Problem Set o. 4 Solutions

Problem No. 1

From the $\frac{1}{2}\,\mathrm{erfc}(x)$ curve for a bit error rate of 10^{-10}, the voltage or current signal/noise ratio must be $1.37\times10^{1} = 13.7$

$$= \frac{2\cancel{\sqrt{2}}\,(I_1 - I_0)}{\cancel{\sqrt{2}}\,(\sigma_1+\sigma_0)} = I_1/\sigma$$

If $\sigma_1 = \sigma_0 = \sigma$ and $I_0 = 0$ (Good for Thermal Noise Limited ASK)

Detection $\eta = .60$

$$\mathcal{R} = \frac{.6\times1.6\times10^{-19}}{2\pi\times10^{-34}\,(3\times10^{10}/1.3\times10^{-4})} = .66$$

Amplifier Noise Figure

$F_n = 2.5$

Thermal + Amplifier Noise (referred to input)

$$= \Delta f\,\frac{4kT}{R}\,F_n\ (\text{amps})^2 = \Delta f\,\frac{4(.026)\times1.6\times10^{-19}\times2.5}{5000}$$

$$= 8.32\times10^{-24}\,(\text{amps})^2\,\Delta f$$

RC time constant $= 5000\times10^{-13} = 5\times10^{-10}$ sec

"corner" frequency $= \dfrac{1}{2\pi\times5\times10^{-10}} = 3.2\times10^{8}$ Hz

Thus it is assumed that the amplifier is "equalized" $G = G_0(1 + j\omega RC)$

(In this case since the noise figure is given the noise remains "white" and integration over the band-width is not necessary)

Shot Noise (Signal)

$$I^2_{shot} = 2\times1.6\times10^{-19}\times.66\ P_{opt}\,\Delta f$$

Signal Current $= I_S = .66\ P_{opt}$

Thus $K = S/N = \dfrac{.66\ P_{opt}}{(8.32\times10^{-24} + 2.11\times10^{-19}\,P_{opt})^{1/2}(\Delta f)^{1/2}}$

(2)

Simplifying

$$K^2 (8.32\times10^{-24} + 2.11\times10^{-19} P_{opt})\,\Delta f = (.66\, P_{opt})^2$$

A quadratic Equation for Popt

$$P_{opt}^2 - K^2(2.29)\times 2.11\times10^{-19}\,\Delta f\, P_{opt} - K^2\times 2.29\times 8.32\times10^{-24}\,\Delta f = 0$$

Normalizing Popt to 10^{-6} watts and Δf to 1 GHz

$$(P_{opt_N})^2 - P_{opt_N} K^2 (2.29)\times 2.11\times10^{-4}\,\Delta f_N - K^2\, 2.29\times 8.32\times10^{-3}\,\Delta f_N = 0$$

($P_{opt_N} = P_{opt}\times10^{6}$ and $\Delta f_N = \Delta f\times10^{-9}$)

If thermal dominates then the second term is negligible (shot noise) and

$$(P_{opt_N}) = K\sqrt{2.29\times 8.32\times10^{-3}\,\Delta f_N}$$

For $K = 13.7$ and $\Delta f_N = 10$ this is

$$P_{opt_N} = 5.98 \quad \text{or} \quad P_{opt} = 5.98\ \mu W$$

Second term is

$$5.98\times(13.7)^2\times 2.11\times10^{-4}\times 10 = 2.37$$

To check that thermal dominates Take Full solution to the quadratic which is

$$P_{opt_N} = \frac{K^2(2.29)\times 2.11\times10^{-4}\,\Delta f_N}{2} + \left(\left(\frac{K^2(2.29)\times 2.11\times10^{-4}\,\Delta f_N}{2}\right)^2 + K^2\, 2.29\times 8.32\times10^{-3}\,\Delta f_N\right)^{1/2}$$

