For Sinusoidal Steady State

$e^{j\omega t - j\bar{k}\cdot r}$ dependence $\quad \boxed{\varphi = \frac{j\bar{k}\cdot\bar{A}}{j\omega\epsilon\mu} = -\frac{\nabla\cdot\bar{A}}{j\omega\mu\epsilon} = +j\omega\frac{\nabla\cdot\bar{A}}{k^2}}$

Far-field $\nabla \to -j\bar{k}$

$$\bar{E} = -j\omega\bar{A} - j\frac{\omega}{k^2}(\nabla)(\nabla\cdot\bar{A}) - \frac{1}{\epsilon}\nabla\times\bar{F}$$

$$= -j\omega\left(1 - \frac{\bar{k}\bar{k}\cdot}{k^2}\right)\bar{A} + \frac{1}{\epsilon}ik\times\bar{F}$$

$\hat{r}\times\hat{\theta} = \hat{\varphi}$

$\hat{r}\times\hat{\varphi} = -\hat{\theta}$

$$A = A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}$$

$$\bar{F} = F_r\hat{r} + F_\theta\hat{\theta} + F_\varphi\hat{\varphi}$$

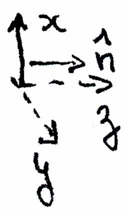
$$\therefore \bar{E} = -j\omega(\bar{A} - A_r\hat{r}) + \frac{1}{\epsilon}ik(F_\theta\hat{\varphi} - F_\varphi\hat{\theta})$$

$$\therefore E_\theta = -j\omega(A_\theta + \eta F_\varphi)$$

$$E_\varphi = -j\omega(A_\varphi - \eta F_\theta)$$

Plane wave Sources

$$\bar{E} = \hat{x}E_{xo} \qquad \bar{H} = \hat{y}H_{yo} = \hat{y}\frac{E_{xo}}{\eta}$$



$$J_x = -H_{yo} = -E_{xo}/\eta$$

$$M_y = \bar{E}\times\hat{n} = -E_{xo}\hat{y}$$

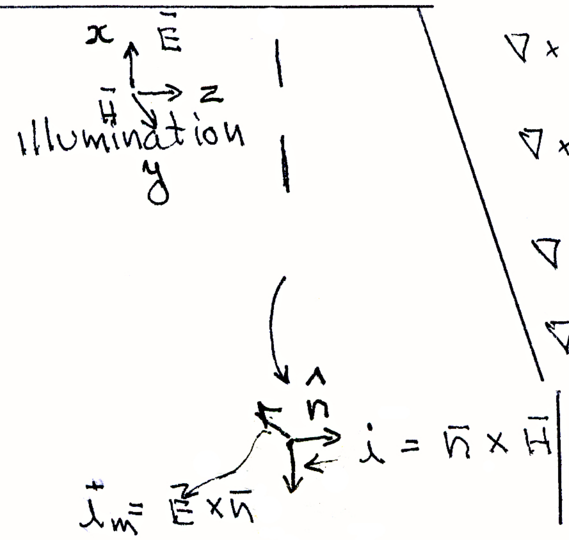
$$\bar{C} = \frac{\epsilon}{4\pi}\int\frac{\bar{E}\times n}{|\bar{r}-\bar{r}'|}e^{j\omega t - k|\bar{r}-\bar{r}'|}d^3\bar{r}'$$

$$\bar{A} = \frac{\mu}{4\pi}\int\frac{\bar{n}\times\bar{H}}{|\bar{r}-\bar{r}'|}e^{j\omega t - k|\bar{r}-\bar{r}'|}d^3\bar{r}'$$

Diffraction **page 608 R. W. V.** We have ① already Treated a modulated susceptibility grating with

$$-\nabla^2 \bar{A} + \mu\epsilon_0 \frac{\partial^2 \bar{A}}{\partial t^2} = \mu_0 \frac{\partial \bar{P}}{\partial t}$$

