Generalization to a Gaussian beam of the A, B, C, D matricies

<u>paraxial rays</u> (φ and h small – all the rays from S are imaged to P)

$\frac{e^{-i\vec{k}\cdot\vec{r}}}{r}$ above is approximately $e^{-ik(\overbrace{x^2+y^2}^{\text{small}}+z^2)^{1/2}}$

$$\simeq \frac{e^{-ikz\left(1+\frac{1}{2}(x^2+y^2)/z^2+\cdots\right)}}{r}$$

$$\simeq \frac{1}{r}\underbrace{e^{-ikz}}_{\text{plane wave}}\ \underbrace{e^{-i\frac{k}{2}(x^2+y^2)/z}}_{\text{spherical curvature}}$$

spherical curvature of phase front. z is the radius of curvature

Generalization : write the exponential as

$$e^{-ikz-i\frac{k}{2}\frac{(x^2+y^2)}{R(z)}-\frac{(x^2+y^2)}{w^2(z)}}$$

we have done two things

a) $z \to R(z)$ (a more general z-dependence for curvature

b) have introduced $e^{\{-(x^2+y^2)/w^2(z)\}}$ to have a Gaussian electric field dependence in the plane ⊥'r to the direction of propagation

can write the field as

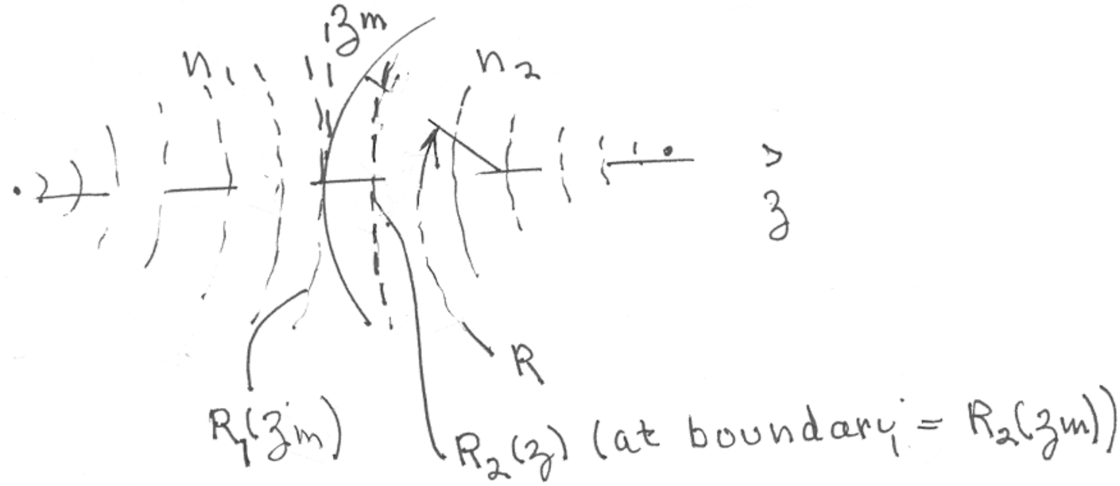
$$e^{-ikz-i\frac{k}{2}\frac{(x^2+y^2)}{q(z)}}$$

with $q(z)$ as a <u>complex curvature</u>

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{2}{k}\frac{1}{w^2(z)}\ ;\quad q(z) = \text{Gaussian beam parameter}$$

We showed that for a curved boundary

Example 1



$$\frac{n_1}{R_1(z_m)} - \frac{n_2}{R_2(z_m)} = \frac{n_2 - n_1}{R\, n_2}$$

but this is

$$\frac{n_1}{q_1} - \frac{n_2}{q_2} \approx \frac{n_2 - n_1}{R\, n_2}$$

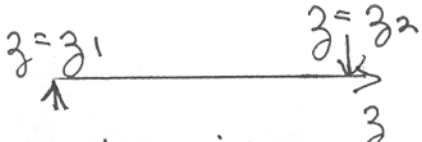
since the "beam width" $w(z)$ is the same on either side of the boundary

The q satisfies the A, B, C, D law

$$q_2 = \frac{q_1 A + B}{q_1 C + D}$$

Simpler Example: Free space propagation of a finite beam

$$q_2 = \frac{q_1 + d}{q_1 \times (0) + 1} = q_1 + d \rightarrow q \text{ increases linearly}$$



$$q_2 = \frac{1}{R(z_2)} - i\frac{2}{k w_2^2(z_2)}$$

$$q_1 = \frac{1}{R_1(z_1)} - i\frac{2}{k w_1^2(z_1)}$$

A gaussian beam with a plane phase front (initially)

generally used for the confocal region

$$\frac{1}{q_1} = \frac{1}{q_0} = \frac{1}{R_0} - \frac{2}{k}\, i/w_0^2 = \frac{2}{k w_0^2}\, i$$

