UNIVERSITY OF CALIFORNIA
College of Engineering
Department of Electrical Engineering
and Computer Scences
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Problem Set No. 3

Problem Number one)Cardinal points of a lens (This is a modification of 6.18 of the text)
A thick biconvex lens has radii of 20 cm and is 5 cm thick. The lens refractive index is 1.5.
a) Find the ABCD parameters (help! The numbers are 1.0133, .0378, .8 and 1.0167)
b) Deduce the cardinal points in terms of the ABCD parameters generally. (I get the nodal points are N_o and N_i (A-1)/C and (1-D)/C, respectively. The ffl= - D/C The unit planes are -(A-1)/C for the image and (-1+D)/C for the object). Of what use are these?
c) What are the cardinal points for the above lens?

Problem Number two) The microscope and the telescope contrasted
a) The pupillary diameter of the typical eye is 2mm and the objective lens of the telescope is imaged on the eye. If the objective is 20 mm in diameter and the focal length is 250 mm What is the magnification of the telescope?
What focal length ocular (eyepiece) should be used?
Find the position of the exit pupil.
What would be the diameter of the exit pupil if the ocular gave a magnification increased by 50 percent ? Decreased by 50 percent ?
b) The first microscope was a clear glass marble resting on the object to be magnified. (van Leeuwenhoek (1632-1723)). What is the power (1/f in m^{-1} of a marble 1cm in diameter (n = 1.5)?

Problem Number three) A re-entrant confocal resonator cavity
Hecht Problem 6.24. Is the cavity stable? The confocal configuration is popular as an optical spectrum analyzer. We hope to discuss this.

Problem Number four) Lens correction of the eye.
A myopic individual has his far-point of best vision at 16.6 cm and his near -point at about 6.5 cm. What is his range of accomodation (in D's)? What spectacle correction will restore his far-point to infinity? What will then be his new near-point? (Comparable to text prob 5.85 and 5.86)

Problem Number five) Oil immersion objective.

Consider a transparent sphere of radius R and index n. Show that a point P at radius R/n is perfectly imaged to a point Q at radius n R and in fact a spherical surface of radius R/n is imaged to a sphere of radius n R. The points P and Q are called aplanatic points. This is the basis of the oil emersion microscope (see page 255 -257 of text) (reference Sears Optics third ed.)

Problem Number six) Off axis focal distances.

a) Show that for a thin lens with an on-axis focal length, for a point off the axis by θ (w.r.t the optic axis), the sagittal and tangential ray bundles focal lengths are $f/cos\theta$ and $fcos\theta$ respectively.

b) Similarly establish that the sagittal and tangential ray bundles traversing a Brewster window travel effective distances given by

$$d_x = t(n^2+1)^{1/2}/n^2$$

and

$$d_y = t(n^2+1)^{1/2}/n^4$$

Anote on $\begin{matrix}A & B\\ C & D\end{matrix}$ Matricies

(1)

Reference : Principles of Applied Optics
Banerjee and Poon

~~axis,~~ It is customary to replace the corresponding θ by $v = n\theta$, where n is the refractive index at the z-constant plane.

ray slope

In Figure 2.7, the ray passes through the input plane with *coordinates* $(x_1,\ v_1 = n_1\theta_1)$, then through the optical system, and finally through the output plane with coordinates $(x_2,\ v_2 = n_2\theta_2)$. In the paraxial approximation, the corresponding output quantities are linearly dependent on the input quantities. We can, therefore, represent the transformation from the input to the output in matrix form as

$$\begin{pmatrix} x_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}. \qquad (2.4\text{-}1)$$

ray slope x n_2 ray slope x n_1

The $\mathscr{ABCD}$ matrix in Eq. (2.4-1) is called the *ray transfer matrix* and, as we shall see later, it can be made up of many matrices to account for the effects of a ray passing through various optical elements. We can consider these matrices as operators successively acting on the input ray coordinate vector. We state here that the determinant of the ray transfer matrix equals unity, that is, $\mathscr{AD} - \mathscr{BC} = 1$. This will become clear after we derive the translation, refraction, and reflection matrices.

Let us now investigate the general properties of an optical system from the $\mathscr{ABCD}$ matrix.