→ Generalize by introducing magnetic charge and current



$$\nabla \times E = -\frac{\partial \bar{B}}{\partial t} - i_m \quad ①$$

$$\nabla \times \bar{H} = \bar{i} + \frac{\partial \bar{D}}{\partial t} \quad ②$$

$$\nabla \cdot \bar{D} = \rho \quad ③$$

$$\nabla \cdot \bar{B} = \rho_m \quad ④$$

Consider For Example Eq 1

Let $\frac{\partial}{\partial t} D = \epsilon \frac{\partial E}{\partial t} + \frac{\partial P_{pert}}{\partial t}$

$$\frac{\partial B}{\partial t} = \mu \frac{\partial \bar{H}}{\partial t} + \frac{\partial \mu_0 \bar{M}_{pert}}{\partial t}$$

$$\boxed{\begin{aligned} \bar{E} &= -\frac{\partial \bar{A}}{\partial t} - \frac{1}{\epsilon} \nabla \times \bar{C} - \nabla \varphi \\ \bar{H} &= -\frac{\partial \bar{C}}{\partial t} + \frac{1}{\mu} \nabla \times \bar{A} - \nabla \varphi_m \end{aligned}}$$

$$-\cancel{\frac{\partial}{\partial t} \nabla \times \bar{A}} - \frac{\nabla \times \nabla \times \bar{C}}{\epsilon} = \mu \frac{\partial^2 \bar{C}}{\partial t^2} - \cancel{\frac{\partial}{\partial t} \nabla \times \bar{A}} + \mu \frac{\partial}{\partial t} \nabla \varphi_m - i_m - \mu_0 \frac{\partial M_{pert}}{\partial t}$$

vector identity

$$-\nabla(\cancel{\nabla \cdot \bar{C}}) + \nabla^2 \bar{C} - \frac{1}{c^2} \frac{\partial^2 c}{\partial t^2} = -\nabla \cancel{\frac{\partial \varphi_m}{\partial t}} \mu\epsilon - \epsilon \bar{i}_m - \epsilon \mu_0 \frac{\partial M_{pert}}{\partial t}$$

cancel for Lorentz Guage $(\nabla \cdot \bar{C} + \mu\epsilon \frac{\partial \bar{C}}{\partial t} = 0)$

Thus the terms remaining are

$$\boxed{\nabla^2 \bar{C} - \mu\epsilon \frac{\partial^2 \bar{C}}{\partial t^2} = -\epsilon \bar{i}_m - \mu_0 \epsilon \frac{\partial M_{pert}}{\partial t}}$$

Similarly ② gives

$$\nabla \times \left(\cancel{\frac{\partial \bar{C}}{\partial t}} + \frac{\nabla \times \bar{A}}{\mu} - \nabla \varphi_m \right) = \bar{i} + \frac{\partial}{\partial t} \epsilon \left(-\frac{\partial \bar{A}}{\partial t} + \cancel{\frac{\nabla \times \bar{C}}{\epsilon}} - \cancel{\nabla \varphi} \right) + \frac{\partial P_{pert}}{\partial t}$$

cancel

$$\frac{\nabla(\cancel{\nabla \cdot A})}{\mu} - \frac{\nabla^2 A}{\mu} = \bar{i} - \epsilon \frac{\partial^2 \bar{A}}{\partial t^2}$$

$\|$

$\nabla\left(-\epsilon \frac{\partial \varphi}{\partial t}\right)$ from the Guage condition $(\nabla \cdot \bar{A} + \mu\epsilon \frac{\partial \varphi}{\partial t} = 0)$

Assume $P_{pert} = M_{pert} = 0$

Thus

$$\boxed{\nabla^2 \bar{A} - \mu\epsilon \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{i} - \mu \frac{\partial \bar{P}_{pert}}{\partial t}}$$

(1)

line drives the dipole and, in parallel electrically, a two-wire transmission line consisting of its own outer conductor and the parallel support for the dipole. By making the distance to the shorting termination $\lambda/4$, the impedance of the two-wire line, as seen at the drive point, is infinite, so the dipole has equal currents in its two arms and is thus balanced with respect to ground. More complicated matching circuits[13] may be necessary to maintain matching over the bandwidth of broad-band systems.

