Notes on P.S. 5

Notes For P.S. 5 EECS 119

Problem No. 1

Diffraction of a Gaussian

Alternative!

Consider now a beam that has $\bar{E} = \hat{x}\, e^{-(x^2+y^2)/w_0^2} e^{i\omega t}$ at $z=0$ (a Gaussian beam - linearly polarized)

Can this propagate in the z-direction without boundaries (answer! yes - but it will spread or diffract in the $r = (x^2+y^2)^{1/2}$ direction)

As with the Gaussian pulse, the phasor representation in the x and y-directions are

$$w_0^2 (\sqrt{\pi})^2 e^{-(k_x^2+k_y^2) w_0^2/4} e^{i\omega t}$$

As it propagates it becomes

$$w_0^2 (\sqrt{\pi})^2 e^{-(k_x^2+k_y^2) w_0^2/4} e^{i\omega t} e^{-ik_x x - ik_y y - ik_z z}$$

and $k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2/c^2$

Assume k_x and k_y small, then

$$k_z = (k^2 - k_x^2 - k_y^2)^{1/2}$$
$$\simeq k\left(1 - \left(\frac{k_x^2 + k_y^2}{2k^2}\right) + \cdots\right)$$

add up all the phasors

$$w_0^2 (\sqrt{\pi})^2 e^{i\omega t - ikz} \int e^{-(k_x^2 + k_y^2) w_0^2/4}\, e^{+i \frac{k_x^2 + k_y^2}{2k} z}$$
$$\times\, e^{-ik_x x}\, e^{-ik_y y} \frac{dk_x\, dk_y}{(2\pi)^2}$$

complete the squares in k_x and k_y

$$k_x^2 w_0^2/4 + i k_x^2 z/2k - i k_x x$$
$$= \underbrace{\left(w_0^2/4 + i/2k\, z\right)}_{A} k_x^2 - i k_x x = A\Big(\overbrace{k_x - \frac{ix}{2A}}^{\eta}\Big)^2 + \frac{x^2}{4A}$$

similarly for k_y terms. Thus (with $\zeta = (k_y - iy/2A)$)

$$w_0^2 \pi\, e^{i\omega t - ikz} \left[\int e^{-A\eta^2 - A\zeta^2} \frac{d\eta\, d\zeta}{(2\pi)^2}\right] e^{-x^2/4A - y^2/4A}$$

$$= w_0^2 \frac{\pi}{4\pi^2} e^{i\omega t - ikz} \frac{1}{A} (\sqrt{\pi})^2\, e^{-(x^2+y^2)/4A}$$

$$= \frac{w_0^2}{4\left(w_0^2/4 + i/2k\, z\right)} e^{-(x^2+y^2)/4A}\, e^{i\omega t - ikz}$$

$$= \frac{1}{\left(1 + 2i/k\, z/w_0^2\right)} e^{-(x^2+y^2)\left[\frac{1}{w_0^2 + i\, 2/k\, z}\right]} e^{i\omega t - ikz}$$

$$\frac{\quad}{\cdot\, 2/k\, z} = \frac{w_0^2}{(w_0^4 + (2/k\, z)^2)} - \frac{i\, 2/k\, z}{(w_0^4 + (2/k\, z)^2)} = \frac{1}{(w(z))^2} - i\frac{k}{2R(z)}$$

$\Rightarrow$ $w(z) = w_0\left(1 + (z/z_0)^2\right)$; $z_0 = \frac{k}{2} w_0^2 = \frac{\pi w_0^2}{\lambda}$ = confocal parameter

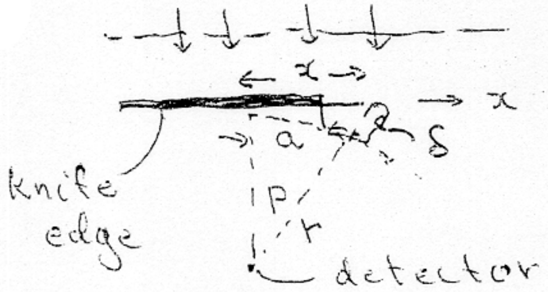
nt
e $R(z) = z\left(1 + (z_0/z)^2\right)$

