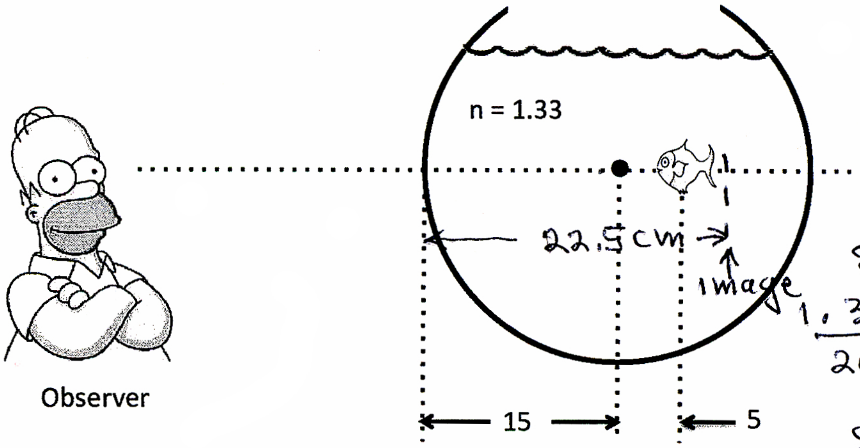
I. Problem 1

A. The paraxial approximation

Briefly explain in words and or pictures what is meant by the paraxial approximation.

B. The fish bowl problem

A spherical fish bowl has a diameter of 30 cm (radius = 15 cm) and is filled with water (n = 1.33). As shown in the diagram below, a small fish (small enough to be considered a point object) is located on-axis 5 cm to the right of the center of the fish bowl . An observer (in air) is looking into the fish bowl from the left. Where is the image of the fish that the observer will perceive as the true object? Give this location with respect to the front surface of the fish bowl.



$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

so

$$\frac{1.33}{20} + \frac{1}{s_i} = \frac{1-1.33}{-15}$$

$$s_i = -22.5 \text{ cm}$$

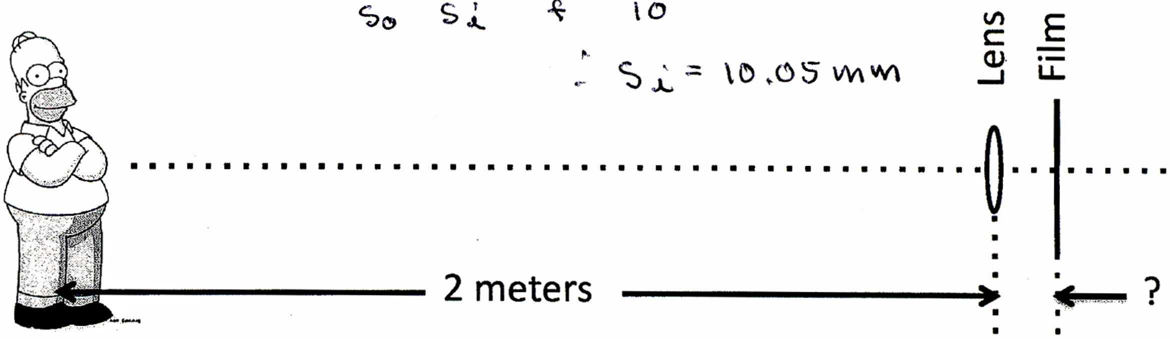
22.5 to the right of front surface

C. Thin lens imaging

Part 1: A thin camera lens with index n = 1.5 has a focal length of 10 mm in air. You are trying to take a picture (in air) of an object that is 2 m away from the camera lens. How far behind the lens should the film be placed? NOTE: you want to place the film at the position where the lens forms an image of the object.

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} = \frac{1}{10} \qquad s_o = 2000 \text{ mm}$$

$$\therefore s_i = 10.05 \text{ mm}$$



10.05 mm

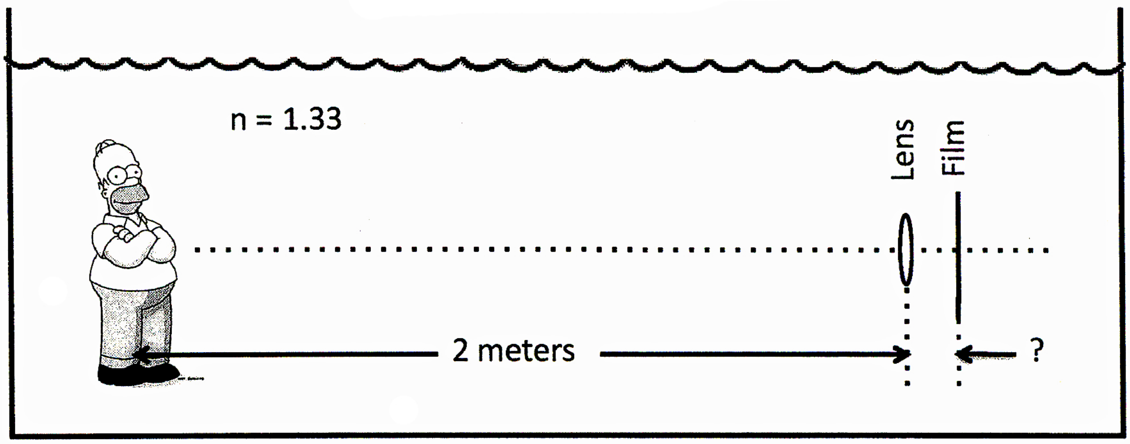
Part 2: Now the camera and the object jump into a pool of water (n = 1.33). The object is still physically 2 m from the thin lens of the camera (i.e., a 2 meter long stick could just be placed between the camera lens and the object). Now how far behind the lens should the (waterproof!) film be placed?