Taken From Fields & Waves in Comm. Electronics (Ramo, Whinnery, Van Duzer 2nd Ed)

RADIATION FROM FIELDS OVER AN APERTURE

12.12 Fields as Sources of Radiation

For wire antennas, it is fairly natural to assume a current distribution over the antenna, and to consider the current elements as the sources of radiation. For other antennas, such as the electromagnetic horn, the slot antennas, the parabolic reflectors, the lens directors, and all optical systems, it is more natural to think in terms of the fields as sources. Huygen's principle states that any wave front can be considered the source of secondary waves that add to produce distant wave fronts. Thus the knowledge (or assumption) of field distribution over an aperture should yield the distant field. We wish now to make a quantitative statement of this general principle. There are several possible approaches, but we shall start with one that considers the fields as arising from equivalent current sheets in the aperture. This approach provides good physical pictures, and builds directly on the formulations already developed in this chapter.

Consider the aperture in a plane, Fig. 12.12, with sources to the left and the field desired in the region to the right. For present purposes the plane may be considered absorbing, so that it has no fields or currents except in the aperture. The exact boundary-value problem can be solved in only a few cases, but it is often possible to make a reasonable estimate of the aperture fields, just as was done for antenna currents in the radiators considered previously. We thus assume that tangential components of fields $E_t(x', y')$ and $H_t(x', y')$ are known in the aperture. Although these fields arise from sources to the left, they may be considered to be produced by equivalent sources located in the aperture plane. In particular, we have seen on many occasions that the relation between tangential magnetic field and a surface current is

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H} \tag{1}$$

Similarly the tangential electric field can be related to a term which we can interpret as a surface magnetic current $\mathbf{M}_s$,

$$\mathbf{M}_s = -\hat{\mathbf{n}} \times \mathbf{E} \tag{2}$$

[13] D. F. Bowman, in *Antenna Engineering Handbook*, H. Jasik, Ed., McGraw-Hill, New York, 1961.

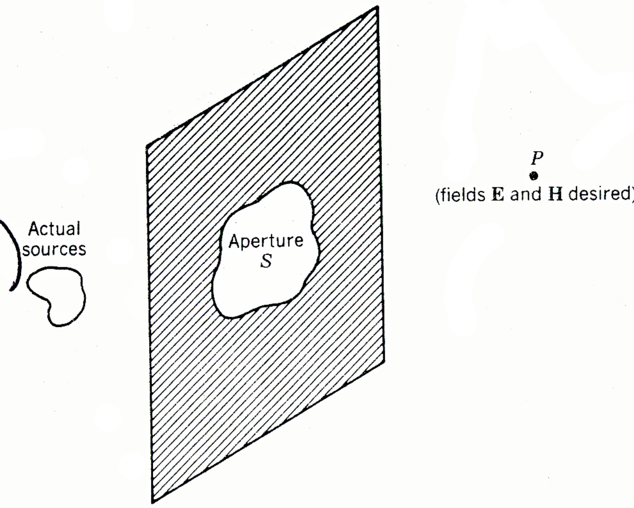


Fig. 12.12 Radiation through an aperature.

Note that (1) and (2) individually assume that the tangential fields H_t and E_t on the left side of the aperture plane are zero. Jointly, $\mathbf{J}_s$ and $\mathbf{M}_s$ produce fields that satisfy that assumption while producing the required fields on the right side. Whether or not true magnetic charges and currents are found in nature, the concept of such equivalent sources is a useful one in a variety of circumstances including this one. Maxwell's equations, augmented by magnetic charge density ρ_m and magnetic surface current $\mathbf{J}_m$, become

$$\nabla \cdot \mathbf{D} = \rho_e$$

$$\nabla \cdot \mathbf{B} = \rho_m$$

$$\nabla \times \mathbf{E} = -\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_e + \frac{\partial \mathbf{D}}{\partial t}$$

A second retarded potential $\mathbf{F}$ can be defined in terms of the magnetic sources as $\mathbf{A}$ was related to electric sources. For a homogeneous isotropic medium, the relations in terms of surface currents are

$$\mathbf{A} = \mu \int_{S'} \frac{\mathbf{J}_s e^{-jkr}}{4\pi r}\, dS', \qquad \mathbf{F} = \varepsilon \int_{S'} \frac{\mathbf{M}_s e^{-jkr}}{4\pi r}\, dS' \tag{4}$$

and the fields in terms of these potentials are

$$\mathbf{E} = -j\omega\mathbf{A} - \frac{j\omega}{k^2}\nabla(\nabla \cdot \mathbf{A}) - \frac{1}{\varepsilon}\nabla \times \mathbf{F} \tag{5}$$

$$\mathbf{H} = -j\omega\mathbf{F} - \frac{j\omega}{k^2}\nabla(\nabla \cdot \mathbf{F}) + \frac{1}{\mu}\nabla \times \mathbf{A} \tag{6}$$

