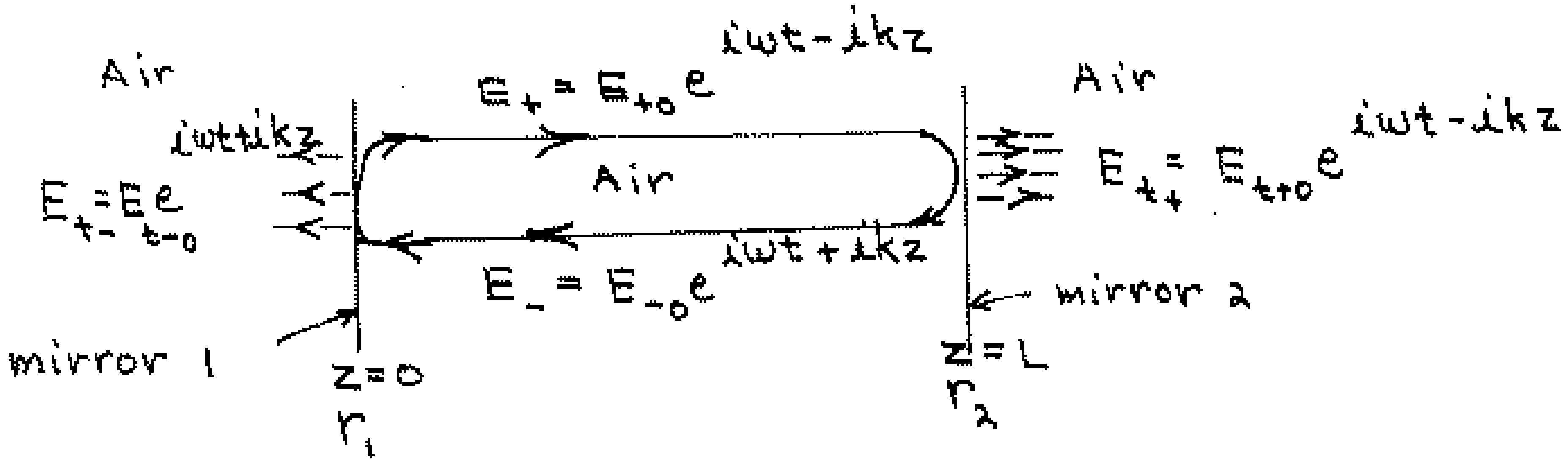
Mode Conditions for a Fabry-Perot Cavity
(Ideal One-Dimensional)



$$r_1 = \left.\frac{E_+}{E_-}\right|_{z=0} \qquad r_2 = \left.\frac{E_-}{E_+}\right|_{z=L}$$

Also

$$t_1 = \left.\frac{E_{t-}}{E_-}\right|_{z=0} \qquad t_2 = \left.\frac{E_{t+}}{E_+}\right|_{z=L}$$

Conservation of energy necessitates $r_2^2 + t_2^2 = 1$ (and $r_1^2 + t_1^2 = 1$)

Consider

$$r_1 r_2 = \left.\frac{E_+}{E_-}\right|_{z=0} \left(\left.\frac{E_-}{E_+}\right|_{z=L} \right) = \frac{\cancel{E_{+0}}}{\cancel{E_{-0}}} \frac{\cancel{E_{-0}}\, e^{ikL}}{\cancel{E_{+0}}\, e^{-ikL}}$$

$$= e^{2ikL}$$

Consequently

$$r_1 r_2 = e^{2ikL}$$

In general this is complex $k = \mathrm{Re}(k) + i\,\mathrm{Im}(k)$

Assuming r_1 and r_2 real; $2\,\mathrm{Re}(k)L$ must $= m(2\pi)$

Real Part of

since

$$e^{im(2\pi)} = +1 \qquad \text{① Longitudinal Mode Condition}$$

And

$$r_1 r_2 = e^{-2\,\mathrm{Im}(k)L} \qquad \text{② } \mathrm{Im}(k) = \text{gain}$$

