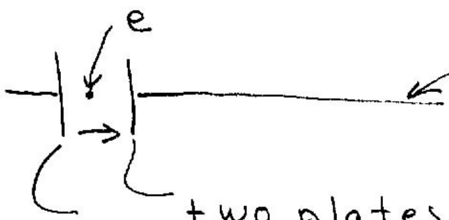
Shot Noise Formula



two plates
(p-n junction)
(etc)

$$i(t) = e\,\delta(t - t_i)$$

$\uparrow$ Random Arrival time

$$= e \int_{-\infty}^{+\infty} e^{i\omega(t-t_i)} \frac{d\omega}{2\pi}$$

Correlation function

$$\lim_{S\to\infty} \int_{-S/2}^{+S/2} i(t)\, i(t+T)\, \frac{dt}{S}$$

Fourier Transform Gives The Current Spectrum

$$F.T. = \int_{-S/2}^{+S/2} \frac{dt}{S} \int \Big[e^{-i\omega T} \int_{-\infty}^{+\infty} I(\omega')\, e^{i\omega' t} \frac{d\omega'}{2\pi} \int_{-\infty}^{+\infty} I(\omega'')\, e^{i\omega''(t+T)} \frac{d\omega''}{2\pi} \times dT \Big]$$

Integral over t gives $2\pi\,\delta(\omega' + \omega'')$, then carry out $\int d\omega''$ leaving

$$F.T. = \int_{-S/2}^{+S/2} \frac{1}{S} \Big[e^{-i\omega T} \int_{-\infty}^{+\infty} \frac{I(\omega')\, I(-\omega')}{2\pi}\, e^{-i\omega' T} d\omega' \Big] dT$$

Integral over T gives $2\pi\,\delta(\omega + \omega')$. Thus

$$F.T.\,(\text{Correlation}) = \frac{1}{S}\, I(\omega)\, I^*(\omega)$$

; S is the time over which the correlation is measured. For single pulse $I(\omega) = e\, e^{-i\omega t_i}$. For random electron stream $I(\omega) = e \sum_i e^{-i\omega t_i}$ t_i - random between $-S/2$ and $S/2$

$$I(\omega)\, I^*(\omega) = e^2 \sum_{i=0}^{\text{Number in S sec}} = e^2 N$$

(cross terms cancel because of randomness. Thus $F.T. = e^2 \frac{N}{S}$

Now the average power

$$= \int_{-S/2}^{+S/2} i(t)\, i(t)\, dt/_S$$

$$= \int_{-S/2}^{+S/2} \iint I(\omega)\, I(\omega')\, e^{i(\omega+\omega')t} \frac{dt}{S} \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi}$$

$$= \int_{-S/2}^{+S/2} \int I(\omega)\, I(\omega')\, 2\pi\, \delta(\omega+\omega') \frac{d\omega}{S} d\omega'$$

$$= \int \frac{I(\omega)\, I(-\omega)}{S\ (2\pi)^2} 2\pi\, d\omega = \int \frac{I(\omega)\, I^*(\omega)}{2\pi\ S} d\omega$$

Thus $\frac{1}{2\pi}$ F.T. (Correlation) = spectral power density. $= \frac{e^2 N}{S} \times \frac{1}{2\pi}$ per unit angular frequency

But $\frac{e\,N}{S}$ = Average Current = I_{Avg}

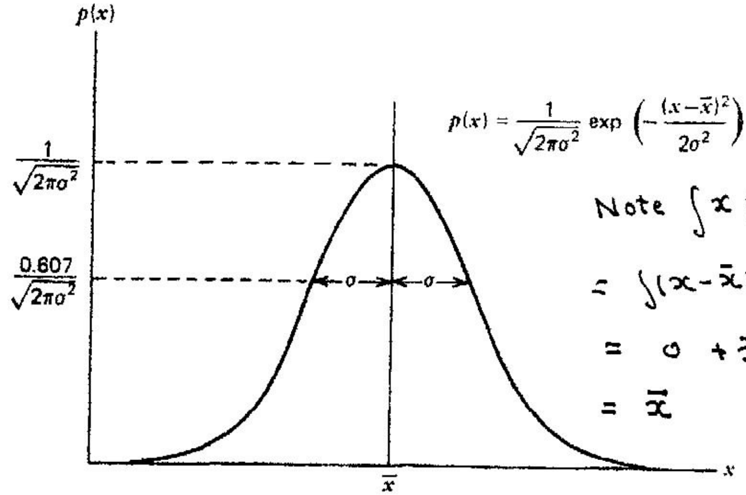
Changing to frequency $\omega = 2\pi f$, we obtain the shot noise formula $\overline{i^2} = 2 e\, I_{Avg} \Delta f$

(where the 2 accounts for negative frequencies)

Note: (We should be calculating

$$\int_{-S/2}^{+S/2} (i(t) - i_{Avg})^2\, dt/_S$$

The result is the same and the proof is included in one of the extra web pages.