In order to calculate fields at any point in the region to the right, the actual sources in the left-hand region are replaced by equivalent sources in the aperture defined by (1) and (2) and the calculation made through the set (4) to (6). If the plane is conducting and not absorbing, any actual currents flowing on the right side of the plane must be added in the calculation of **A** in addition to the equivalent sources in the aperture, although this correction is often negligible.

If we are concerned only with the radiation field, the usual approximations appropriate to great distances can be employed, and the general formulation of Sec. 12.4 extended. A magnetic radiation vector **L** may be related to vector potential **F** as **N** was to **A**. Thus, consistent with the assumptions listed previously,

$$\mathbf{A} = \mu\frac{e^{-jkr}}{4\pi r}\mathbf{N}, \qquad \mathbf{F} = \varepsilon\frac{e^{-jkr}}{4\pi r}\mathbf{L} \tag{7}$$

where

$$\mathbf{N} = \int_{S'} \mathbf{J}_s e^{jkr'\cos\psi}\, dS', \qquad \mathbf{L} = \int_{S'} \mathbf{M}_s e^{jkr'\cos\psi}\, dS' \tag{8}$$

Here r' and ψ are as defined in Sec. 12.4.

If electric and magnetic field components are now written in the usual way in terms of these two vector potentials, the only components not decreasing faster than $(1/r)$ are

$$E_\theta = \eta H_\phi = -j\frac{e^{-jkr}}{2\lambda r}(\eta N_\theta + L_\phi) \tag{9}$$

$$E_\phi = -\eta H_\theta = j\frac{e^{-jkr}}{2\lambda r}(-\eta N_\phi + L_\theta) \tag{10}$$

So the radiation intensity, or power per unit solid angle, is

$$K = \frac{\eta}{8\lambda^2}\left[\left|N_\theta + \frac{L_\phi}{\eta}\right|^2 + \left|N_\phi - \frac{L_\theta}{\eta}\right|^2\right] \tag{11}$$

In optics, this far-zone field is called the region of *Fraunhofer diffraction*. For the near zone, or region of *Fresnel diffraction*, better approximations to r are necessary.[14]

[14] M. Born and E. Wolf, *Principles of Optics*, 6th ed., Pergamon Press, Oxford, 1980, p. 660.

One does not need to use the concept of magnetic currents explicitly since the expressions (1) and (2) may be substituted in (4) and then one works with the aperture fields directly. In fact Stratton and Chu derived such expressions by direct integration of the field equations.[15] For many purposes, however, the physical pictures inherent in the formulation with magnetic currents is helpful, because of the duality with fields from electric currents.

12.13 Plane Wave Sources

The radiation vector and radiation intensity may be calculated for a differential surface element on a uniform plane wave. Such an element might be considered the elemental radiating source in radiation calculations from field distributions, as was the differential current element for radiation calculations from current distributions.

The plane wave source, that is, one that produces **E** and **H** of constant direction, normal to each other, and in the ratio of magnitudes η over the area of interest may be replaced by equivalent electric and magnetic current sheets over that area, Fig. 12.13*a*.