Source

S

knife-edge diffraction

wavefront (approximate as plane)



$$E = \sum dE$$ Scalar theory (parax approx)

$$= \frac{E_0}{r}\int_a^{\infty} e^{-jk(r+\delta)}\,dx$$

$$(\delta + r)^2 = x^2 + r^2 \rightarrow 2r\delta \simeq x^2 \qquad \delta << r$$

$$\therefore \delta \simeq \frac{x^2}{2r} \rightarrow \text{Fresnel.}$$

$$\therefore E = \frac{E_0}{r}\int e^{-j(k x^2/2r)}\,dx\; e^{-jkr} \qquad \text{Let} = \pi u^2/2 \qquad u^2 = \frac{k x^2}{r\pi} = \varkappa^2 x^2$$

$$dx\, k\frac{2x}{2r} = \pi\, 2u\, du$$

$$= e^{-jkr}\frac{E_0}{\varkappa r}\int_{\varkappa a}^{\infty} e^{-j\pi u^2/2}\,dx \qquad dx = r\frac{\pi u}{k x}\,du = \sqrt{\frac{r\pi}{k}}\,du = \left(\frac{r\pi}{2}\right)^{1/2} du = \frac{du}{\varkappa}$$

$$= \frac{E_0}{\varkappa r}e^{-jkr}\left[\frac{1}{2} + i\frac{1}{2} - [C(\varkappa a) + i\,S(\varkappa a)]\right]$$

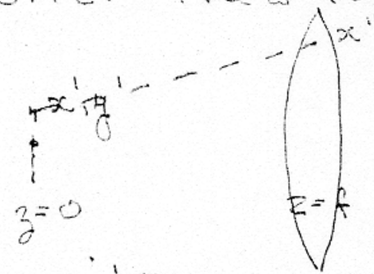
$$S(\varkappa a) = \int_0^{\varkappa a}\sin\frac{\pi u^2}{2}\,du = \text{Fresnel Sine Int}$$

$$\int_0^{\varkappa a}\cos\frac{\pi u^2}{2}\,du = \text{Fresnel Cos Int}$$

rest is looking up Cornu spiral values.

Problem 2

Fourier transform of a lens



Lens-curvature changer

$e^{-i\frac{k}{2S_1}(x^2)} \rightarrow e^{+i\frac{k}{2S_2}x^2}$

must be

$e^{-i\frac{k}{2}\left(-\frac{1}{S_1}-\frac{1}{S_2}\right)x}$

$'' \quad e^{+i\frac{k}{2}+\frac{1}{f}x^2}$

$\frac{e^{-ikr}}{r} f(x', y') \qquad r = \sqrt{z^2+(x-x')^2}$

$\simeq z + \frac{1}{2}\frac{(x-x')^2}{z}$

$\frac{1}{z}\int dx' \; e^{-ikf - i\frac{k}{2z}(x''-x')^2} \quad dx'$

Lens $\quad e^{+ik\,(x'')^2/2f}$

$\frac{1}{f}\int dx' \; e^{-ikf \; - i\frac{k}{2f}[(x''-x')^2 - (x'')^2]} \quad dx'$

$\times \; \frac{e}{f}^{-ikf - i\frac{k}{2f}[(x-x'')^2]} \quad dx''$

$((\cancel{x''} - x') - \cancel{x''})(x'' - x' + x'') + (x - x'')^2$

$= \; -x'(x'' - x' + x'') + (x - x'')^2$

$= \; -2x'x'' + (x')^2 + x^2 - 2xx''$

$+ (x'')^2$

$= [x'' - (x+x')]^2 - (x+x')^2$

$+ (x')^2 + x^2 .$

$= \underbrace{[x'' - (x+x')]^2}_{} - 2xx'$

change integration variable

from x'' to $x'' - (x+x')$

to get Gaussian integral

Problem No. 3

For a solid state material

$$Re(\bar{P}) = \sum_{k_1,k_2} \frac{q^2}{m}\frac{f}{2\omega_0}\,\pi\,\delta(\omega-(E_2-E_1)/\hbar)$$ in the limit as damping $\to 0$

continuous limit. – 3d – bulk solids.