Note $\int x\, p(x)\, dx$

$= \int (x-\bar{x})\, p(x)\, dx + \bar{x} \int p(x)\, dx$

$= 0 + \bar{x}$

$= \bar{x}$

FIGURE 4.6. Gaussian probability density function.

obtained whenever a large number of independent random causes produces a cumulative effect.* This is the case with thermal noise which is the superposition of the effects of a very large number of electrical charges in spontaneous and random motion within a conductor. Similarly, the shot noise (Section 2.6) follows the gaussian distribution since it, too, is generated by a large number of electrons emitted from the cathode at random times.

The gaussian† probability density is defined as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\bar{x})^2}{2\sigma^2}\right) \tag{4-28}$$

and is plotted in Fig. 4.6. The terms $\bar{x}$ and σ are the mean and the standard deviation defined in Section 4.4. The curve has an even symmetry about $\bar{x}$ which implies

$$P(X \leqslant \bar{x}) = P(X > \bar{x}) = 0.5 \tag{4-29}$$

Since thermal noise has a zero average value,‡ we shall from now on set $\bar{x} = 0$, and use

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{4-30}$$

*More formally, this follows from the central limit theorem of statistics, which states that if there are N independent random variables, and if we designate M as their sum, then the distribution of M approaches the gaussian distribution as M becomes very large [3].

†The equation was first derived by DeMoivre in 1733, but named in honor of K. F. Gauss (1777–1855) who used it while studying the statistics of errors.

‡The shot noise does have a dc component—the anode current of the tube. This is subtracted out in the analysis.

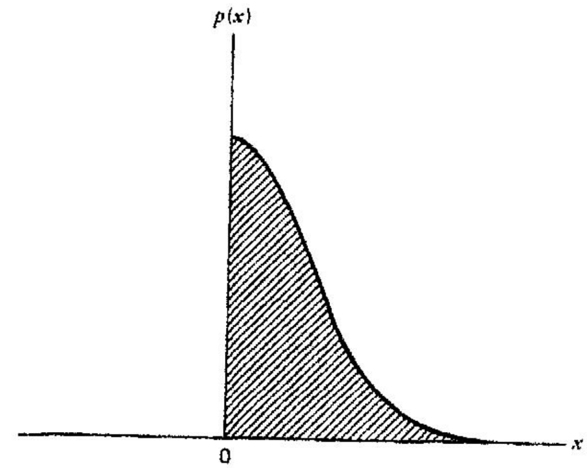


FIGURE 4.7. Idealized representation of rectified gaussian noise.

As an exercise, let us apply (4-22) to (4-30). The answer, after some alge is quite as expected. The variance of the gaussian distribution is σ^2, whic why the symbol is already included in (4-28). If (4-18) is tried on (4-30), w $\bar{x} = 0$. Again, this is expected and not very informative. Let us, however, p gaussian-noise voltage through a half-wave linear detector. By definition means that the output contains only those noise peaks which are, say, in positive direction (see Fig. 4.7). We now have for gaussian noise:

$$\text{Average} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

$$= \frac{\Gamma(1)}{\sigma\sqrt{2\pi}\, 2(1/2\sigma^2)} = \frac{\sigma}{\sqrt{2\pi}} \qquad (4\text{-}$$

Since the rms value of a gaussian waveform is σ, the ratio of the rms to average value of thermal noise is

$$(\text{Ratio})_{\text{noise}} = \frac{\text{rms}}{\text{average}} = \sqrt{2\pi} = 2.507 \qquad (4\text{-}$$

For a pure sine wave, $V \sin\theta$, the rms value is $V/\sqrt{2}$, and the average va (for half-wave rectification) is

$$\frac{1}{2\pi} \int_0^\pi V \sin\theta\, d\theta = \frac{V}{\pi} \qquad (4\text{-}$$

Hence

$$(\text{Ratio})_{\text{sine wave}} = \frac{\pi}{\sqrt{2}} = 2.221 \qquad (4\text{-}$$

which is different by $20\log(2.507/2.221) = 1.05$ dB. The significance of this will be seen in Chapter 11.