gain = mirror loss

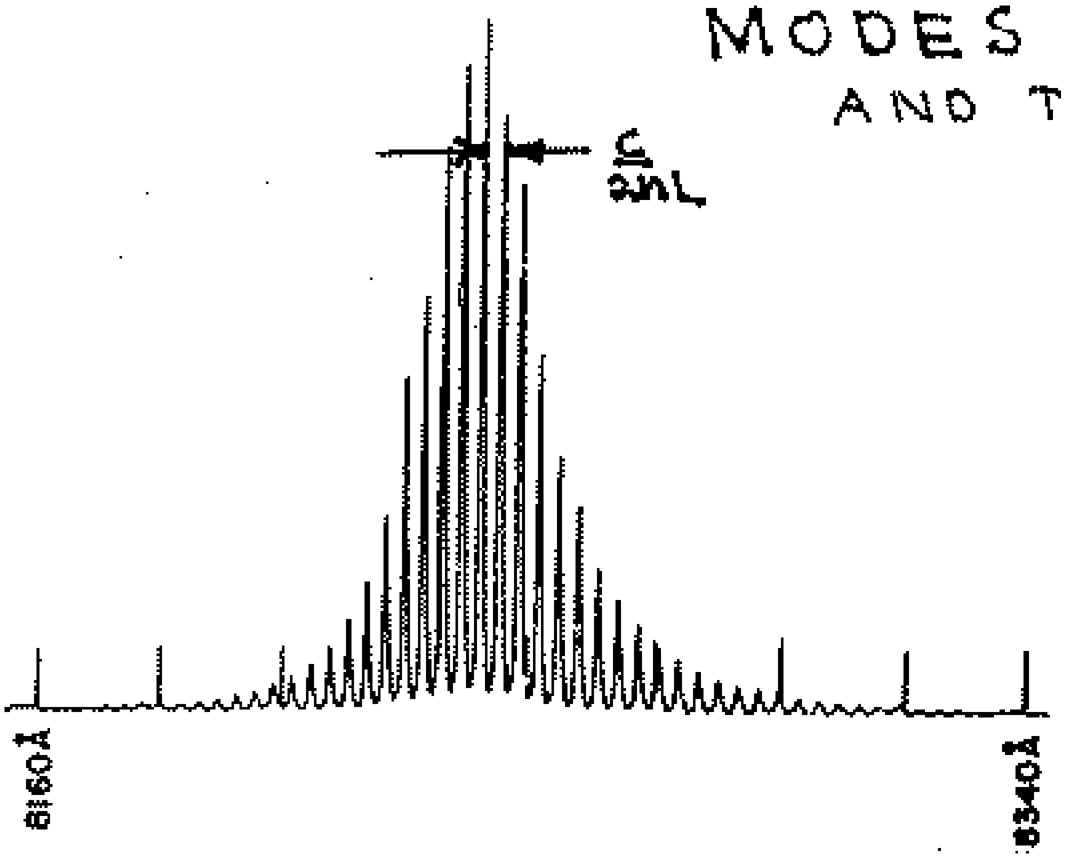
Thus $(\mathrm{Re}\,k)L = m\pi$; and Since $\mathrm{Re}(k) = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$

So $\frac{2\cancel{\pi} f_m L}{c} = m\cancel{\pi}$, or $f_m = \left(\frac{m}{2}\frac{c}{L}\right)$ or $L = \frac{m}{2}\left(\frac{c}{f_m}\right)$

For such a resonance situation $L = m \times \frac{\lambda_m}{2}$ and

$$f_{(m+1)} - f_m = \frac{m+1}{2}\frac{c}{L} - \frac{m}{2}\frac{c}{L} = \frac{c}{2L} \longleftarrow \text{mode Spacing}$$