If

$$\mathbf{E} = \hat{\mathbf{x}}E_{x0} \quad \text{and} \quad \mathbf{H} = \hat{\mathbf{y}}H_{y0} = \frac{\hat{\mathbf{y}}E_{x0}}{\eta} \tag{1}$$

the equivalent current sheets are

$$J_x = -H_{y0} = -E_{x0}/\eta, \qquad M_y = -E_{x0} \tag{2}$$

If this is a source of infinitesimal area dS (actually it need only be small compared with wavelength for following results to hold true), the radiation vectors **N** and **L** given by Eqs. 12.12(8) are

$$N_x = -\frac{E_{x0}\, dS}{\eta}, \qquad L_y = -E_{x0}\, dS$$

since the source element is at the origin of coordinates. The components in spherical coordinates are from Eq. 12.4(14):

$$N_\theta = -\frac{E_{x0}\, dS}{\eta}\cos\phi\cos\theta, \qquad N_\phi = \frac{E_{x0}\, dS}{\eta}\sin\phi \tag{3}$$

$$L_\theta = -E_{x0}\, dS\sin\phi\cos\theta, \qquad L_\phi = -E_{x0}\, dS\cos\phi$$

from which far-zone fields, by Eqs. 12.12(9)–(10) are

$$E_\theta = \frac{jE_{x0}\, dS\, e^{-jkr}}{2\lambda r}(1 + \cos\theta)\cos\phi \tag{4}$$

$$E_\phi = -j\frac{E_{x0}\, dS\, e^{-jkr}}{2\lambda r}(1 + \cos\theta)\sin\phi \tag{5}$$

[15] J. A. Stratton, Ref. 8, Sec. 8.14.

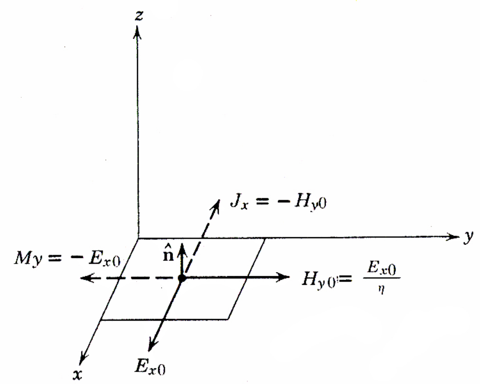


Fig. 12.13*a* Small plane wave source and equivalent current sheets.

and radiation intensity, by Eq. 12.12(11) is

$$K = \frac{E_{x0}^2(dS)^2}{8\eta\lambda^2}[(-\cos\phi\cos\theta - \cos\phi)^2 + (\sin\phi + \sin\phi\cos\theta)^2]$$

$$K = \frac{E_{x0}^2(dS)^2}{2\eta\lambda^2}\cos^4\frac{\theta}{2} \tag{6}$$

Paraxial Approximation for Large Apertures In most cases of radiation from large apertures and horns, we are interested in small values of θ (*paraxial approximation*) so that $\cos\theta$ may be replaced by unity. Let us also convert to rectangular components in the radiation field,

$$E_x = E_\theta\cos\theta\cos\phi - E_\phi\sin\phi \approx E_\theta\cos\phi - E_\phi\sin\phi \tag{7}$$

$$E_y = E_\theta\cos\theta\sin\phi + E_\phi\cos\phi \approx E_\theta\sin\phi + E_\phi\cos\phi \tag{8}$$

Substituting (4) and (5) with $\cos\theta = 1$,

$$E_x \approx j\frac{E_{x0}\,dS\,e^{-jkr}}{\lambda r}, \qquad E_y \approx 0 \tag{9}$$

Thus in the paraxial approximation, the radiation field is in the direction of the source field and we may drop the subscript on E. In order to integrate over the aperture shown in Fig. 12.13*b*, we must first adapt the above results to an element at an arbitrary location x', y' in the aperture. This can be done using Eqs. 12.12(8) but a direct approach is more convenient here. Within the paraxial approximation, (9) gives the field produced by the element at x', y' if E_{x0} is replaced by $E(x', y')$ and r by r''. Now integrating over the given aperture distribution,

$$E(x, y, z) = \frac{j}{\lambda}\int_{S'}\frac{E(x', y')e^{-jkr''}}{r''}\,dx'\,dy' \tag{10}$$