$$\sum_{k_1,k_2} \to \frac{d^3\bar{k}_1 V}{(2\pi)^3}\frac{d^3\bar{k}_2 V}{(2\pi)^3}$$

$$\boxed{f = \frac{2\omega_0 m}{\hbar q^2}|p_{21}|^2} \to \text{oscillator strength}$$

The matrix element:

$$p_{21} = \int \tilde{\varphi}_2^* \, e\bar{r}\, \tilde{\varphi}_1 \, dV \qquad \varphi_2 = \frac{U_2(\bar{r})}{\sqrt{V}}e^{-i\frac{E_2}{\hbar}t - i\bar{k}_2\cdot\bar{r}}$$

$$\varphi_1 = \frac{U_1(\bar{r})}{\sqrt{V}}e^{-i\frac{E_1}{\hbar}t - i\bar{k}_1\cdot\bar{r}}$$

<u>Sum over periodic cells</u> and integrate over unit cell gives

V = crystal volume

$$\boxed{|p_{21}|^2 = |M_{12}|^2\,\delta(\bar{k}'-\bar{k}'')\frac{(2\pi)^3}{V}}$$

$$|M_{12}|^2 = \left[-\frac{i\hbar}{\Delta}\int_{\text{Cell}} U_c^*(\bar{k}')\,\hat{e}\cdot\nabla U_c(\bar{k}'')\,d\bar{r}'\right]^2$$

Note: Have used $\bar{p}\cdot\bar{E} \to (q\bar{r})\cdot i\omega\bar{A}$; $m\bar{r}i\omega = \bar{p}$

$\to \frac{q}{m}(\bar{A})\cdot\hbar\nabla$

This generalizes the interaction from dipole to $H' = \frac{q}{m}\bar{A}\cdot\bar{p} \to$ <u>all multipoles</u>

Don't forget population difference (Fermi)

Thus

$$\mathrm{Im}\, P = \frac{q^2}{m^2\hbar} |M_{21}|^2 \int \frac{d^3k}{(2\pi)^3} \pi\, \delta(\omega - (E_2 - E_1)/\hbar)$$

$$E_2 - E_1 = \frac{\hbar^2}{m} k^2 + E_G$$

$$\therefore\; d(E_2 - E_1) = \frac{2\hbar^2}{2m}\, dk\, k$$ → allows one to carry out integral

$$4\pi k^2\, dk = \frac{2m}{2\hbar^2} \cdot 4\pi\, k\, d(E_2 - E_1) \qquad d(E_2 - E_1) = d\left(\frac{E_{21}}{\hbar}\right)\hbar$$

$$\therefore\; \mathrm{Im}\, P = \frac{q^2}{m^2\hbar} |M_{21}|^2 \frac{2m}{\hbar^2} \cdot 4\pi^2 \frac{k}{(2\pi)^3}\hbar \Big|_{(E_2 - E_1) = \hbar\omega}$$

$$= \frac{q^2}{m\hbar^2} |M_{21}|^2 \frac{1}{2\pi} \cdot k$$

In terms of dipole matrix element

$$= \frac{q^2}{m\hbar^2} m^2 |x_{cv}|^2 \frac{1}{2\pi} k \quad = \mathrm{Im}\, \chi \epsilon_0 E$$

$$\alpha = 2\frac{\omega}{c} \cdot \frac{1}{2} \chi = \frac{\omega}{c\epsilon_0} \frac{m}{\hbar^2} q^2 |x_{cv}|^2 \frac{1}{2\pi} \sqrt{\frac{m}{\hbar^2}} (\hbar\omega - E_G)^{1/2}$$

Yariv Quantum Electronics 2'nd Ed Eq. 10.7-13

The matrix element can be related to the fluorescent lifetime (8.3-7)

$$q^2 |x_{cv}|^2 = \left(\frac{1}{t_{spon}}\right) \frac{\epsilon \hbar c^3}{2 n^3 \omega^3} (2\pi)$$

$$\alpha = \frac{1}{t_{spon}} \cdot \frac{m}{4\pi} \frac{2\pi}{\omega^2} \frac{c^2}{n^3} \sqrt{\frac{m}{\hbar^2}} \frac{1}{\hbar^2} (\hbar\omega - E_G)^{1/2}$$ — Eq. 15.2-13 Opt Electr.