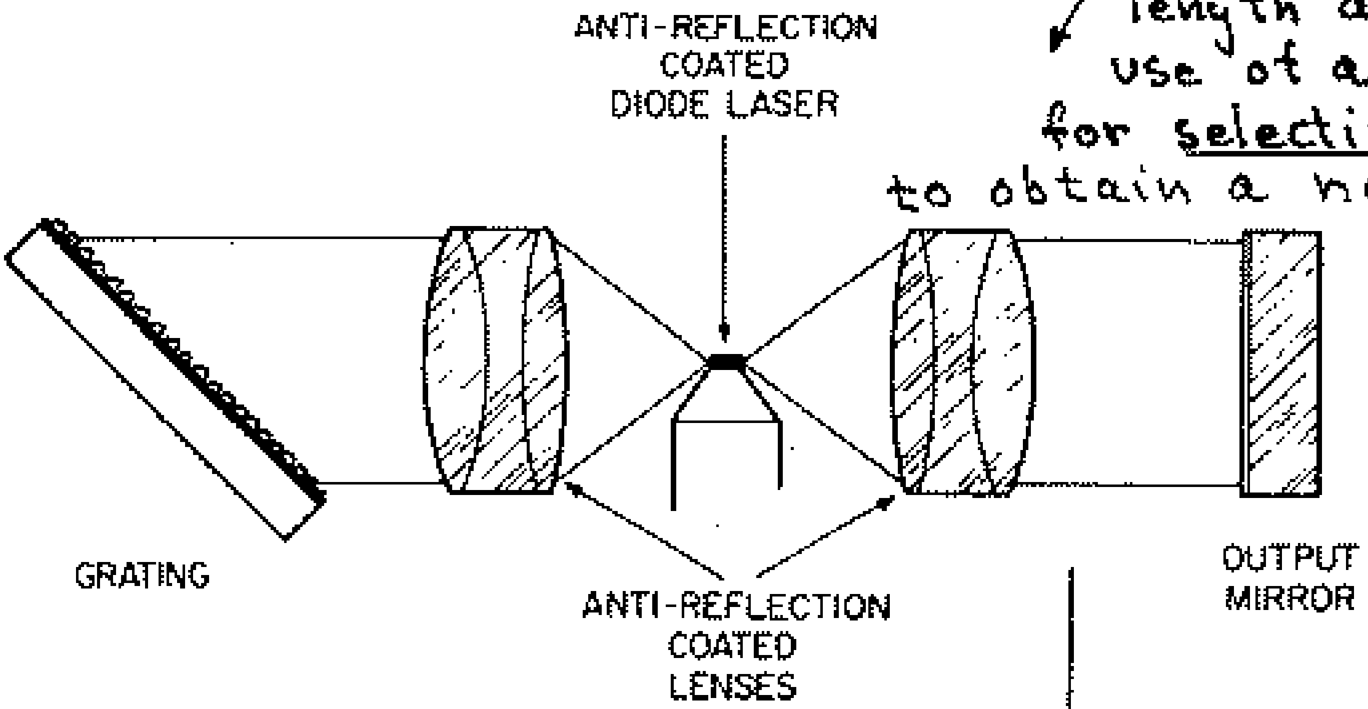
MODES (Longitudinal) AND THEIR CONTROL



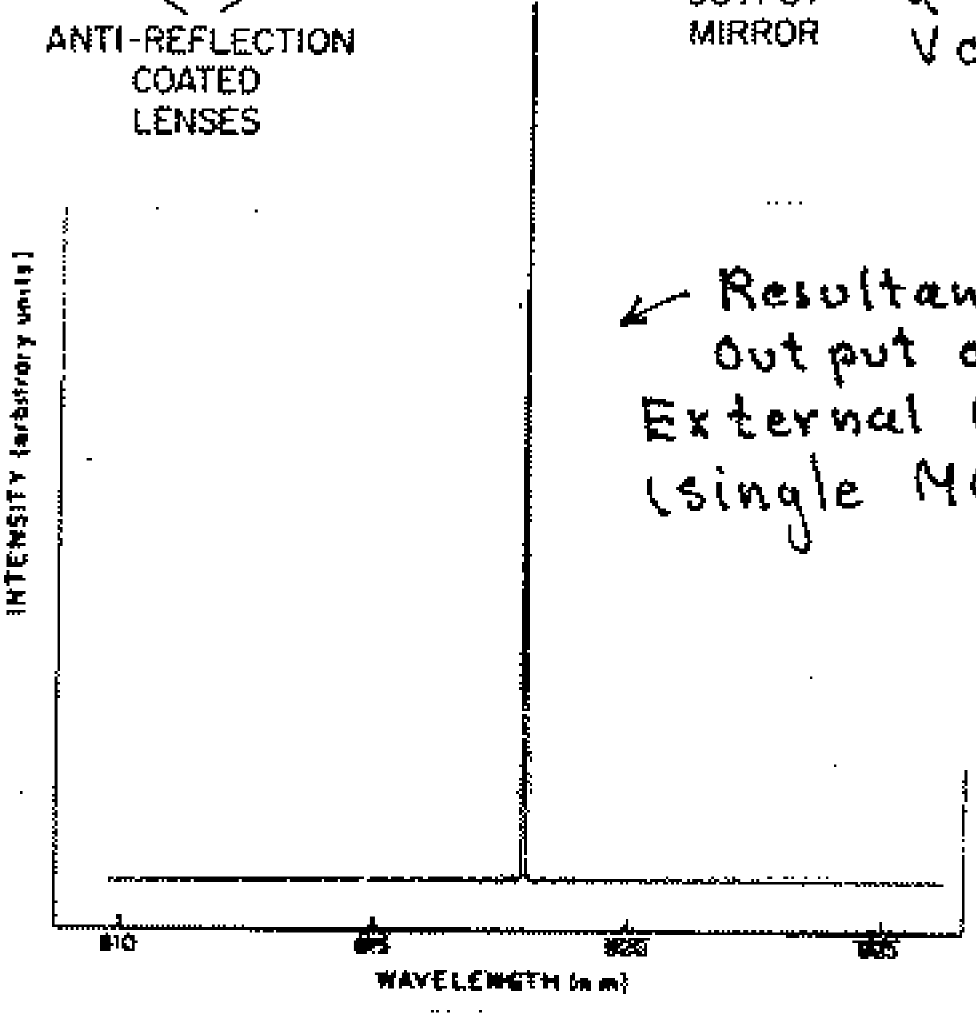
Typical Longitudinal Mode Structure of a semiconductor laser using facet reflectivity (Multi-mode)

External Cavity to increase length and to allow use of a grating for selectivity (and to obtain a narrow linewidth $\Delta\phi = \frac{1}{2\bar{n}}$