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{n_g - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \frac{1}{f} = \frac{n_g - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{2000} + \frac{1}{s_i} = \frac{1.5-1.33}{1.33}\left(\frac{1}{10} \times \frac{1}{.5}\right) \qquad \frac{1}{10} = \left(\frac{1.5-1}{1}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

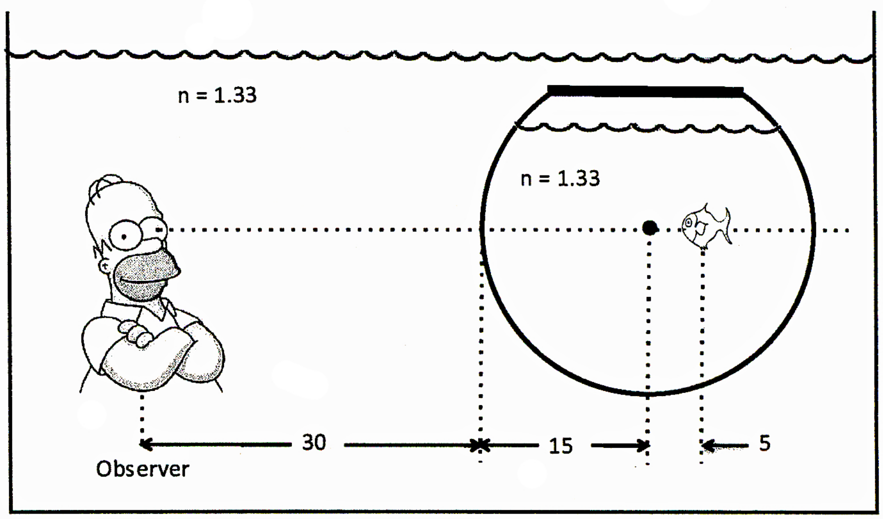
substitute

$$s_i = 39.9 \text{ mm}$$



D. Revisiting the fish bowl problem

You now put a lid on the fish bowl so it is sealed. The object and fish bowl jump into a large pool of water (the same type of water in the fish bowl, n = 1.33). The fish is lazy and has not moved from it's original location in part B. The front surface of the fish bowl is 30 cm from the observer's eye. Where is the image of the fish that the observer perceives as the true object? Give this location with respect to the front surface of the fish bowl.



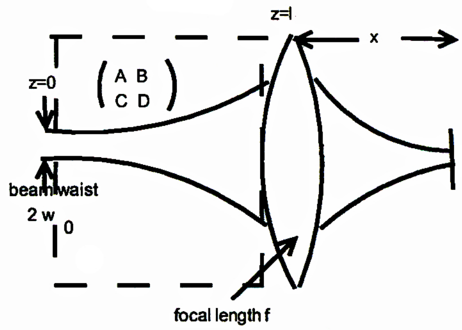
Index matched Homer sees fish bowl where it is 20 cm to right of front surface of bowl.

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R} = 0 \qquad n_1 = n_2$$

$$\therefore s_2 = -s_1$$

II. Problem Number Two : Objective (Lenses and Gaussian Beams)

A Gaussian beam has its focal region at z= 0 where the Gaussian beam radius is w_0.



A.

What is the vaue of the complex Gaussian beam parameter, $q(0)$ at z=0 in terms of w_0?

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{2}{k}\frac{1}{\omega^2(z)}$$

$$@\ z=0,\quad R(0)=\infty,\quad \omega^2(0)=\omega_0^2$$

$$\frac{1}{q(0)} = \frac{1}{\infty} - i\frac{2}{k}\frac{1}{\omega_0^2}$$

$$q(0) = i\frac{k}{2}\omega_0^2 = i z_0 \qquad z_0 = \pi\omega_0^2/\lambda$$

B.

The Gaussian beam traverses a medium of length l with given A,B,C,D parameters. What is the Gaussian beam parameter at z =l in terms of $q0$ and the A,B,C, and D?

$$q_1 = \frac{A q_0 + B}{C q_0 + D}$$

C.

What is the Gaussian beam parameter at $z = l$ in terms of $q(0)$ and l if the medium is free space ?

$$q(l) = q(0) + l = iz_0 + l$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

D.