Property 1: If $\mathscr{D} = 0$, we have from Eq. (2.4-1) that $v_2 = \mathscr{C}x_1$. This means that all rays crossing the input plane at the same point, namely, x_1, emerge at the output plane making the same angle with the axis, no matter at what angle they enter the system. The input plane is called the *front focal plane* of the optical system [see Figure 2.8(a)].

Property 2: If $\mathscr{B} = 0$, $x_2 = \mathscr{A}x_1$ [from Eq. (2.4-1)]. This means that all rays passing through the input plane at the same point (x_1) will pass through the same point (x_2) in the output plane [see Figure 2.8(b)]. The input and output planes are called the *object* and *image planes*, respectively. In addition, $\mathscr{A} = x_2/x_1$ gives the *magnification* produced by the system.

Furthermore, by inverting the $\mathscr{ABCD}$ matrix and from the fact that $\mathscr{AD} - \mathscr{BC} = 1$, we note from Eq. (2.4-1) that $x_1 = \mathscr{D}x_2 = (1/\mathscr{A})x_2$, because $\mathscr{B} = 0$. The implication of this is that the point x_2 is imaged at x_1 with magnification $1/\mathscr{A}$. Hence, the two planes containing x_1 and x_2 are called *conjugate planes*. Moreover, if $\mathscr{A} = 1$, that is, the magnification between the two conjugate planes is unity, these two planes are called the *unit*, or *principal*, *planes*. The points of intersection of the unit planes with the optical axis are the *unit*, or *principal*, *points*. The principal points constitute one set of *cardinal points*.

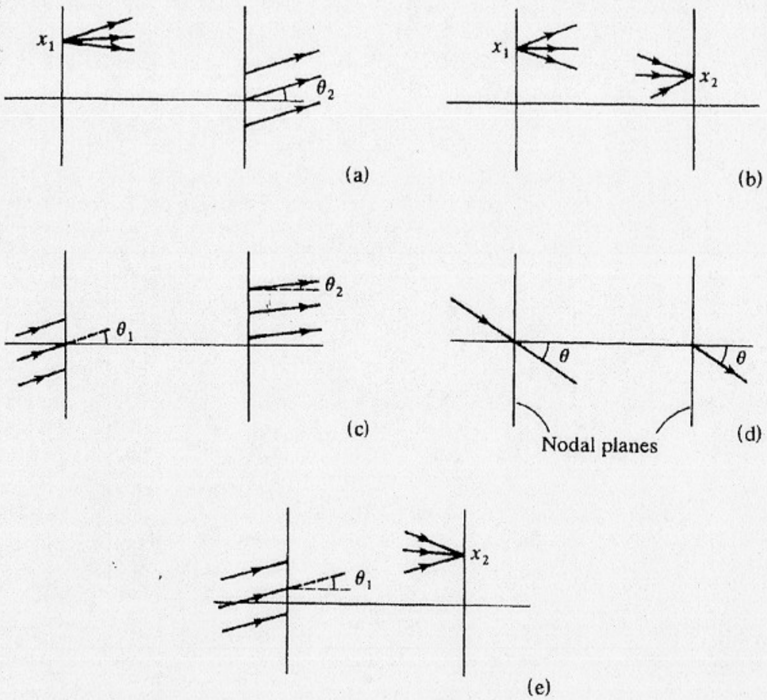


Figure 2.8 Rays at input and output planes for (a) $\mathscr{D} = 0$, (b) $\mathscr{B} = 0$, (c) $\mathscr{C} = 0$, (d) the case when the planes are nodal planes, and (e) $\mathscr{A} = 0$.

Property 3: If $\mathscr{C} = 0$, $v_2 = \mathscr{D}v_1$. This means that all the rays entering the system parallel to one another will also emerge parallel, albeit in a new direction [see Figure 2.8(c)]. In addition, $\mathcal{D}(n_1/n_2) = \theta_2/\theta_1$ gives the *angular magnification* produced by the system.

If $\mathcal{D} = n_2/n_1$, we have unity angular magnification, that is, $\theta_2/\theta_1 = 1$. In this case, the input and output planes are referred to as the *nodal planes*. The intersections of the nodal planes with the optical axis are called the *nodal points* [see Figure 2.8(d)]. The nodal points constitute the other set of cardinal points.

Property 4: If $\mathcal{A} = 0$, $x_2 = \mathcal{B} v_1$. This means that all rays entering the system at the same angle will pass through the same point at the output plane. The output plane is the *back focal plane* of the system [see Figure 2.8(e)].