Returning to (4-30), we note that (4-15) and (4-16) cannot be applied directly because the gaussian function (4-30) is not integrable in closed form. The expression for the gaussian PDF is, however, related to the error function defined as

$$\operatorname{erf}(u) \equiv \frac{2}{\sqrt{\pi}} \int_0^u e^{-\lambda^2}\, d\lambda \tag{4-35}$$

and to the complementary error function

$$\operatorname{erfc}(u) \equiv \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-\lambda^2}\, d\lambda = 1 - \operatorname{erf}(u) \tag{4-36}$$

Using these functions it can be shown [4] that the probability $P(X \leqslant x)$ is given by

$$P(X \leqslant x) = 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2\sigma^2}}\right)$$

$$= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2\sigma^2}}\right)\right] \tag{4-37}$$

for $x \geqslant 0$. For $x \leqslant 0$, the proper expression is

$$P(X \leqslant x) = \frac{1}{2}\operatorname{erfc}\left(\frac{|x|}{\sqrt{2\sigma^2}}\right) \tag{4-38}$$

A short table of the erfc(x) is included in Appendix A.

A frequent practical problem is to calculate the probability of an outcome falling within a given range centered around the mean (Fig. 4.8). In particular,

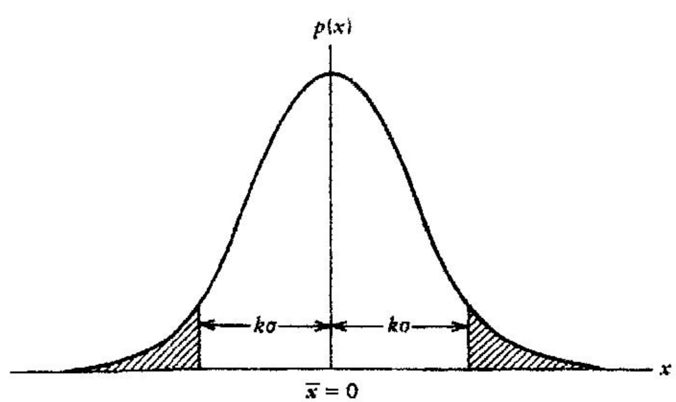


FIGURE 4.8. Definition of $\pm k\sigma$ limits.

TABLE 4.2. Probability of an Outcome Falling Within $\pm k\sigma$

k	$P_{\pm k\sigma}$
0.5	0.383
1.0	0.683
1.5	0.866
2.0	0.955
2.5	0.988
3.0	0.997
3.5	0.9995
4.0	0.99994

a commonly encountered range is $\pm k\sigma$, where k is some positive integer. thus seek the probability $P(-k\sigma \leqslant X \leqslant +k\sigma)$. This is calculated most ea by utilizing the symmetry of the gaussian PDF (see Fig. 4.6) and excluding two areas outside the $\pm k\sigma$ limits:

$$P_{\pm k\sigma} = P(-k\sigma \leqslant X \leqslant +k\sigma)$$

$$= 1 - 2P(X > k\sigma) = 1 - 2\int_{k\sigma}^{\infty} \frac{\exp(-x^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}\, dx \qquad (4\text{-}$$

Now make a change in variable, $u = x/\sqrt{2}\,\sigma$ and $du = dx/\sqrt{2}\,\sigma$. Then

$$P_{\pm k\sigma} = 1 - \int_{k/\sqrt{2}}^{\infty} \frac{2e^{-u^2}}{\sqrt{\pi}}\, du = 1 - \operatorname{erfc}\left(\frac{k}{\sqrt{2}}\right) = \operatorname{erf}\left(\frac{k}{\sqrt{2}}\right) \qquad (4\text{-}$$

Table 4.2 shows $P_{\pm k\sigma}$ for several values of k. We can see from the that the familiar $\pm 3\sigma$ limit does indeed include nearly all of the poss outcomes.

4.6. PEAK FACTOR

A useful application of the preceding material is the evaluation of the p factor for gaussian noise. For a sinusoidal wave, $V_p \cos \omega t$, the peak fac defined as the ratio

$$\frac{\text{peak value}}{\text{rms value}} = \frac{V_p}{V_p/\sqrt{2}} = \sqrt{2} \qquad (4\text{-}$$

is (in decibels) $20\log\sqrt{2} = 3$ dB.

Example Binomial Distribution For

$m = 6 \quad p = \frac{1}{2}$ (Random Walk $\longleftrightarrow$ Phase Fluctuations in a Laser)

$$P_m(n) = \begin{Bmatrix} m \\ n \end{Bmatrix} p^n (1-p)^{n'}$$

As discussed in class n = number of steps to the right

n' = number of steps to the left.

Let $N = n - n'$ be the <u>net</u> no of steps to the right after m steps.