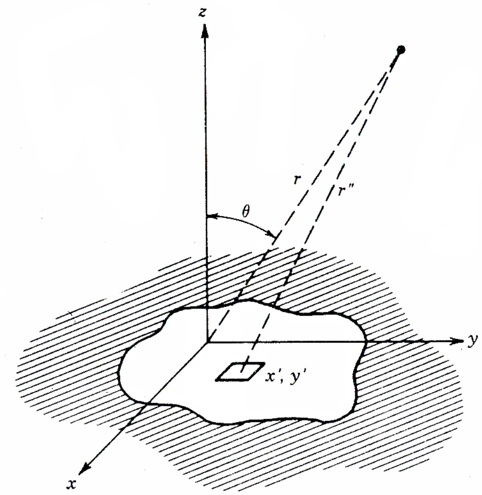


Fig. 12.13*b* Plane wave element as part of a plane aperature located at $z = 0$.

where r'' is distance between the element and the field point and may be approximated for the far zone by the binomial expansion,

$$r'' = [(x - x')^2 + (y - y')^2 + z^2]^{1/2} \approx r - (xx' + yy')/r \tag{11}$$

Also, as in other radiation problems, the difference between r and r'' is important only to phase and not to amplitude. Thus (10) becomes

$$E(x, y, z) = \frac{je^{-jkr}}{\lambda r}\int_{S'} E(x', y')e^{jk(xx' + yy')/r}\,dx'\,dy' \tag{12}$$

This is a standard form of diffraction for the Fraunhofer region. Moreover, (12) can be recognized as a two-dimensional Fourier integral (Sec. 7.11), so in the paraxial approximation, the *far-zone field is the Fourier transform of the aperture field.*

12.14 Examples of Radiating Apertures Excited by Plane Waves

The expressions developed in the preceding section will now be applied to several important examples. It should be remembered that **E** and **H** at any point in the aperture are assumed to be related as in a plane wave (though strength may vary over the aperture), that we are ignoring contributions from any induced currents outside of the aperture, and that we are restricting ourselves to angles near the polar axis so the paraxial approximation can be used. These assumptions are best satisfied for apertures large compared to wavelength.

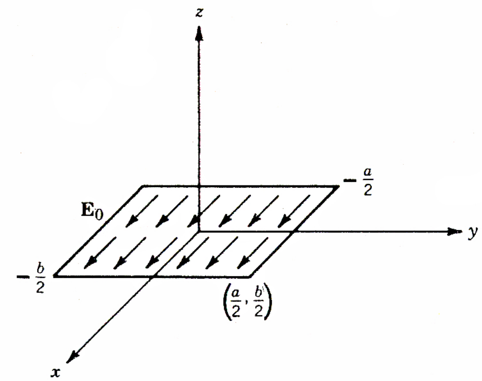


Fig. 12.14*a* Uniformly illuminated aperture.

Rectangular Aperture with Uniform Illumination Consider first a rectangular aperture as in Fig. 12.14*a* with uniform illumination E_0 over the aperture. Equation 12.13(12) becomes

$$E(x, y, z) = \frac{je^{-jkr}}{\lambda r} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_0 \, e^{jkxx'/r} e^{jkyy'/r} \, dx' \, dy' \tag{1}$$

The integrations are readily performed to give

$$E(x, y, z) = \frac{je^{-jkr}}{\lambda r} E_0 ab \operatorname{sinc}\left(\frac{kax}{2r}\right) \operatorname{sinc}\left(\frac{kby}{2r}\right) \tag{2}$$

where sinc $X \triangleq (\sin X)/X$. The pattern in the plane $y = 0$ has nulls of the field at

$$(\theta)_{\text{null}} \approx \left(\frac{x}{r}\right)_{\text{null}} = \frac{m\lambda}{a}, \qquad m = 1, 2, 3, \ldots \tag{3}$$

The pattern in the $x = 0$ plane is of the same form with y replacing x and b replacing a. Thus as expected, the primary radiating lobe becomes narrower as the aperture dimensions become larger in comparison with wavelength.

Directivity of this aperture is readily calculated since the maximum field is that at $x = 0$, $y = 0$ so that power radiated by an isotropic radiator of this field strength is

$$(W)_{\text{isotropic}} = 4\pi r^2 \frac{|E_{\max}|^2}{2\eta} = \frac{2\pi E_0^2 a^2 b^2}{\lambda^2 \eta} \tag{4}$$

Actual power radiated may be calculated from the fields in the aperture neglecting currents in the surrounding aperture plane:

$$W = \frac{E_0^2}{2\eta} ab \tag{5}$$

Thus directivity is

$$(g_d)_{\max} = \frac{W_{\text{isotropic}}}{W} = \frac{4\pi ab}{\lambda^2} = \frac{4\pi(\text{area})}{\lambda^2} \tag{6}$$

which is most accurate for large apertures. The relation between directivity and area is very important and will be found for other aperture radiators.