2.4.2 Translation and Refraction Matrices

When a ray passes through an optical system, there are usually two types of processes, translation and refraction (and, sometimes, reflection; this is treated later), that we need to consider in order to determine the progress of the ray. As the rays propagate through a homogeneous medium, they undergo a translation process. In order to specify the translation, we need to know the thickness of the medium and its refractive index. However, when a ray strikes an interface between two regions of different refractive indices, it undergoes refraction. To determine how much bending the ray undergoes, we need to know the radius of curvature of the boundary and the values of the refractive indices of the two regions. We shall investigate the effect each of these two processes has on the coordinates of a ray between the input and the output planes. In fact, we will derive the ray transfer matrices for the two processes.

Figure 2.9 shows a ray travelling a distance d in a homogeneous medium of refractive index n. Because the medium is homogeneous, the ray travels in a straight line. The set of equations of translation by a distance d is

$$x_2 = x_1 + d \tan\theta_1 \approx x_1 + \theta_1 d, \tag{2.4-2a}$$

$$n\theta_2 = n\theta_1 \quad \text{or} \quad v_2 = v_1. \tag{2.4-2b}$$

These equations relate the output coordinates of the ray with its

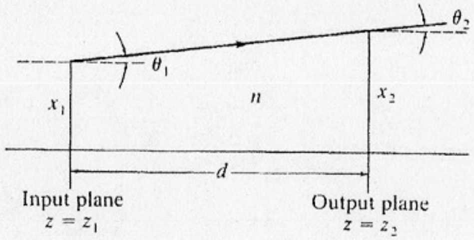


Figure 2.9 A ray in a homogenous medium of refractive index n_o.

input coordinates. We can express this transformation in matrix form as

$$\begin{pmatrix} x_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}. \tag{2.4-3}$$

The 2×2 ray transfer matrix, called the *translation matrix* $\mathcal{T}$, is defined as

$$\mathcal{T} = \begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix}. \tag{2.4-4}$$

Note that its determinant is unity.

[redacted]

Cardinal Points
Banerjee and Poon

Example 2.8 Imaging by a Thick Lens; Cardinal Points

In Example 2.4, we wrote down the system matrix $\mathcal{S}$ for a thick lens in terms of the radii of curvature R_1 and R_2 of both surfaces and the thickness of the lens. In the discussion that follows, we will symbolically write the elements of $\mathcal{S}$ in Eq. (2.4-14) as

$$\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4-25}$$

As in the thin-lens case, we can study the imaging properties of a thick lens in air by considering the input and output planes to be located at distances d_1 and d_2 from the front and back refracting surfaces, respectively, as shown in Figure 2.16. We can find the output ray coordinates $(x_2, v_2)^T$ in terms of the input ray coordinates $(x_1, v_1)^T$ as

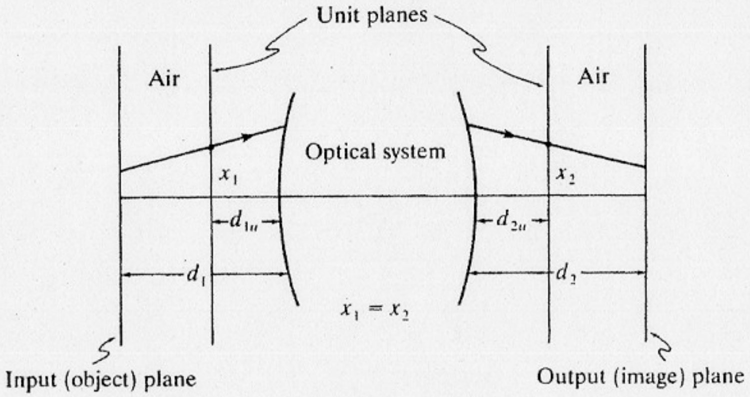


Figure 2.16 Unit planes for an optical system. A ray starting from any height x_1 from the input unit plane will cross the output unit plane at $x_2 = x_1$.

$$\begin{pmatrix} x_2 \\ v_2 \end{pmatrix} = \underset{\text{translation}}{\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}} \underset{\text{thick lens}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underset{\text{translation}}{\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} \tag{2.4-26a}$$

$$= \begin{pmatrix} a + c d_2 & a d_1 + b + c d_1 d_2 + d d_2 \\ c & c d_1 + d \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}. \tag{2.4-26b}$$