$$q_0 = i\left(\frac{\pi w_0^2}{\lambda}\right) = i z_0 \qquad z_0 = \pi w_0^2/\lambda = k w_0^2/2$$

known as the confocal parameter

Now one can obtain the freely propagating beam,

$$E = e^{-i\frac{k}{2}\frac{(i q_0^* + d)}{|q_0|^2 + d^2}(x^2+y^2)}\; e^{i\omega t - ikz}$$

$$= e^{-i\frac{k}{2}\times\frac{1}{R(z)}(x^2+y^2)}\; e^{-(x^2+y^2)/w^2(z)}$$

$$\times\, e^{i\omega t - ikz}$$

$$R(z) = \frac{|q_0|^2 + d^2}{d}$$

$$= d + z_0^2/d = d\left(1 + z_0^2/d^2\right)$$

$$w^2(z) = \frac{2}{k}\frac{(|q_0|^2 + d^2)}{z_0} = \frac{w_0^2(|z_0|^2 + d^2)}{z_0^2}$$

$$= w_0^2\left(1 + d^2/z_0^2\right)$$

An example of diffraction

Define a diffraction angle (far-field)

$$d \gg z_0 \qquad \left(\frac{w(z)}{d}\right) = w_0/z_0 = \lambda/\pi w_0 = \theta_d$$

So a Gaussian beam in free space has the form

$$\bar{E} = \frac{\bar{E}_0}{r}\, e^{i(\omega t - kz) - \frac{i}{q_0+d}(x^2+y^2)\frac{k}{2}}$$

There are <u>two corrections</u> to be <u>made</u>

1) $\frac{1}{r}$ applies for a spherically symmetric beam.
It is necessary to conserved total power

$$EE^* \propto \int \frac{1}{r^2}\, ds = \int_0^{2\pi}\int_0^{\pi} \frac{1}{r^2}\, r^2 \sin\theta \, d\phi = 4\pi = \underline{\text{constant}}$$

But now we have $e^{-(x^2+y^2)/w^2(z)}$ and in the paraxial approximation it propagates in the z-direction. Thus the power is proportional to

$$EE^* \propto \int e^{-2(x^2+y^2)/w^2(z)}\, \overbrace{2\pi r\, dr}^{\text{cylindrical coordinates}} \qquad r = \sqrt{x^2+y^2}$$

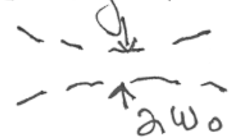
$$= \int e^{-\xi^2}\, 2\pi\, \xi\, d\xi\, \frac{w^2(z)}{2} \qquad \xi = \frac{r}{\sqrt{2w(z)}}$$

$$= \pi\, w^2(z)/2$$

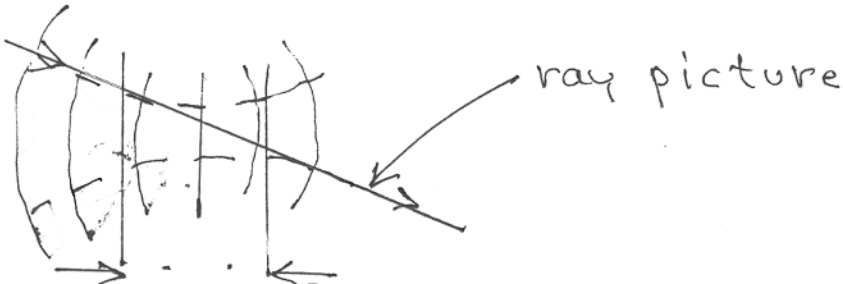
(Something like $\pi w^2(z)$ is to be expected)

<u>Thus</u> to conserve power the $\frac{1}{r}$ is replaced by

$\frac{w_0}{w(z)}$ where w_0 is the radius at the beam waist $(z=0)$
Once again $w(z) = w_0\left(1+\left(\frac{z}{z_0}\right)^2\right)^{1/2}$



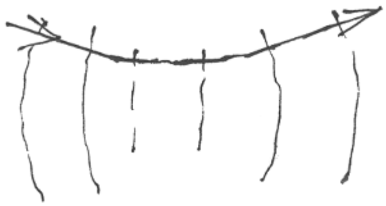
2) In the focal region rays cross



$z_0 = \underline{\pi w_0^2}$

diffraction limit takes over

In actuality we would say ray is $\perp$'r to phase front so why isn't is



The answer is that the <u>phase suffers a π shift</u> through the confocal region. Thus the corrected phase is

$$e^{-\frac{i}{R(z)}(r^2) - i\tan^{-1}(z/z_0)} \quad e^{-i\,s(r,z)}$$

The π phase shift can be interpreted as a crossing of the rays in the confocal region