For $\Delta f_N = 10$ and $K = 13.7$ this is

$$P_{opt}(\mu w) = 1.2 + ((1.2)^2 + 35.7)^{1/2} \simeq 5.98$$

Thus Thermal Dominates

Problem No. 2

a) Linear gain $G_0 = 100$

Saturated Gain given by

$$\frac{dP}{dz} = \left(\frac{g_0}{1 + P/P_s} \right) P$$

Integrate

$$\int_0^L \left(1 + P/P_s \right) dP = g_0 P \, dz$$

or

$$\int_0^L \left(\frac{1}{P} + 1/P_s \right) dP = g_0 \, dz$$

$$P_0/P_{in} + \frac{P_0 - P_{in}}{P_s} = g_0 L \qquad \begin{array}{l} P_0 = P(L) \\ P_{in} = P(0) \end{array}$$

$$\therefore \; P_0 = P_{in} (e^{g_0 L}) \, e^{-\frac{P_0 - P_{in}}{P_s}}$$

$$= P_{in} \, G_0 \, e^{-(P_0 - P_{in})/P_s}$$

$$= G \, P_{in} \qquad G = G_0 \, e^{-(P_0 - P_{in})/P_s}$$

To obtain rapid convergence take ln

$$P_s \ln \frac{P_0}{P_{in} G_0} = -(P_0 - P_{in}) = -(P_0 - 1) \quad (\text{in mW})$$

$$10 \ln \frac{P_0}{100} = -(P_0 - P_{in}) \quad \text{or} \quad P_0 = P_{in} - 10 \ln P_0/100$$

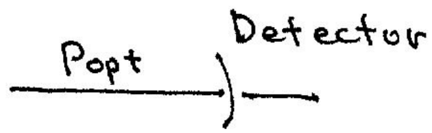
① try $P_0 = 40$ (mW) $\rightarrow$ $P_0 = 1 - 10 \ln .4 = 8.16$

② try $P_0 = 20$ (mW) $\rightarrow$ $P_0 = 15$ $\rightarrow$ close

③ try $P_0 = 17$ (mW) $\rightarrow$ $P_0 = 16.7$

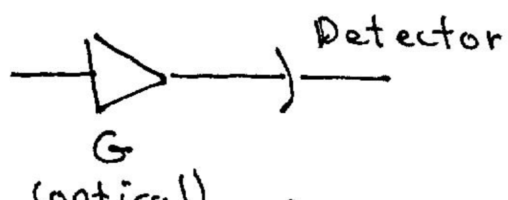
Could keep going but this is close!

b) Without ne amplifier th noise is given by shot noise



$$\overline{i_N^2} = 2e\mathcal{R} P_{opt}\Delta f \qquad\qquad \text{Signal } P_{opt}\mathcal{R}$$

With the Amplifier → shot + Sp-Sig beat noise



$$i_{N_G}^2 = 4G P_{opt}\mathcal{R}(G-1)hf\, n_{sp}\Delta f\,\mathcal{R} + 2eG\mathcal{R}P_{opt}\Delta f$$

<u>Signal</u> is $P_{opt}\mathcal{R}G$

$$\left(\frac{S}{N}\right)_{\text{with Amp}} = \frac{P_{opt}\mathcal{R}G}{i_{N_G}} = K_G$$

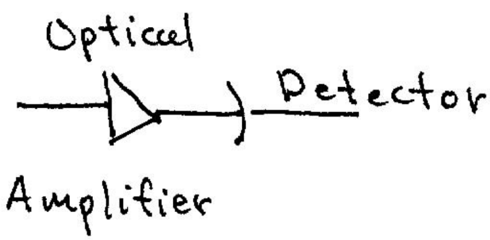
$$\left(\frac{S}{N}\right)_{\text{without Amp}} = \frac{P_{opt}\mathcal{R}}{i_N} = K$$

