Basic Postulates of Quantum Mechanics and Dirac Notation

Postulates (Ref. Margeneau & Murphy Math. of Physics and Chemestry)

1) There exist a wave function (Dirac ket $|\psi\rangle$)
2) To every observable there correspouds an operator
3) The only possible measurements of an observable are the eigen values

$$P\psi = P_{num}\psi \qquad P|\psi\rangle = P_{num}|\psi\rangle$$

4) when a system is in a state ϕ the expected mean of a sequence of measurements whose operator is given by P is

$$\bar{P} = \int \phi^* P \phi \, dV$$

Some of the intricicies of Dirac Notation (Reference Messiah QM Vol I page 377)

$|\psi\rangle$ - ket or state-vector

- wave function coord space $\langle x|\psi\rangle = \psi(x)$
- wave function momentum space $\langle p|\psi\rangle = \psi(p)$

Dirac Notation $|\psi\rangle \rightarrow$ specifies the state of the system

Examples $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ for spin "up" and spin "down"

position eigenket $|x'\rangle$

$$x_{op}|x'\rangle = x'_{num}|x'\rangle$$

$$\langle x|x'\rangle = \delta(x-x') \leftarrow \text{a diagonal continuous matrix}$$

Discrete

$$x_{op}|x_j\rangle = x_{j\,num}|x_j\rangle$$

$$\langle x_i|x_j\rangle = \delta_{ij}$$

Similarly: For Momentum States

$$\langle p | p' \rangle = \delta(p - p')$$

Displacement operator

$$e^{i p/\hbar\, \epsilon} | x \rangle = | x + \epsilon \rangle$$ - by Taylor expansion with $p = \hbar/i \frac{\partial}{\partial x}$

$$\langle p | x + \epsilon \rangle = \langle p | e^{i p/\hbar\, \epsilon} | x \rangle$$ - Take ϵ small and expand

$$= \langle p | 1 + i p/\hbar\, \epsilon | x \rangle$$

$$\therefore \frac{\langle p | x+\epsilon \rangle - \langle p | x \rangle}{\epsilon} = i/\hbar \langle p | p | x \rangle = \frac{\partial(\langle p | x \rangle)}{\partial x}$$

Also

$$\frac{\langle x | \psi(x+\epsilon) \rangle - \langle p | \psi(x) \rangle}{\epsilon} = i/\hbar \langle x | p | \psi(x) \rangle$$

$$\boxed{\frac{\hbar}{i} \frac{\partial(\psi(x))}{\partial x} = \langle x | \frac{i}{\hbar} p | \psi(x) \rangle}$$

In addition

$$\langle p | x | \psi \rangle$$

$$= \int \langle p | x' \rangle dx' \langle x' | x | \psi \rangle$$

$$= \int e^{i p/\hbar\, x'} \langle x' | x | x'' \rangle dx'' \langle x'' | \psi \rangle$$

$$= \int e^{i p/\hbar\, x'} x' \delta(x' - x'') \psi(x'')\, dx'\, dx''$$

$$= \int e^{i p/\hbar\, x'} \psi(x')\, x'\, dx'$$

$$= \hbar/i \frac{\partial}{\partial p} \int e^{i p/\hbar\, x'} \psi(x')\, dx'$$

$$= \hbar/i \frac{\partial}{\partial p}(\psi(p))$$

Representations - Wave mechanics

For $H = \frac{p^2}{2m} + V(q)$ ← potential which is a function of position

↑ Messiah often uses q for coordinates

completeness relation

$$\langle q' | V(q) | \psi \rangle = \int \langle q' | V(q) \boxed{|q''\rangle dq'' \langle q''|} \psi \rangle$$

operator ↓ $V(q_{op}) |q''\rangle$ ← eigenket of position

$= V(q'') |q''\rangle = |q''\rangle V(q'')$

↑ Number

$$= \int \langle q' | q'' \rangle V(q'') dq'' \langle q'' | \psi \rangle$$

$$= \int \delta(q' - q'') V(q'') dq'' \psi(q'')$$

$q|q''\rangle = q''|q''\rangle \Rightarrow$ diagonal in q

$\langle q' | q | q'' \rangle = q'' \delta(q' - q'') \Rightarrow$ representation in which

↑ number

q is diagonal

Thus $\langle q' | V(q) | \psi \rangle = V(q') \psi(q')$

Wave equations follow immediately

$$H(p, x) |\psi\rangle = E_{op} |\psi\rangle = \hbar i \frac{\partial}{\partial t} |\psi\rangle$$

↑ momentum

$$\langle x | H(p, x) |\psi\rangle = \hbar i \frac{\partial}{\partial t} \langle x | \psi \rangle = \hbar i \frac{\partial}{\partial t} \psi(x)$$

From the above and from bottom $\frac{1}{3}$ of page 2

$$\langle x | H(p, x) |\psi\rangle = H\left(p = \frac{\hbar}{i} \nabla, x\right) \underbrace{\langle x | \psi \rangle}_{= \psi(x)} = \hbar i \frac{\partial \psi(x)}{\partial t}$$

Similarly if we project to $\langle p|$

$$\langle p | H(p, x) |\psi\rangle = \hbar i \frac{\partial \psi(p)}{\partial t}$$

$$= H\left(p, x = \frac{\hbar}{i} \frac{\partial}{\partial p}\right) \psi(p)$$

Verification of the commutation relations.

$$\langle q' | qp - pq | q'' \rangle = (q' - q'') \langle q' | p | q'' \rangle$$

$$= (q' - q'') \frac{\hbar}{i} \frac{\partial}{\partial q'} \langle q' | q'' \rangle$$

$$= (q' - q'') \hbar/i \frac{\partial}{\partial q'} \delta(q' - q'')$$

$$= \hbar i \, \delta(q' - q'')$$

where we have used the δ function property

$$(q' - q'') \frac{\partial}{\partial q'} \delta(q' - q'') = -\delta(q' - q'')$$

Thus since $\langle q' | q'' \rangle = \delta(q' - q'')$; $qp - pq = i\hbar$

Quantization of the e.m. field. Follows in exactly the same manner (one way to do this)

$$H = \tfrac{1}{2} \epsilon E^2 + \tfrac{1}{2} \mu H^2$$

Introduce a state vector $|\psi\rangle$, obtain the wave function either in E space or H space by $\langle E | \psi \rangle = \psi(E)$ and $\langle H | \psi \rangle = \psi(H)$

Dynamical Equation

$$H |\psi\rangle = i\hbar \frac{\partial |\psi\rangle}{\partial t} = E_{en} |\psi\rangle$$

eigen energy states

The projection onto E or H can be made once the displacement operator is known

$$e^{i\left(\frac{EH}{2} \frac{VK}{\hbar\omega^2}\right)} \leftarrow \text{Dimensionless}$$

V = mode volume

Complications:

a) Polarization

b) Definition of mode

but the principle is the same