From II-C deduce the phase-front curvature and beams radius at $z = l$ (in terms of l and w_0)?

$$\frac{1}{q_1} = \frac{1}{iz_0 + l} = \frac{l - iz_0}{z_0^2 + l^2} = \frac{l}{z_0^2 + l^2} - i\frac{z_0}{z_0^2 + l^2}$$

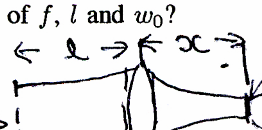
$$= \frac{1}{R(l)} - i\frac{2}{k} \times \frac{1}{\omega^2(z)}$$

Thus $R(l) = (z_0^2 + l^2)/l = l\left(1 + \left(\frac{z_0}{l}\right)^2\right)$; $z = l$

$$\omega^2(l) = \frac{2}{k}\left(\frac{z_0^2 + l^2}{z_0}\right) = \frac{\lambda z_0}{\pi}\left(1 + (l/z_0)^2\right) = \omega_0^2\left(1 + (l/z_0)^2\right)$$

E.

A thin convex lens of focal length f is placed at $z = l > f$. Determine how far from the lens, x the beam is focussed in terms of f, l and w_0?



plane phase front

Steps:

① $q_0 = -i\frac{2}{kw_0^2}$ $\quad q_1 = q_0 + l$ ②

$q_2 = \frac{q_1}{-\frac{1}{f}q_1 + 1}$ ③

④ $q_3 = q_2 + x$

So $q_3 = q_2 + x = \frac{q_1}{-\frac{1}{f}q_1 + 1} + x$

$$= \frac{q_0 + l}{-\frac{1}{f}(q_0 + l) + 1} + x = \frac{q_0 + l + (-\frac{1}{f}(q_0 + l) + 1)x}{-\frac{1}{f}(q_0 + l) + 1} = q_3$$

Now $\frac{1}{q_3} = -i\frac{2}{k\omega_{03}^2}$ at focus

$$\frac{1}{q_3} = \frac{-\frac{1}{f}(q_0 + l) + 1}{q_0 + l - \frac{q_0 x}{f} - (l/f - 1)x}$$

imaginery q_0 ; $-i\frac{2}{kw_0^2}$

$$= \frac{(-\frac{1}{f}q_0 + (1 - l/f))}{q_0(1 - \frac{x}{f}) + l + (1 - l/f)x}$$

$Real\left(\frac{1}{q_3}\right) = 0$ implies

$$\boxed{-|q_0|^2\frac{1}{f}\left(1 - \frac{x}{f}\right) + \left(1 - \frac{l}{f}\right)\left[l + (1 - l/f)x\right] = 0}$$

x can be written as

$$x = f + \frac{f^2}{\left(1 + \frac{|q_0|^2}{(l-f)^2}\right)(l-f)}$$

F.

What is the beam radius at the focal region of II-E ?

The remaining imaginery part with x substituted gives the focussed beam radius.

Use $$q_3 = \frac{q_0 + l}{-\frac{1}{f}(q_0 + l) + 1} + x = \frac{q_0 + l}{-\frac{1}{f}(q_0 + l) + 1} + f + \frac{(f)^2}{\left[1 + \frac{|q_0|^2}{(l-f)^2}\right](l-f)}$$

$$= \frac{(q_0 + l)\left(\left(1 - \frac{l}{f}\right) - \frac{q_0^*}{f}\right)}{\left(1 - \frac{l}{f}\right)^2 + \frac{|q_0|^2}{f}} + f + \frac{(l-f)}{\underbrace{\frac{(l-f)^2}{f^2}}_{\left(1-\frac{l}{f}\right)^2} + \frac{|q_0|^2}{f^2}}$$

rationalized

$\longleftrightarrow$ same

multiplied top & bottom by $\frac{(l-f)^2}{f^2}$

$$= \frac{q_0\left(1 - \frac{l}{f}\right) - \cancel{\frac{|q_0|^2}{f}} + l\left(1 - \frac{l}{f}\right) - l\frac{q_0^*}{f} + \frac{(l-f)}{f\left(-1 + \frac{l}{f}\right)} + \frac{(l-f)^2}{f} + \cancel{\frac{|q_0|^2}{f}}}{\frac{(l-f)^2}{f^2} + \frac{|q_0|^2}{f^2}}$$

$$= \frac{q_0\left(1 - \cancel{\frac{l}{f}}\right) + \cancel{(l-f)\left(1 - \frac{l}{f}\right)} - l\cancel{\frac{q_0^*}{f}} + \cancel{\frac{(l-f)^2}{f^2}}}{\frac{(l-f)^2}{f^2} + \frac{|q_0|^2}{f^2}}$$

$$q_3 = \frac{q_0 \frown -i\frac{2}{k\omega_0^2}}{\left(\frac{(l-f)}{f}\right)^2 + \underbrace{\left(\frac{|q_0|}{f}\right)^2}_{\frac{2}{k\omega_0^2}}} = -i\frac{2}{k(\omega_{03})^2}$$