Once again, to find the location of the image (d_2), we consider a point object located on the axis in the input plane, that is, take $x_1 = 0$. Because the image should also be a point on the axis, we can set $x_2 = 0$ in Eq. (2.4-26b). This gives

$$a d_1 + b + c d_1 d_2 + d d_2 = 0, \tag{2.4-27}$$

from which we can easily solve for d_2 in terms of d_1. Also, the image magnification can be found using Eqs. (2.4-26) and (2.4-27) as

$$\frac{x_2}{x_1} = (a + c d_2). \tag{2.4-28}$$

It is instructive, at this point, to determine the cardinal points of our thick-lens imaging system. By setting $x_2/x_1 = 1$ and making use of the fact that the determinant of the matrix in Eq. (2.4-26b) is equal to unity, we find the locations of the two unit planes d_{1u} and d_{2u} as

$$d_{1u} = \frac{1 - d}{c}, \qquad d_{2u} = \frac{1 - a}{c}. \tag{2.4-29}$$

The two unit planes are shown in Figure 2.16. Next, to find the nodal points, we set $v_1 = v_2$. From Eq. (2.4-26b), we find the locations of the two nodal points d_{1n} and d_{2n} as

$$d_{1n} = \frac{1 - d}{c} = d_{1u}, \qquad d_{2n} = \frac{1 - a}{c} = d_{2u}. \tag{2.4-30}$$

Note that the two nodal planes coincide with the unit planes when the indicies are the same

The cardinal points of a system (4)

1) The focal planes. f_i and f_o
which are conjugate with the two infinity
two points of the system

2) Unit or principle planes (H_1 + H_2) (or U_o and U_i), the two planes between which magnification
two is unity

3) Nodal points (N_1 + N_2) (N_o + N_i)
two points lying on the optic axis between which the incoming object ray and the outgoing image ray make equal angles

In thick lens systems, the focal lengths (effective focal lengths) are measured from the unit planes. The distances from the surfaces are the back and front focal lengths.

The A, B, CD matricies are given by

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$

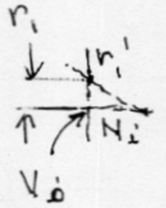
Nodal points. If $r_2' = +r_1'$

$$r_2 = A r_1 + B r_1'$$

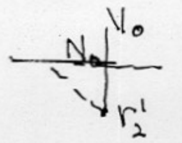
$$r_1' = C r_1 + D r_1'$$ (Schafer – a = +C b = D)

schaf ↓ Yariv

$$r_1' = \frac{C r_1}{1-D}$$



$r_1' N_i = r_1$ $N_i = r_1/r_1' = (1-D)/C$

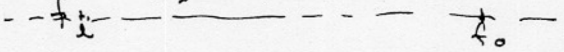


$r_2' N_o = r_2$ $N_o = r_2/r_2' = r_2/r_1' = \frac{A r_1 + B}{r_1'}$

$$= A(1-D)/C + B = \frac{A - AD + BC}{C} = \frac{A-1}{C}$$

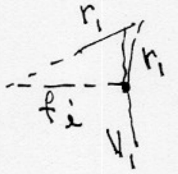
(usual convention is to take $-N_i = (-A+1)/C$ as in Bo

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} \begin{pmatrix} A, & B \\ C, & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix}$$



ffl $\quad r_2' = 0$

$$r_2 = A r_1 + B r_1'$$
$$0 = C r_1 + D r_1'$$



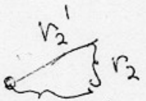
$$r_1' f_i = r_1$$

$$\therefore f_i = r_1/r_1' = -D/C = \text{ffl}$$

$$= +\left(b/a\right) \leftarrow \text{Schaffer uses } \left(-b/a\right)$$

bfl $\quad r_1' = 0 \qquad r_2 = A r_1$

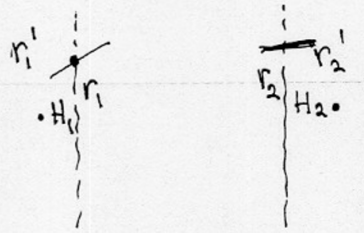
$$r_2' = C r_1$$



$$-\cancel{r_2'}\ \text{bfl} = r_2 = A r_1 = \frac{A}{C} \cancel{r_2'}$$

$$\therefore \text{bfl} = -A/C = +\frac{c}{e}$$

The <u>unit</u> planes.



