Ray Equation - What if the refractive index is a function of position (Chapt III - Born + Wolfe)

$$E = E_o(\bar{r})\, e^{-i\omega t}$$

$$E_o(\bar{r}) = \bar{e}\, e^{i k_o S(\bar{r})}$$

$$H_o(\bar{r}) = \bar{h}(\bar{r})\, e^{i k_o S(\bar{r})}$$

Maxwell - first order (neglect $1/k_o$ terms)

$$\nabla(S) \times \bar{h} + \epsilon \bar{e} = 0 \quad (1) \quad - \text{cgs}$$

$$\nabla S \times \bar{e} - \mu \bar{h} = 0 \quad (2) \quad - \text{cgs}$$

$$\bar{e} \cdot \nabla S = 0 \quad (3)$$

$$\bar{h} \cdot \nabla S = 0 \quad (4)$$

Example $\nabla \times (\bar{E}) = -\dfrac{\partial \bar{B}}{\partial t}$

$$\underset{\uparrow\; \omega/c}{ik_o} \nabla S \times \bar{e} + \nabla \times \bar{e} = +i\omega \bar{h}\, \mu$$

drop

keep

$$\nabla S \times \bar{e} - c\mu h = 0$$

$$\therefore \nabla S \times \bar{e} - \sqrt{\frac{\mu_0}{\epsilon_0}}\, h = 0 \quad - \text{MKS} \quad (1)$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = -i\omega\epsilon \bar{E}$$

$$+ik_o \nabla S \times \bar{h} = -i\omega\epsilon\, \bar{e}$$

$$\nabla S \times \bar{h} + n^2 \sqrt{\frac{\epsilon_0}{\mu_0}}\, \bar{e} = 0 \quad (2)$$

(1) in (2)

$$\nabla S \times (\nabla S \times \bar{e}) - n^2 \bar{e} = 0$$

$$\nabla S\, (\nabla S \cdot \bar{e}) - \bar{e}\, (\nabla S \cdot \nabla S) - n^2 \bar{e} = 0$$

$$\nabla \cdot \bar{D} = 0$$

$$\therefore (\nabla S \cdot \nabla S) = n^2 \rightarrow \text{eikonal equation}$$

The ray equation

$\nabla S \rightarrow$ gives the ray

$$\therefore \left(\frac{d\bar{r}}{ds}\right) = \frac{\nabla S}{|\nabla S|} = \frac{\nabla S}{n}$$

$$\therefore n\frac{d\bar{r}}{ds} = \nabla S$$

$$\frac{d}{ds}\left(n\frac{d\bar{r}}{ds}\right) = \frac{d}{ds}(\nabla S) = \left(\frac{d\bar{r}}{ds}\cdot\nabla\right)(\nabla S)$$

ordinary derivative because $y = y(s)$ only

$$\left(\frac{dx}{ds}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{dy}{ds} + \frac{\partial}{\partial z}\frac{dz}{ds}\right) = \frac{d\bar{r}}{ds}\cdot\nabla$$

$$= \frac{1}{n}\nabla S\cdot\nabla(\nabla S)$$

$$= \frac{1}{2n}\nabla[\nabla S\cdot\nabla S]$$

$$= \frac{1}{2n}\nabla[\overbrace{\nabla S^2}^{n^2}]$$

$$= \frac{1}{\cancel{n}}(\nabla n)\cancel{n}$$

$$\therefore \frac{d}{ds}n\frac{d\bar{r}}{ds} = \nabla n$$

Example $\quad n = n_0 - n_2(x^2+y^2)$

paraxial $\quad ds \Rightarrow dz$

$$n_0\frac{d^2\bar{r}}{ds^2} = -2n_2(x\hat{x} + y\hat{y})$$

$$\therefore \frac{d^2x}{ds^2} = -n_2 2x \qquad \frac{d^2y}{dz^2} = -n_2 2y$$

$$x = A\cos\sqrt{2n_2}\,z + B\sin\sqrt{2n_2}\,z \qquad A = x_0$$

$$x' = -A\sqrt{2n_2}\sin\sqrt{2n_2}\,z + B\sqrt{2n_2}\cos\sqrt{2n_2}\,z \qquad B = x_0'/\sqrt{2n_2}$$

$$\therefore \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{2n_2}\,z\,, & \sin\sqrt{2n_2}\,z \big/ \sqrt{2n_2} \\ -\sqrt{2n_2}\sin\sqrt{2n_2}\,z\,, & \cos\sqrt{2n_2}\,z \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$= \begin{pmatrix} A\,, & B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Gaussian beams in graded index materials

