings Formula

$$P(N) = \frac{m!}{\left(\frac{m-N}{2}\right)!} \times \frac{1}{\left(\frac{m+N}{2}\right)!}\left(\frac{1}{2}\right)^m \quad \text{if } p = \tfrac{1}{2}$$

$$\ln P(N) = \ln m! - \ln\left(\frac{m-N}{2}\right)! - \ln\left(\frac{m+N}{2}\right)! - m\ln 2$$

use stirlings formula

Example of Central Limit Theorem

$$\ln P(N) = \left(m+\tfrac{1}{2}\right)\ln m - \cancel{m} + \tfrac{1}{2}\ln(2\pi)$$
$$-\left(\frac{m-N}{2}+\tfrac{1}{2}\right)\ln\left(\frac{m-N}{2}\right) + \frac{m-N}{2} - \frac{1}{2}\ln(2\pi)$$
$$-\left(\frac{m+N}{2}+\tfrac{1}{2}\right)\ln\left(\frac{m+N}{2}\right) + \frac{m+N}{2} - \frac{1}{2}\ln(2\pi)$$
$$- m\ln 2$$

$\ln m - \ln 2$

$$\ln\left(\frac{m-N}{2}\right) = \ln\frac{m}{2}\left(1-\frac{N}{m}\right) \simeq \ln\frac{m}{2} + \left(-\frac{N}{m} - \frac{1}{2}\left(\frac{N}{m}\right)^2\right)$$

$$\ln\left(\frac{m+N}{2}\right) \quad \ln\frac{m}{2}\left(1+\frac{N}{m}\right) \simeq \ln\frac{m}{2} + \left(\frac{N}{m} - \frac{1}{2}\left(\frac{N}{m}\right)^2\right)$$

$\ln m - \ln 2$

Thus

$-\frac{1}{2}\ln m$

$$\ln P(N) = +\tfrac{1}{\cancel{2}}\ln m + m\cancel{\ln} 2 - \cancel{\ln m} + \ln 2 - \cancel{m\ln 2}$$
$$- \frac{N^2}{m} + \frac{1}{2}\frac{N^2}{m} + \cancel{\frac{1}{2}\left(\frac{N}{m}\right)^2} \quad \text{higher order}$$
$$-\frac{1}{2}\ln(2\pi)$$

Taking Antilog

$$P(N) = \frac{2}{\sqrt{2\pi m}}\, e^{-\frac{1}{2}\frac{N^2}{m}} \qquad N > 0$$

(since N was net No. of steps right)

(If N can be negative as well (steps to left; divide by 2)

To Relate this to diffusion let $m = qt$ where q = no of steps/unit time ; $x = N\ell$ where ℓ = distance between steps. Let $D = \frac{q\ell^2}{2}$ the Diffusion coefficient

$$P(N) \to \frac{dP}{dx}dx = \frac{1}{\sqrt{\pi D t}}\, e^{-\frac{x^2}{4Dt}}\, \ell$$

$dx = (N+1)\ell - N\ell = \ell$

This satisfies the diffusion equation $D\frac{d^2 n}{dx^2} = +\frac{dn}{dt}$