$$r_1 + H_1 r_1' = r_2 + r_2' H_2$$
$$= A r_1 + B r_1' + (C r_1 + D r_1') H_2$$

$$r_1 (1 - A \mp C H_2) + r_1' (H_1 - B - D H_2) = 0$$

$$H_i = H_2 = \frac{A-1}{-C} = \boxed{\frac{c-1}{+a} \text{ – Schaffer}}$$

$$H_1 - B - D\left(\frac{A-1}{-C}\right) = 0$$

$$H_o = H_1 = \frac{BC - AD \pm D}{-C} = \frac{-1 + D}{-C} \boxed{= \frac{-1\ b}{a}} \text{ Schaffer}$$

This agrees with Banerjee

If you have to use these be clear on what you mean by signs – just be consistent

Problem 1 - Cardinal Points - Taken From Schaefer (6)

It is important to note that these elements are arranged in the inverse order of the system, or in a right-to-left order. The system matrix then takes the form

$$S = \begin{bmatrix} b & -a \\ -d & c \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \quad [4\text{-}7]$$

Yariv & Banerjee

and this set of four numbers contains the essential first-order properties of the system.

A useful property of the system matrix is its determinant. Because the determinant of the system matrix is always unity, it provides a way of checking the system matrix for arithmetic errors. It will not, however, detect an error in the ordering of or construction of the matrices making up the system.

XAMPLE 4.5 A thick biconvex lens has radii of 20 cm and is 5 cm thick, as shown in Figure 4.6. Find its system matrix. The lens refractive index is 1.5.

OLUTION: The matrices are found using equations (4-4) and (4-5).

$-n_2$ n_1

$$\mathfrak{R}_1 = \begin{bmatrix} 1 & -\dfrac{1.5 - 1.0}{20} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.025 \\ 0 & 1 \end{bmatrix}$$

$$\mathfrak{T}_1 = \begin{bmatrix} 1 & 0 \\ \dfrac{5}{1.5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3.33 & 1 \end{bmatrix}$$

$\frac{d}{n}$

$$\mathfrak{R}_2 = \begin{bmatrix} 1 & -\dfrac{1.0 - 1.5}{-20} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.025 \\ 0 & 1 \end{bmatrix}$$

this defines Schaffers matricies as

$$\begin{pmatrix} r_2' \\ r_2 \end{pmatrix} = \begin{pmatrix} b, & -a \\ -d, & c \end{pmatrix} \begin{pmatrix} r_1' \\ r_1 \end{pmatrix}$$

$$\begin{pmatrix} r_2' \\ r_2 \end{pmatrix} = \begin{pmatrix} D, & C \\ B & A \end{pmatrix} \begin{pmatrix} r_1' \\ r_1 \end{pmatrix}$$

Banerjee and Yariv

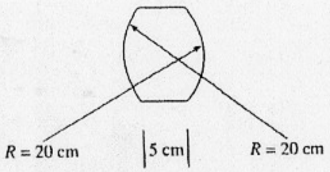


FIGURE 4.6. Example 4.5

Note in $\mathfrak{R}_2$ the order of the refractive indices: the ray goes from the lens with index 1.5 into air with index 1.0. Also, the center of curvature for $\mathfrak{R}_2$ is to the left of the surface, so that its radius is negative.

The system matrix is given by

$$\mathbf{S} = \mathfrak{R}_2 \mathfrak{T}_1 \mathfrak{R}_1$$

Note the ordering of the matrices. When we r trices using the rules in Appendix I, we get

$$\mathbf{s} = \begin{bmatrix} 0.9168 & -0.0479 \\ 3.33 & 0.9168 \end{bmatrix}$$

which has determinant 1.0001. The values of th

ABCD Coeffs for Prob 1

$a = 0.0479 \simeq -C$

$b = 0.9168 = D$

$c = 0.9168 = A$

$d = -3.33 = -B$

Note that the signs here refer to equation (4-7) d terms of the matrix are negative.