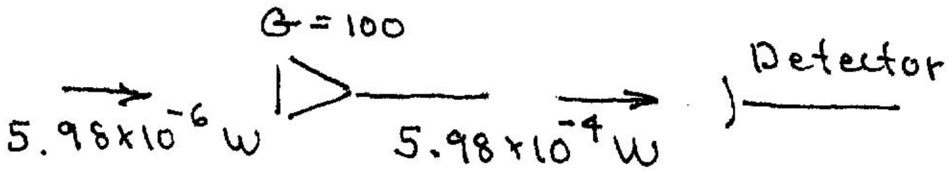
$$\text{Noise Figure} = \frac{K}{K_G} = \Big(2e\frac{\mathcal{R}}{G}P_{opt}\Delta f + 4P_{opt}\mathcal{R}^2\left(1-\frac{1}{G}\right)hf\,n_{sp}\Delta f\Big)^{1/2}\Big/\left(2eP_{opt}\mathcal{R}\Delta f\right)^{1/2}$$

$$= \left(\frac{1}{G} + \frac{\cancel{4}^{2}\,\cancel{e}h}{\cancel{2}\cancel{e}\,\cancel{\Delta f}\,\cancel{hf}}\left(1-\frac{1}{G}\right)\cancel{hf}\,n_{sp}\,\cancel{\Delta f}\right)^{1/2}$$

We see $\left(\frac{K}{K_G}\right)^2 \rightarrow 2\eta\, n_{sp}\left(1-\frac{1}{G}\right)$ which has a value of 2 if η and $n_{sp} = 1$

c)





* Additional Noise Expected to be primarily Spontaneous-signal beat noise
* Amplifier not saturated so power into detector is 5.98×10^{-4} W.
* Thermal + Amplifier Noise is the same

$$\Delta f \; 8.32\times10^{-24}\,(\text{Amps})^2 \qquad (\Delta f = \text{bandwidth})$$

* Signal Shot noise

$$2e\times.6\times5.98\times10^{-4}\,\Delta f = 1.15\times10^{-22}(\Delta f)\,(\text{Amps})^2$$

We see that this now dominates the thermal noise

* Spont - Signal Beat Noise

$$4\,G\,(G-1)\,\mathcal{R}^2\,\Delta f\,(5.98\times10^{-6})\,hf\,n_s$$

$$= 4\cdot100\cdot(99)(.44)\times5.98\times10^{-6}\times2\pi\times10^{-34}\times\frac{3.\times10^{10}}{1.3\times10^{-4}}\,\Delta f\times\underset{\substack{\uparrow\\ n_s}}{\frac{1}{2}}$$

$$= \frac{1}{2}\times(1.46)\times10^{-20}\,\Delta f \;\text{amps}^2$$

Thus we see that the Sp - Beat Noise Dominates and since it is larger than the thermal noise the signal/noise ratio increases. For $\Delta f = 10$ GHz

$$S/N = \frac{.66(5.98\times10^{-4})}{(.7\times10^{-10})^{1/2}} = 46.5 \text{ for the 1-bit}$$

To obtain the bit error rate we need $x = \dfrac{I_1 - I_0}{\sqrt{2}(\sigma_1+\sigma_0)}$

Now to dete mine the bit erro rate

$$x = \frac{I_1 - I_0}{\sqrt{2}(\sigma_1 + \sigma_0)} = \frac{I_1}{\sqrt{2}(\sigma_1+\sigma_0)}$$ (Ref. Agrawal page 172)

$\sigma_1 = (6.2 \times 10^{-21} \Delta f)^{1/2} \rightarrow$ the noise current for the one bit

$\sigma_0 = (8.32 \times 10^{-24} \Delta f)^{1/2}$ - the noise current for the zero bit (thermal + amplifier)

Thus $$x = \frac{.66 \times (5.98 \times 10^{-4})}{\sqrt{2}((.62 \times 10^{-10})^{1/2} + (.832 \times 10^{-13})^{1/2})}$$