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\,\frac{2}{k}\cdot\frac{1}{w(z)}$$

As previously for Gausian beam propagation

$$q(z) = \frac{A(z)\,q(z_0) + B(z)}{C(z)\,q(z_0) + D(z)}$$

(the analytical development of this is in Yariv (Optical Electronics in Modern Communications page 53 in particular)

Example 1): for stationary evolution $q(z) = q(z_0) = q_0$

$$q_0\,(C\,q_0 + D) = A\,q_0 + B$$

$$q_0^2 + q_0\left(\frac{D}{C} - \frac{A}{C}\right) + (\;\; - \frac{B}{C}) \;; \qquad D = A$$

$$q_0^2 + (\;\; - \frac{B}{C}) = 0$$

$$q_0^2 + \frac{-(\cancel{\sin\sqrt{2n_2}\,z})/\sqrt{2n_2}}{-\sqrt{2n_2}\,\cancel{\sin\sqrt{2n_2}\,z}} = 0$$

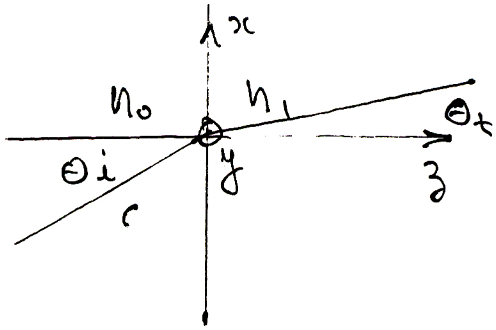
Solution $\quad q_0 = -\dfrac{1}{i\sqrt{2n_2}} = +i/\sqrt{2n_2} = +i\,\dfrac{k}{2}\cdot w_0^2$

beam radius $= w_0 = \sqrt{\dfrac{\sqrt{2}}{k\sqrt{n_2}}}$; Typical numbers

$n = 1.5$, $n_2 \simeq 5\times10^2\ \mathrm{cm}^{-2}$ $\quad k \simeq 2\pi\times10^4$

$$\therefore\; w_0 \simeq \left(\frac{1.4}{2\pi\times10^4\ 25}\right)^{1/2} \mathrm{cm} \simeq \underline{10\ \mu\mathrm{m}} \rightarrow$$

Example 2)



$$\frac{d}{ds}\, n \frac{d\bar{r}}{ds} = \nabla n = (n_1 - n_0)\,\delta(z)\,\hat{z}$$

$\therefore$ for $\hat{x}$ and $\hat{y}$ directions

$$\frac{d}{ds}\left(n \frac{dx}{ds}\right) = 0$$

Integrate across boundary

$$n_0 \frac{dx}{ds}\Big|_{z=0^-} - n_1 \frac{dx}{ds}\Big|_{z=0^+} = 0$$

$n_0 \sin\theta_i - n_1 \sin\theta_t = 0 \rightarrow$ Snell.

<u>Proof of the A B C D law</u> for a Gaussian in a quadratic index media

$$\nabla^2 \tilde{E} - \frac{\omega^2}{c^2}\epsilon\, \tilde{E} = 0$$

Drop

$$\cancel{-k^2 E} + \cancel{\frac{\partial^2 E}{\partial z^2}} - 2ik\frac{\partial E}{\partial z} + \nabla_T^2 E \cancel{- \frac{\omega^2}{c^2}\epsilon_0} + \frac{\omega^2}{c^2}\epsilon_2 (x^2+y^2)\tilde{E} = 0 \qquad k^2 = \frac{\omega^2}{c^2}\epsilon_0$$

$$E \simeq E_0\, e^{-i\frac{k}{2}\frac{1}{q(z)}(x^2+y^2) + i\omega t - ikz - iP(z)}$$

$$\frac{\partial E}{\partial x} \simeq \left(-i\frac{k}{2}\frac{1}{q(z)}\,2x\right)E$$

① $$\frac{\partial^2 E}{\partial x^2} \simeq \left(-i\frac{k}{2}\frac{1}{q(z)}\right)E - k^2\frac{1}{q^2}x^2 E - i2k\frac{1}{q(z)}E$$