EXAMPLE 4.6 A negative lens has the following parameters: = 1.50, $t = 1.2$ cm, and $R_2 = 30$ cm. Find th

SOLUTION: The system matrix is given by

$$\begin{bmatrix} 1 & -\dfrac{1.0 - 1.50}{30} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1.2}{1.50} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\dfrac{1.5}{} \\ 0 & \end{bmatrix}$$

which after multiplication gives

$$\begin{bmatrix} 1.0133 & 0.0378 \\ 0.8 & 1.0167 \end{bmatrix}$$

THE CARDINAL POINTS

We noted previously that the system matrix contains the ess erties of the lens system. These properties are called the *ca* sist of pairs of conjugate elements. First are the *focal plane* conjugate with the two infinity points of the system, one to t the right. Second are the *unit planes* or *principal planes* u_o an the system between which the magnification is unity. Final

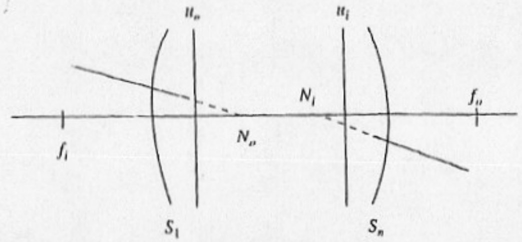


FIGURE 4.7. The cardinal points of an optical system.

points N_o and N_i, two points lying on the optic axis between which the incoming object ray and the outgoing image ray make equal angles. The cardinal points are illustrated in Figure 4.7. It is important to note that although the unit planes and the nodal points in the figure are shown within the system, in some cases one or more of them may be external to the system.

In thick lens systems, the focal lengths, which we call the *effective focal lengths*, are measured from the unit planes. The distances from the surfaces to the focal points are called the *back focal length*, bfl, and the *front focal length*, ffl. One must be clear when specifying the focal length of a system to identify which one is being stated. The common practice is to give the effective focal length of a lens system, so that if one simply has the focal length, one can generally assume it to be the effective focal length. In terms of the matrix elements the effective focal length is given by

$$f = \frac{1}{a} = -\frac{1}{C} \quad \text{(Yariv)} \qquad [4\text{-}8]$$

so that we immediately see that in the system matrix the element a is the power of the system. The back focal length is given by

$$\text{bfl} = \frac{c}{a} \qquad [4\text{-}9]$$

The bfl is the conjugate with the object-side infinity and is the distance from the last surface of the system to the focal plane. The effective focal length conjugate with the image-side infinity is $-1/a$, and the front focal length is given by

$$\text{ffl} = -\frac{b}{a} \qquad [4\text{-}10]$$

The unit planes are given by

$$\frac{u_i}{n_i} = \frac{c-1}{a}$$

and

$$\frac{u_o}{n_o} = \frac{1-b}{a}$$

where u_o is the object-side unit plane, and u_i the image-
the refractive index of the object's medium and n_i that
see that the unit planes are helpful in sketching systems
images.

The *nodal points* are given by

$$\frac{N_o}{n_o} = \frac{\overset{A}{c} - \frac{n_o}{n_i}}{a} \;\; -B$$

and

$$\frac{N_i}{n_i} = \frac{\frac{n_i}{n_o} - \overset{D}{b}}{a} \rightarrow -B$$

where n_i is the medium to the right (image) side of the syst
If these media are the same—for example, in the typical c
air—then the nodal points coincide with the intersection o
optic axis. The nodal points come into play only when tl
ferent from that of the object, as is the eye, for example.
points but unit *planes* and focal *planes*, since an extende
aged at the focal plane, which includes the focal point wh
axis.

EXAMPLE 4.7 Find the cardinal points and the bfl of the l

SOLUTION: The matrix elements for the lens are

$$a = 0.0479$$
$$b = c = 0.9168$$
$$d = -3.33$$

The focal length is given by equation (4-8)

$$f = \frac{1}{a} = \frac{1}{0.0479} = 20.88 \text{ cm}$$

Because the lens is in air, the nodal points and the principal planes coincide, and these are given by equations (4-11) and (4-12):

$$u_l = \frac{c - 1}{a} = \frac{0.9168 - 1}{0.0479} = -1.74 \text{ cm}$$

$$U_o = \frac{1 - b}{a} = \frac{1 - 0.9168}{0.0479} = 1.74 \text{ cm}$$

and the principal and nodal planes lie within the lens itself. Note that u_l is measured from the last vertex, and u_o from the first vertex of the system.