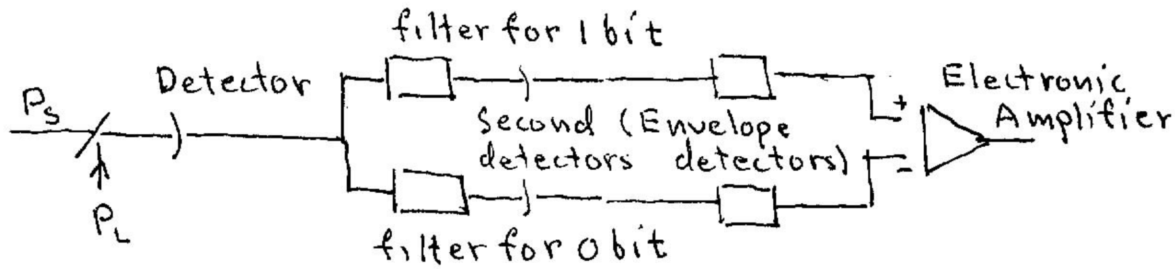
$$= \frac{2.79 \times 10^{-4}}{8.16 \times 10^{-6}} = 34.$$

Thus the bit-error-rate is $\frac{1}{2}\,\mathrm{erfc}(x) \simeq \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x}$

Thus ln B.E.R. $\simeq -x^2/2 = 480$

$\lg_{10}$ B.E.R. $\simeq$ 208 And BER $\simeq 0$ $(1 \cdot 10^{-208})$

Problem 3) Heterodyne F.S.K.



Configuration

$$P = P_S + P_L + 2\sqrt{P_S P_L}\cos((\omega_L - \omega_S)t)$$

$\omega_L - \omega_S = \omega_1$ for 1 bit

$\omega_L - \omega_S = \omega_0$ for 0 bit

The calculation is the same as homodyne ASK (as follows) since when a 1 bit is transmitted there is a local oscillator signal for the 0 bit channel.

Also since photons are transmitted for both the 0 and 1 bits the average photon No/bit = 36 as well.

Note that we use $x = \frac{12}{2\sqrt{2}}$ for a BER of 10^{-9}

Example of Homodyne ASK

$$N = \frac{R}{e}\left(P_s + P_L + 2\sqrt{P_L P_s}\cos((\omega_L - \omega_s)t + \varphi_L - \varphi_s)\right)$$

$$= \frac{R}{e}\left(P_s + P_L + 2\sqrt{P_L P_s}\right)$$

$$= N_s + N_L + 2\sqrt{N_s N_L}$$

For B.E.R = 10^{-9}

$$\frac{I_1 - I_0}{2\sqrt{2}\,\sigma} = \frac{N_1 - N_0}{2\sqrt{2}\sqrt{N_{Lo}}} = \frac{12}{2\sqrt{2}}$$

from $\frac{1}{2}\operatorname{erfc}\left(\frac{I_1 - I_0}{2\sqrt{2}\,\sigma}\right)$

$$N_1 = \underbrace{\cancel{N_s}}_{\text{small}} + N_L + 2\sqrt{N_s N_L}$$

$$N_0 = N_L = N_{Lo}$$

$$\therefore \quad N_1 - N_0 = 2\sqrt{N_s N_L}$$

so $\dfrac{\sqrt{N_s N_{Lo}}}{\sqrt{2}\sqrt{N_{Lo}}} = \dfrac{12}{2\sqrt{2}}$ $\Rightarrow$ $N_s = 36$ photo-electrons.