Long Slit A long slit is often used to demonstrate diffraction effects with visible light, and if uniformly illuminated, is just a special case of the above. With b very long in comparison with wavelength, one would expect an extremely thin pattern in the plane $x = 0$, and this is the case if the light used is coherent (in phase) over the length of the slit. In many demonstrations this is not the case, so that there are no appreciable interferences and, therefore, no variations in the long y direction, but only over the shorter x direction. The resulting radiation intensity can be found as proportional to $|E|^2$ using only the variable x in (2). Since this problem has only two-dimensional symmetry, it is advantageous to express it in cylindrical coordinates with ϕ measured about the y axis and $\phi = 0$ along the x axis. Then in the approximation $x/r \approx \cos\phi$ radiation intensity is

$$K = A\left\{\frac{\sin[(ka/2)\cos\phi]}{(ka/2)\cos\phi}\right\}^2 \tag{7}$$

where A is a constant related to the illumination of the aperture. The resulting pattern is shown in Fig. 12.14*b*. We will return to this point, showing patterns using visible light when we consider multiple slits in Sec. 12.20.

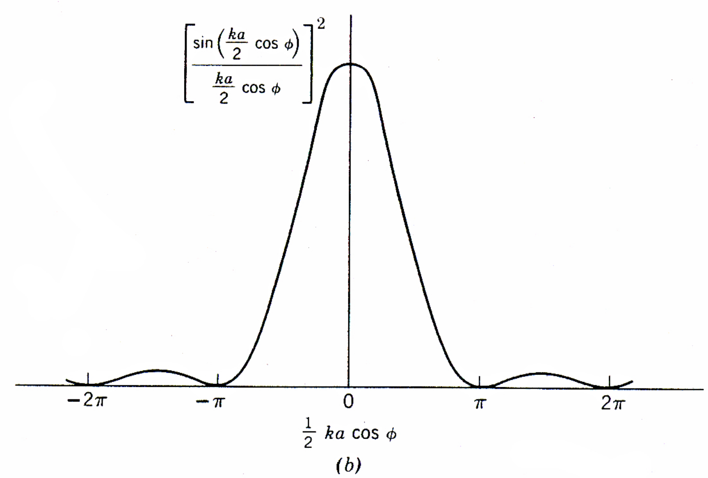


Fig. 12.14*b* Radiation intensity variation in one coordinate for a rectangular aperture.

Circular Aperture with Uniform Illumination In considering the circular aperture, let us first transform to polar coordinates. Let

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad x' = r'\cos\phi', \qquad y' = r'\sin\phi'$$

Then Eq. 12.13(12) becomes

$$E(r, \theta, \phi) = \frac{je^{-jkr}}{\lambda r}\int_0^{2\pi}\int_0^a E(r', \phi')e^{jkr'\sin\theta\cos(\phi-\phi')}r'\,dr'\,d\phi' \tag{8}$$

If $E(r', \phi')$ is independent of ϕ', $E(r, \theta, \phi)$ is independent of ϕ, so we may take $\phi = 0$ and for the ϕ' integration may use the integral[16]

$$\int_0^{2\pi} e^{jq\cos\psi}\,d\psi = 2\pi J_0(q) \tag{9}$$

where $J_0(q)$ is a zeroth order Bessel function. Then

$$E(r, \theta) = \frac{2\pi je^{-jkr}}{\lambda r}\int_0^a E(r')J_0(kr'\sin\theta)r'\,dr' \tag{10}$$

If $E(r')$ is a constant E_0, Eq. 7.15(20) may be utilized to give

$$E(r, \theta) = \frac{2\pi je^{-jkr}}{\lambda r}E_0 a^2\frac{J_1(ka\sin\theta)}{ka\sin\theta} \tag{11}$$