Classical Radiating Dipole

9.12 of Ulaby

$$W = \sqrt{\frac{\mu}{\epsilon}} \frac{k^2}{32\pi^2} \omega^2 p^2 \times \frac{4}{3} 2\pi = \frac{dE}{dt} = \hbar\omega \frac{dN}{dt} = \frac{\hbar\omega}{t_{spon}}$$

$$\therefore \frac{1}{t_{spon}} = \sqrt{\frac{\mu}{\epsilon}} \frac{\omega^2 \mu\epsilon}{\cancel{32}_{\cancel{8}\,4}\,\pi^{\cancel{2}}} \frac{\omega^{\cancel{2}} p^2}{\hbar\cancel{\omega}} \frac{\cancel{4}}{3} \cdot \cancel{2}\cancel{\pi}$$

$$\frac{1}{t_{spon}} \frac{c^3 \epsilon \hbar}{\omega^3} 12\pi = p^2 \Rightarrow 12\, p^2_{qm}$$

Problem No 4

is certainly true for a pulse after propagation through a fiber, but before compression),

$$a = \frac{1}{2B} \tag{10.6.35}$$

The bandwidth of the spectrally broadened pulse emerging from the fiber can be estimated by:

$$(\Delta\omega)^2 = 4t_p^2(\frac{1}{t_p^4} + B^2) \qquad or \qquad \Delta\omega = 2Bt_p \tag{10.6.36}$$

if it is assumed that $B >> \frac{1}{t_p^2}$, that is that the chirp is large compared to the initial bandwidth of the pulse divided by initial pulse duration. It now remains to be shown, how a quadratic phase distortion of correct sign can be obtained in a dispersive element. .NH 2

10.7 Theory of Compression by Dispersive Elements- Gratings

.LP The compression of pulses with spectra broadened by SPM was first accomplished by Treacy using the grating pair, which has since seen broad application in the optical signal processing area. The anomalous dispersion characteristics arises from a combination of a spectrally dependent path delay and phase shift due to the grating momentum component. It is anomalous because the spectral rate of change of the phase is negative as we will show. The general grating arrangement is illustrated in fig.(9), taken from the initial development given by Tracey [19]. Phase delay arising from different optical paths traveled by the different frequencies is given by:

$$\Phi(\omega) = \frac{\omega}{c}P - 2\pi\frac{d}{\Lambda}tan\theta_r \tag{10.7.1}$$

The second term accounts for a 2π phase shift per ruling in the first order diffraction. This term can be deduced by referring to ??. The incident angle on the first grating is $\theta_i = \gamma$ and the reflected angle is $\theta_r = \theta - \gamma$ for wavelength λ. For a wavelength shift of $d\lambda$ the reflection angle is shifted by $d\theta_r = d\theta$. The path length between the gratings is $G/cos(\theta_r)$. The distance between the reflection points on the second grating at λ and $\lambda + d\lambda$ is then $dl = (G/cos(\theta_r)d\theta)/cos(\theta_r)$. The phase shift $d\phi$ is thus $-dl\frac{2\pi}{d}$. Thus

$$d\phi/d\theta_r = -\frac{2\pi d}{G}/(cos(\theta_r)^2) \tag{10.7.2}$$

Integrating this gives the required extra phase term above. The optical path P can be expressed as,

Work in the -1 order

$$P = AD + DE = \frac{d}{cos\theta_r}(cos(\theta_r + \theta_i) + 1) \tag{10.7.3}$$