We find the bfl using equation (4-8):

$$\text{bfl} = \frac{c}{a} = \frac{0.9168}{0.0479} = 19.14 \text{ cm}$$

EXAMPLE 4.8 What is the bfl of the lens in Example 4.6?

SOLUTION: The bfl is given by equation (4-8) as

$$\frac{c}{a} = \frac{1.0133}{-0.0378} = -26.81 \text{ cm}$$

Note the signs of *a* and of the bfl. The bfl is measured from the last vertex of the system through the lens and falls 26.81 cm to the left of the last vertex of the system.

Sears Problem 2 Notes & Solution.

by the objective O. The image I serves as a virtual object for the ocular E. The final image I' is virtual and erect, as indicated. The angular magnification of this telescope is also given by

$$\gamma = -\frac{f_1}{f_2},$$

but since f_2 is negative, γ is a positive quantity and the image is erect. The distance between objective and ocular is the difference between (the absolute values of) their focal lengths. Consequently this instrument may be made much more compact than the astronomical type. Its chief disadvantage is that it cannot cover as wide a field of view without the use of objectives of unduly large diameter. The "opera glass" is a Galilean telescope.

6-8 Normal magnification. Thus far nothing has been said regarding the diameters of the lenses in a telescope; the magnification involves only the ratio of focal lengths. To see how the diameter of the objective sets a limit to the useful magnification, let us consider the optical system of a refracting telescope from a somewhat different viewpoint.

The ocular of a telescope, as well as imaging the *image* formed by the objective, also forms a real, reduced image of the objective lens itself in

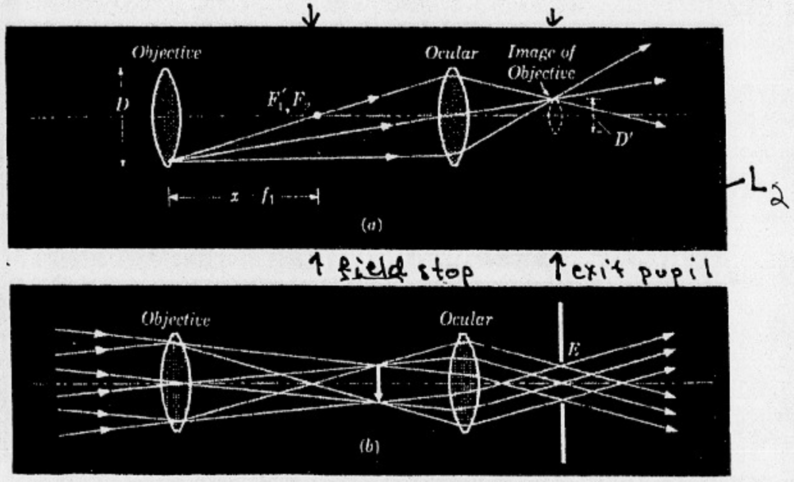


FIG. 6-23. (a) The ocular of a telescope forms a real, reduced image of the objective lens. (b) Rays from a distant object are refracted by a telescope and pass through the exit pupil E. The exit pupil lies at the same point, and has the same diameter, as the image of the objective lens.

Problem 2 - Taken From Sears. (9)

the space beyond the ocular. All of the light that enters the object and is refracted by the ocular, must pass through this image of the jective, which is called the *exit pupil* of the telescope. The diamete the transmitted beam is a minimum in the plane of the exit pupil. Fig. 6-23 (b).) If all of the transmitted light is to enter the pupil of observer's eye, the diameter of the exit pupil of the telescope should be no larger than the pupillary diameter of the eye. In practice, the eye is usually placed at the exit pupil which is also called the *eye point* of telescope.

Let us assume that the object being viewed and the virtual image formed by the ocular are both at infinity. If the objective lens is considered as the object for the ocular, the object distance x in Fig. 6-23 measured from the first focal point F_2 of the ocular, is

$$\boxed{x = f_1,}$$ Depends upon

where f_1 is the focal length of the objective lens. Let D be the diameter of the objective and D' the diameter of its image. By definition, the lateral magnification of the image of the objective is

$$m = \frac{D'}{D},$$

and from Eq. (4-15),

$$m = -\frac{f_2}{x},$$

where f_2 is the focal length of the ocular. Then (disregarding algebra signs),

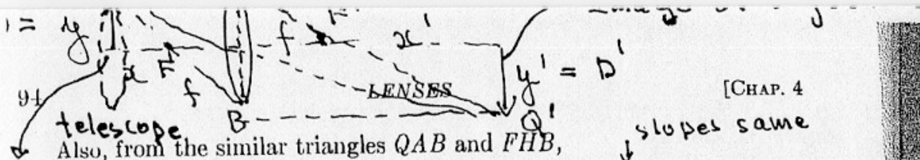
$$\frac{D}{D'} = \frac{x}{f_2} = \frac{f_1}{f_2}.$$