The magnitude of the last term in $E(r, \theta)$ is plotted as a function of $ka\sin\theta$ in Fig. 12.14*c*. The first null is reached at an angle

$$\theta_0 = \sin^{-1}\left(\frac{3.83\lambda}{2\pi a}\right) \approx \frac{0.61\lambda}{a} \tag{12}$$

Directivity of the circular aperture may be calculated in the same fashion as for the rectangular aperture, taking the ratio of power from an isotropic radiator having intensity the same as that at $\theta = 0$, to that of the plane wave power through the actual aperture,

$$(g_d)_{\max} = \frac{(4\pi r^2/2\eta)(2\pi E_0 a^2/2\lambda r)^2}{(\pi a^2 E_0^2/2\eta)} = \left(\frac{4\pi}{\lambda^2}\right)(\pi a^2) \tag{13}$$

Directivity is thus related to aperture area exactly as for the rectangular aperture in (6).

Circular Aperture with Gaussian Illumination In order to compare the effects of a tapered illumination with a uniform illumination, we next consider a gaussian distribution of field over the circular aperture,

$$E(r', \phi') = E_0 e^{-(r'/w)^2} = E_0 e^{-(x'^2+y'^2)/w^2} \tag{14}$$

[16] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 4th ed., Academic Press, New York, 1965, 8.411(7).

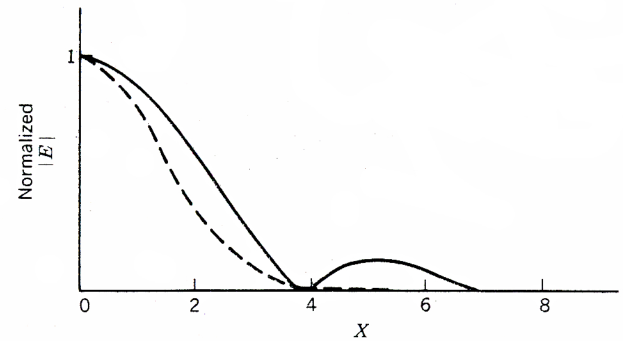


Fig. 12.14*c* Normalized electric field magnitude for circular aperture with uniform illumination (solid) and gaussian illumination (dashed). $X = ka\sin\theta$ for uniform case and $X = kw\sin\theta$ for gaussian illumination, with a and w defined in text.

but assume that **E** and **H** are related as in a plane wave. The quantity w is beam radius to the $1/e$ point in field. We assume that w is small compared with aperture radius a so that the latter may be assumed effectively infinite. Since field is a function of r' only, we could use (10) directly, but it is simpler to return to the rectangular form, Eq. 12.13(12):

$$E(x, y, z) = \frac{je^{-jkr}}{\lambda r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0 e^{-(x'^2/w^2 - jkxx'/r)}e^{-(y'^2/w^2 - jkyy'/r)}\,dx'\,dy' \tag{15}$$

The imaginary parts of the complex exponentials are odd functions and integrate to zero, so

$$E(x, y, z) = \frac{4je^{-jkr}}{\lambda r}\int_0^{\infty}\int_0^{\infty} E_0\left[e^{(-x'^2/w^2)}\cos\frac{kxx'}{r}\right]\left[e^{(-y'^2/w^2)}\cos\frac{kyy'}{r}\right]dx'\,dy' \tag{16}$$

This integral is tabulated[17] and gives

$$E(x, y, z) = \frac{je^{-jkr}}{\lambda r}E_0\pi w^2 e^{(-k^2(x^2+y^2)w^2/4r^2)} = \frac{je^{-jkr}}{\lambda r}E_0\pi w^2 e^{-(k^2w^2\sin^2\theta)/4} \tag{17}$$

The angle θ_0 in the direction where the radiation field is e^{-1} of its value at $\theta = 0$ is

$$\theta_0 = \sin^{-1}\left(\frac{2}{kw}\right) \approx \frac{2}{kw} = \frac{\lambda}{\pi w} \tag{18}$$

The pattern as a function of $kw\sin\theta$ is compared with that for the case of uniform illumination in Fig. 12.14*c*. It is seen that the tapered illumination eliminates the side lobes present in the uniformly illuminated example.

[17] Gradshteyn and Ryzhik, Ref. 16; 3.896(4).