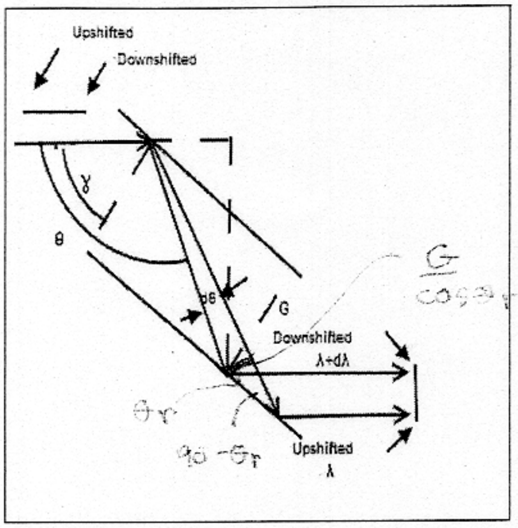
$\theta_i = \gamma$
$\theta_r = \theta - \gamma$

The grating formula reveals that θ_r and θ_i are related through the following equation:

$$sin\theta_r = sin\theta_i + \frac{n\lambda}{\Lambda} \tag{10.7.4}$$

where n is the order of the grating reflection and Λ is the spatial grating period, but here $n = -1$. Since we are interested in dispersion parameter $a = -\frac{\partial^2\Phi}{\partial\omega^2}$, taking second derivative of $\Phi(\omega)$ in eq.(2.2.1) yields:

$$\frac{\partial\Phi}{\partial\omega} = \frac{P}{c} + \frac{\omega}{c}\frac{\partial P}{\partial\omega} - 2\pi\frac{d}{\Lambda}\frac{\partial}{\partial\omega}tan\theta_r = \frac{P}{c} \tag{10.7.5}$$



$\frac{G}{\cos\theta_r} d\theta$

Figure 10.7.1: The Treacy Grating Pair

$\sin\theta_r = \sin\theta_i - \frac{\lambda}{\Lambda}$

$$\frac{\partial^2\Phi}{\partial\omega^2} = \frac{1}{c}\frac{\partial P}{\partial\omega} = \frac{d}{c}\left(\frac{\lambda}{cos^2\theta_r\Lambda}\right)\frac{\partial\theta_r}{\partial\omega} \tag{10.7.6}$$

taking the derivative of the grating equation (eq.(2.2.2)),

$$\frac{\partial\theta_r}{\partial\omega} = -\frac{\lambda}{\omega\Lambda cos\theta_r} \tag{10.7.7}$$

plugging the above result into eq.(2.2.4), yields an expression for the parameter a:

Λ = periodicity of grating

$$\boxed{a = \frac{d\lambda^2}{c\omega\Lambda^2\left(cos^2\theta\right)^{\frac{3}{2}}} = \frac{4\pi^2 dc}{\omega^3\Lambda^2}\left(1 - (sin\theta_i + \frac{\lambda}{\Lambda})^2\right)^{-\frac{3}{2}}} \tag{10.7.8}$$

d = grating separation

A similar expression for the compression parameter a in a prism pair is developed next [20], [21]. A desired prism pair arrangement is shown in fig.(10). Again, the equivalent optical path P is given by an equation similar to the one for a grating pair,

$$P = \Phi\frac{\omega}{c} = n_2 l cos\theta \tag{10.7.9}$$

There is no need for a term accounting for a phase shift unlike the previous case. Parameter a is obtained in a usual manner, by taking the second derivative of the phase:

$$a = \frac{\partial^2\Phi}{\partial\omega^2} = \frac{\lambda^3}{2\pi c^2}\frac{\partial^2 P}{\partial\lambda^2} \tag{10.7.10}$$

Some further manipulation yields a following expression for $\frac{\partial^2 P}{\partial\lambda^2}$:

$$\frac{\partial^2 P}{\partial\lambda^2} = -l\left(cos\theta\left(\frac{\partial\theta}{\partial\lambda}\right)^2 + sin\theta\left(\frac{\partial^2\theta}{\partial\lambda^2}\right)\right) \tag{10.7.11}$$

a must cancel the <u>opposite</u> <u>chirp on</u> the <u>pulse</u> to eliminate phase <u>modulation</u>