Photon Number $\propto$ Cavity Volume



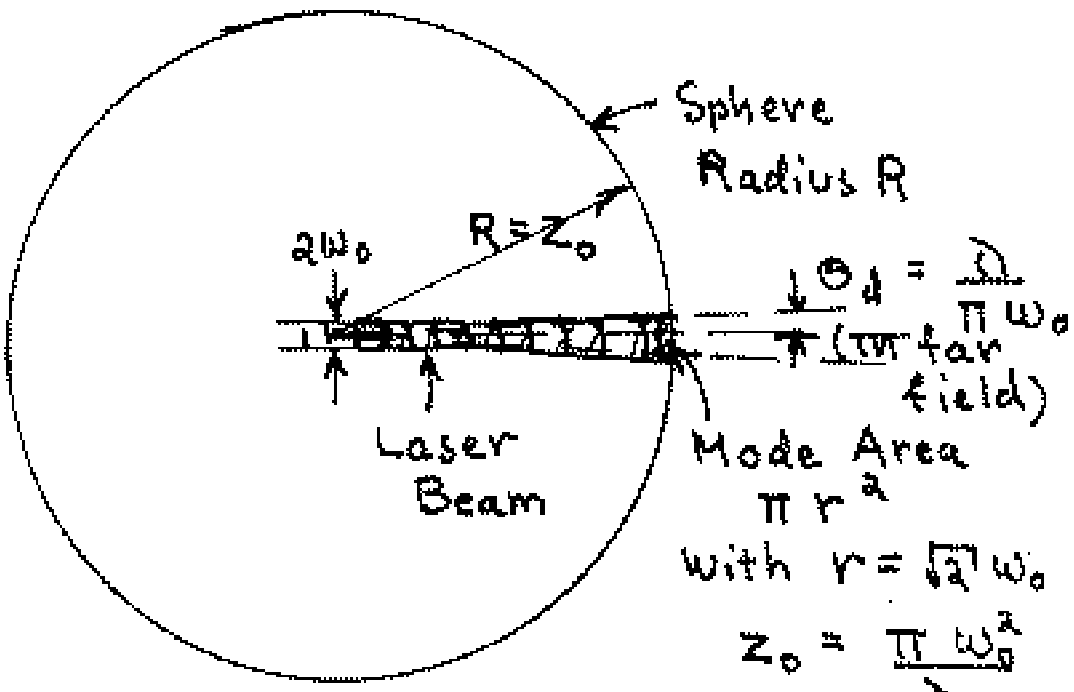
The other means to control the mode structure is the DFB (Distributed Feed back Laser) (Grating in the active Layer)
However the linewidth is not as narrow as for the external cavity structure because $\bar{n}$ is so much less ($\Delta^2\phi = 1/2\bar{n}$)