Also have $m = n + n'$

Thus eliminating n and n' above we obtain

$$P_m(N) = \begin{Bmatrix} m \\ \frac{N+m}{2} \end{Bmatrix} (p)^{\frac{N+m}{2}} (1-p)^{\frac{m-N}{2}} = \frac{m!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!} p^{\frac{N+m}{2}} (1-p)^{\frac{m-N}{2}}$$

Numbers for $m = 6$

n	n'	$n-n' = N$
6	0	6
5	1	4
4	2	2
3	3	0
2	4	−2
1	5	−4
0	6	−6

$P_m(N)$ Prob Dist^n of Net No to Right

$$P_6(0) = \frac{6!}{(3!)^2}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{20}{64}$$

$$P_6(2) = \frac{6!}{(4!)(2!)}\left(\frac{1}{6}\right)^6 = \frac{15}{64}$$

$$P_6(4) = \frac{6}{64}$$

$$P_6(6) = \frac{1}{64}$$

Probability $P_m(n)$ of No to the right

$$P_6(0) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$P_6(1) = \frac{6}{64}$$

$$P_6(2) = \frac{15}{64}$$

$$P_6(3) = \frac{20}{64}$$

$$P_6(4) = \frac{15}{64}$$

$$P_6(5) = \frac{6}{64}$$

$$P_6(6) = \frac{1}{64}$$

Notes

$$2\left(\sum_{2,4,6} P_m(N)\right) + P_m(0) = 1$$

$$P_m(-N) = P_m(N)$$

$$\sum_{n=0}^{6} P_m(n) = 1$$

Review Questions!

a) What is $\bar{N} = \sum P(N)\, N$?

b) What is $\bar{n}$

c) What are $\overline{(n-\bar{n})^2}$ and $\overline{(N-\bar{N})^2}$

$$\ln P(N) = m! - \ln\left(\frac{m-N}{2}\right)! \;, \left(\frac{m+N}{2}\right)! - m\ln 2$$

Use stirlings formula

Example of Central Limit Theorem

$$\ln P(N) = \left(m+\tfrac{1}{2}\right)\ln m - \cancel{m} + \tfrac{1}{2}\ln(2\pi)$$

$$- \left(\frac{m-N}{2} + \frac{1}{2}\right)\ln\left(\frac{m-N}{2}\right) + \cancel{\frac{m-N}{2}} - \frac{1}{2}\ln(2\pi)$$

$$- \left(\frac{m+N}{2} + \frac{1}{2}\right)\ln\left(\frac{m+N}{2}\right) + \cancel{\frac{m+N}{2}} - \frac{1}{2}\ln(2\pi)$$

$$- m\ln 2$$

$\ln m - \ln 2$

$$\ln\left(\frac{m-N}{2}\right) = \ln \frac{m}{2}\left(1-\frac{N}{m}\right) \simeq \ln\frac{m}{2} + \left(-\frac{N}{m} - \frac{1}{2}\left(\frac{N}{m}\right)^2\right)$$

$$\ln\left(\frac{m+N}{2}\right) \quad \ln\frac{m}{2}\left(1+\frac{N}{m}\right) \simeq \ln\frac{m}{2} + \left(\frac{N}{m} - \frac{1}{2}\left(\frac{N}{m}\right)^2\right)$$

$\ln m - \ln 2$

$-\frac{1}{2}\ln m$

Thus

$$\ln P(N) = +\frac{1}{\cancel{2}}\ln m + m\cancel{\ln}2 - \ln m + \ln 2 \quad - \cancel{m\ln}2$$

$$- \frac{N^2}{m} + \frac{1}{2}\frac{N^2}{m} + \cancel{\frac{1}{2}\left(\frac{N}{m}\right)^2} \quad - \frac{1}{2}\ln(2\pi)$$

higher order

Taking Antilog

$$P(N) = \frac{2}{\sqrt{2\pi m}}\, e^{-\frac{1}{2}\frac{N^2}{m}} \qquad N > 0$$

(since N was net No. of steps right)

(If N can be negative as well (steps to left; divide by 2)

To Relate this to diffusion let $m = qt$ where q = no of steps/unit time ; $x = N\ell$ where ℓ = distance between steps. Let $D = \frac{q\ell^2}{2}$ the Diffusion coefficient

$$P(N) \to \frac{dP}{dx}dx = \frac{1}{\sqrt{\pi D t}}\, e^{-\frac{x^2}{4Dt}}\,\ell$$

$dx = (N+1)\ell - N\ell = \ell$

This satisfies the diffusion equation $D\frac{\partial^2 n}{\partial x^2} = \frac{\partial n}{\partial t}$