Example of Homodyne P.S.K.

$$N_1 = N_s + N_L + 2\sqrt{N_s N_L}$$

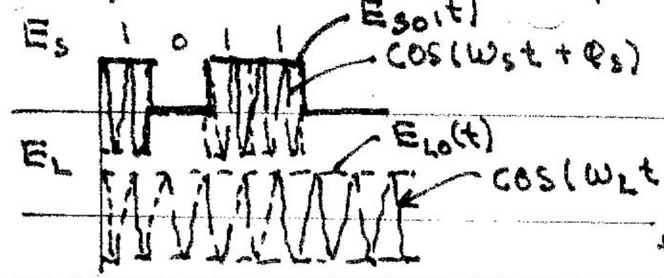
$$N_0 = N_s + N_L - 2\sqrt{N_s N_L}$$

$$\therefore \quad N_1 - N_0 = 4\sqrt{N_s N_L}$$

so $\dfrac{4\sqrt{N_s N_L}}{\sqrt{N_L}} = 12$

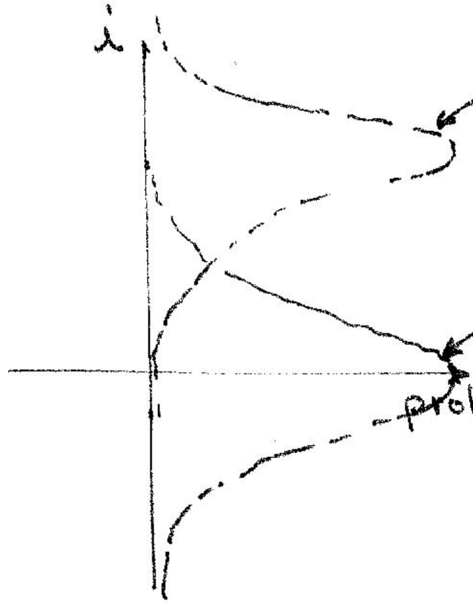
And Thus $N_s = 9$ photons/bit

Example Homodyne A.S.K (Amplitude Shift 8
Keying) (NRZ)



Assume E_{Lo} sufficient to have noise due solely to shot noise

Assume 1-bit and 0-bit described by Gaussian Distributions (C.L.T)



$$p(i) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-\frac{(i-I_1)^2}{2\sigma^2}}$$

σ due to shot noise from I

$$p(i) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-\frac{(i-I_0)^2}{2\sigma^2}}$$

Can express currents in terms of photo-electron number per bit

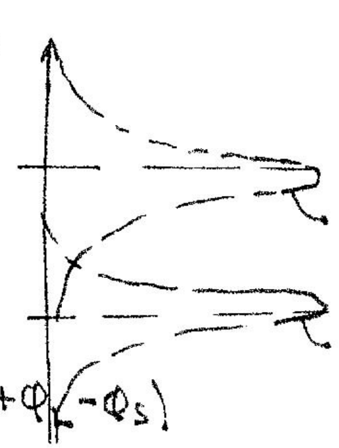
$$\frac{(i-I_1)^2}{2\sigma^2} = \frac{\left(\frac{iT}{e} - \frac{I_1T}{e}\right)^2}{2\left(\frac{\sigma T}{e}\right)^2} = \frac{(N - N_1)^2}{2N_{Lo}}$$

$$R = \frac{e}{h\nu}$$

$$\left(\frac{\sigma T}{e}\right)^2 = \left(2eRP_{Lo}\frac{\Delta f T^2}{e^2}\right) = \left(RP_{Lo}\frac{T}{e}\right) = N_1$$

$$N_{Lo} = \frac{R}{e}P_{Lo}\frac{T}{e}$$

$$N = \frac{R}{e}(P_s + P_L + 2\sqrt{P_L P_s} \times \cos((\omega_L - \omega_s)t + \varphi_L - \varphi_s)$$



$$p(N) = \frac{1}{(2\pi)^{1/2} N_{Lo}} e^{-\frac{(N-N_1)^2}{2N_{Lo}}} ; \quad \int_{-\infty}^{+\infty} p(N)\,dN =$$

$$p(N) = \frac{1}{(2\pi)^{1/2} N_{Lo}} e^{-\frac{(N-N_0)^2}{2N_{Lo}}} \quad \int_{-\infty}^{+\infty} p(N)\,dN =$$