Resultant Output of the External Cavity (single MODE)

"Counting" Modes

Number of Modes per Unit Volume per Hz



$z_0 = \frac{\pi w_0^2}{\lambda}$

<u>Number of modal Spots on the Sphere</u>

$$= \frac{4\pi R^2}{\pi r^2} = \frac{4\pi z_0^2}{\pi\, 2 w_0^2}$$

$$= \frac{4\pi^2 w_0^2}{2\lambda^2} \quad \text{(substituting for } z_0^2\text{)}$$

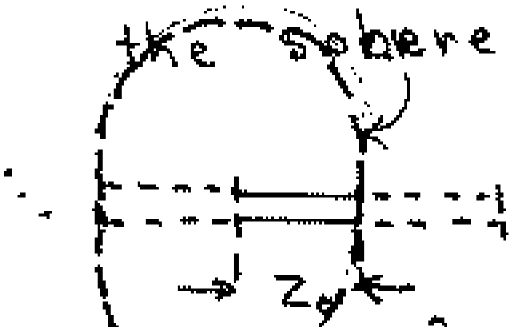
<u>Depth of focus for the laser</u> $= \frac{\pi w_0^2}{\lambda} = z_0 = R$

<u>Volume of the mode</u>

$$= \pi w_0^2 \text{ (Depth of Focus)}$$

$$= (\pi w_0^2)^2/\lambda$$

For a range of frequencies for each spots on the sphere, how many modes are there?



$z_0 = \pi w_0^2/\lambda$ = Depth of Focus

Let $2z_0$ contain m <u>wave lengths</u>/2 $\left(\frac{\lambda}{2}\right)$

$$z_0 = m\frac{\lambda}{4}$$

Thus $m = \frac{4}{\lambda} z_0$

$$= 4\frac{f}{c}(z_0)$$

Thus $\Delta m = 4 . z_0 \frac{\Delta f}{c}$

(for z_0 const.)

Consequently The $\frac{\text{Number of Modes}}{(\text{Volume})(\text{Hz})}$

Only $\frac{1}{2}$ the sphere since each mode contains forward and backward waves.

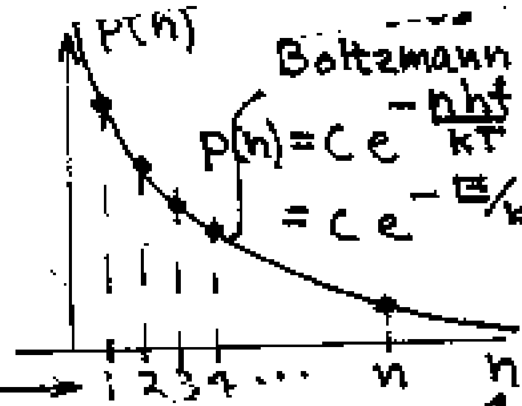
$$= \left(\frac{1}{2}\right)\frac{(4\pi^2 \cancel{w_0^2})}{\lambda^2} \times \frac{1}{(\pi \cancel{w_0^2})^2/\cancel{\lambda}} \times 4\pi\frac{f}{c^2}\cancel{w_0^2}\frac{\Delta f}{\Delta f}$$

$$\frac{\text{Number}}{\text{Unit Volume Hz}} = \left(\frac{4\pi f^2}{c^3}\right) \quad [\mathrm{m^{-3}\ Hz^{-1}}]$$

Black Body Radiation and Modes (Thermal)

p(n) = probability of having n-photons at frequency f in one direction.