② $$\frac{\partial E}{\partial z} \simeq +i\frac{k}{2}\times\frac{1}{q^2(z)}\frac{dq}{dz}x^2 E - i\frac{dP}{dz}E$$ (taken care of ikE term)

Equate x^2 terms $$-\frac{k^2x^2}{q^2(z)} - k^2\times\frac{1}{q^2(z)}\frac{dq}{dz} + \frac{\omega^2}{c^2}\epsilon_2 = 0$$

Thus $$\left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' + \frac{\epsilon_2}{\epsilon_0} = 0$$ $\frac{1}{q} - q = -z \mp q(0)$

Equate x^0 terms $$\frac{dP}{dz} + \frac{i}{q} = 0 \rightarrow P \text{ is a "slave" to } q(z)$$

To Solve : Let

$$q = \frac{s}{s'}$$ (question? Is s the ray coordinate and s' the slope?)

$$\cancel{\left(\frac{s'}{s}\right)^2} + \frac{s''}{s} - \cancel{\frac{1}{s^2}(s')^2} + \frac{\epsilon_2}{\epsilon_0} = 0$$

$$\therefore \; s = a \sin\sqrt{\epsilon_2/\epsilon_0}\, z + b \cos\sqrt{\epsilon_2/\epsilon_0}\, z$$

$$s' = a\sqrt{\epsilon_2/\epsilon_0} \cos\sqrt{\epsilon_2/\epsilon_0}\, z - b\left(\sin\sqrt{\epsilon_2/\epsilon_0}\, z\right)\sqrt{\epsilon_2/\epsilon_0}$$

$$s(0) = b$$

$$s'(0) = a\sqrt{\epsilon_2/\epsilon_0}$$

so as for rays

$$s = s(0)\,A + B\,s'(0)$$

$$s' = s(0)\,C + D\,s'(0)$$

Note on ϵ_2 and n_2

$$n = n_0 - n_2 r^2 \qquad \epsilon = \epsilon_0 - \epsilon_2 r^2$$

$$n^2 = \epsilon/\epsilon_0 \cong n_0^2 - 2 n_0 n_2 r^2 = \frac{\epsilon_0 - \epsilon_2 r^2}{\epsilon_0}$$

$$\therefore \; n_0^2 = 1 \qquad \underline{n_2 = \frac{1}{2}\frac{\epsilon_2}{\epsilon_0 n_0}}$$

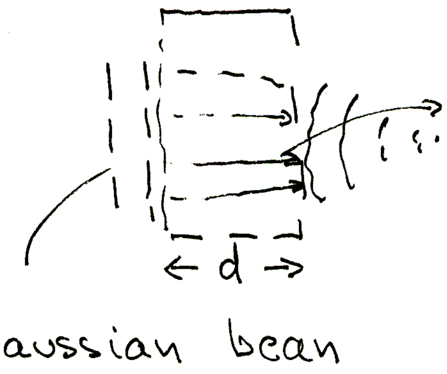
Note on the propagation constant for the stationary wave

$$k = \omega/c \; n_0 \qquad \frac{dP}{dz} = -\frac{i}{q} = -\frac{2}{k w_0^2} = -\sqrt{\frac{2 n_2}{n_0}}$$

so the wave propagates as

$$e^{i\omega t - ikz + iP(z)} = e^{i\omega t - i\beta z}$$

$$\beta = k - \sqrt{\frac{2 n_2}{n_0}}$$



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\frac{\epsilon_2}{\epsilon_0}}\, d, & \sin\sqrt{\frac{\epsilon_2}{\epsilon_0}}\, d \Big/ \sqrt{\frac{\epsilon_2}{\epsilon_0}} \\ -\left(\sin\sqrt{\frac{\epsilon_2}{\epsilon_0}}\, d\right)\sqrt{\frac{\epsilon_2}{\epsilon_0}}, & \cos\sqrt{\frac{\epsilon_2}{\epsilon_0}}\, d \end{pmatrix}$$

$$\frac{1}{f} = \left(\sin\sqrt{\frac{\epsilon_2}{\epsilon_0}}\, d\right)\sqrt{\frac{\epsilon_2}{\epsilon_0}}$$

$$\simeq (25)\sin(25\, d)\ \text{cm}^{-1}$$

$$\therefore f \simeq \frac{1}{25\sin(25\, d)}\ \text{cm}$$

Simplified Derivation of the Ray Equation

(reference Verdyne Lasers)