"number of photons at frequency f"



Boltzmann

$$p(n) = Ce^{-\frac{nhf}{kT}} = Ce^{-E/kT}$$

(In contrast to laser radiation which has a Poisson distribution)

And $E = hfn$ number of photons at frequency f ($\therefore E = nhf$)

Geometric series

Normalization $\sum_n Ce^{-\frac{nhf}{kT}} = 1 = \sum_n p(n)$

$$= \frac{C}{e^{-\frac{hf}{kT}}-1} = 1$$

Defines C

Average energy (per "Mode")

$$= \sum (hfn)\, Ce^{-\frac{nhf}{kT}} = C \frac{d}{d(-\frac{1}{kT})} \sum e^{-\frac{nhf}{kT}}$$

frequency f travelling in one direction

$$= C \frac{d}{d(-\frac{1}{kT})} \frac{1}{(1-e^{-\frac{hf}{kT}})} = C \frac{1}{(1-e^{-\frac{hf}{kT}})^2}\, hf$$

$$= \frac{hf}{(e^{hf/kT}-1)}$$

At One Frequency f

Total Average Energy (per unit volume per Hz) (Spectral Energy Density) $= \dfrac{hf}{e^{hf/kT}-1} \times \dfrac{\text{No of Modes}}{(\text{Unit Volume})(\text{Hz})}$

See the following for a discussion

"Noisy" Photons per mode = 1

Consider two energy levels (one in the conduction band and one in the valence band) populated according to the Boltzmann distribution (Fermi-Dirac will work as well!). Let these be in thermal equilibrium with one mode of the radiation field.

$\hbar\omega = E_2 - E_1$

Thus the energy (average) is

$$(V\,df)\,\rho(f) = \frac{\hbar\omega}{e^{\hbar\omega/kT}-1} \quad \leftarrow 1 \text{ mode } (\omega = 2\pi f)$$

Volume of the mode; Hz; J/m³/Hz

Thus $(e^{\hbar\omega/kT} - 1)\,V\,df\,\rho(f) = \hbar\omega = hf$

Substitute $e^{\hbar\omega/kT} = e^{(E_2-E_1)/kT} = \left(\frac{N_1}{N_2}\right)$ (Boltzmann population)

Thus multiplying through by N_2

$$N_1\,(V\,df\,\rho(f)) = \hbar\omega N_2 + \rho(f)(V\,df\,N_2)$$

If the N_1 term is proportional to absorption. and the N_2 terms are proportional to emission, then this simply states that in thermal equilibrium

rate of absorption = Total Rate of Emission of photons. -- just conservation of energy.

Let the constant of proportionality be $B(f)$ at frequency f. Then the net rate of photon emission is in general

$$\frac{1}{\hbar\omega}\frac{d\rho(f)V\,df}{dt} = \underbrace{\hbar\omega N_2 \overset{\text{Joules}^{-1}\text{m}^{-1}\text{sec}^{-1}}{B(f)}}_{\text{No. of Photons/sec}} + \underbrace{\overset{\text{Joules}}{\rho(f)\,V\,df}\,(N_2 - N_1)\,B(f)}$$

and this is 0 for thermal equilibrium.

The second (R.H.S.) term is "gain" when $N_2 > N_1$. This rate is "stimulated" by the field which it amplifies (N_2 is the electron density in the C.B and $N_1 \simeq 0$).

The first term is field independent (random). It is equivalent to the emission part of the second term $(\rho(f)\,V\,df\,N_2 B(f))$ with $\rho(f)\,V\,df = \hbar\omega = hf$, that is one photon in the mode.