But $\frac{f_1}{f_2}$ equals the angular magnification of the telescope, γ. Hence

$$\gamma = \frac{D}{D'}, \qquad (6-$$

and the angular magnification equals the ratio of the diameter of the ol jective to the diameter of its image formed by the ocular, or the exit pupi

Incidentally, Eq. (6-4) indicates a convenient method for measurin the angular magnification of a telescope. The instrument may be directe toward a bright sky and a screen moved along the axis until the minimur diameter of the transmitted beam is found. The magnification is then th ratio of the diameter of the objective to the minimum diameter of th transmitted beam.



slopes same

Also, from the similar triangles QAB and FHB,

$$\frac{y-y'}{s} = \frac{-y'}{f}, \qquad \frac{y}{x} = -\frac{y'}{f} \tag{4-10}$$

and from the triangles ABQ' and $AH'F'$,

$$\frac{y-y'}{s'} = \frac{y}{f}. \qquad \frac{y}{f} = -\frac{y'}{x'} \tag{4-11}$$

When Eqs. (4-10) and (4-11) are added, we get

$$\frac{y-y'}{s} + \frac{y-y'}{s'} = \frac{y}{f} - \frac{y'}{f} = \frac{y-y'}{f},$$

or

$$\boxed{\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}.} \tag{4-12}$$

When Eq. (4-8) is divided by Eq. (4-9) we get

$$\frac{f}{x} = \frac{x'}{f},$$

or

$$\boxed{xx' = f^2.} \tag{4-13}$$

The lateral magnification m is the ratio of y' to y. Dividing Eq. (4-10) by Eq. (4-11) gives

$$\boxed{m = -\frac{s'}{s}.} = \frac{y'}{y} \tag{4-14}$$

Also, from Eqs. (4-8) and (4-9),

$$\boxed{m = -\frac{f}{x} = -\frac{x'}{f}.} \tag{4-15}$$

Eq. (4-12) is known as the *Gaussian* form of the lens equation, after the mathematician Karl F. Gauss. (Gauss's law in electrostatics, as well as the unit of magnetic flux density in the electromagnetic system of units, the gauss, take their names from him also.) Eq. (4-13), first derived by Sir Isaac Newton, is the *Newtonian* form of the lens equation. The Gaussian form is probably more familiar, but the Newtonian equation is algebraically simpler. Notice carefully that in the former equation object and image distances s and s' are measured from the *principal points* H and H' respectively (or from the center of a thin lens), while in the latter, object and image distances x and x' are measured from the *focal points* F and F'.

For object at ∞ $[x = f_1]$ $\therefore$ $m = -\frac{f}{f_1}$

but f/f_1 = angular magnification of the telscope

The lateral magnification m can be expressed either in terms of s and s', by Eq. (4-14), or in terms of x, x', and f, by Eq. (4-15).

Example of Gaussian Form ~~of Lens~~ Eqn.

Example. An object is located 30 cm to the left of a thin lens of focal length 20 cm, as in Fig. 4-6. Find the position and lateral magnification of its image, using both the Gaussian and Newtonian forms of the lens equation.

The object distance s, measured from the center of the lens, is +30 cm. From the Gaussian equation.

$$\frac{1}{30} + \frac{1}{s'} = \frac{1}{20},$$

$$s' = +60 \text{ cm}.$$

The image is real (s' is positive) and lies 60 cm to the right of the center of the lens.

The object distance x, measured from the first focal point, is +10 cm. From the Newtonian equation,

$$10x' = (20)^2,$$

$$x' = +40 \text{ cm}.$$

This is evidently in agreement with the answer above. The lateral magnification, by Eq. (4-14), is

$$m = -\frac{60}{30} = -2.$$

The image is inverted (m is negative) and twice the height of the object.

From Eq. (4-15),

$$m = -\frac{20}{10} = -\frac{40}{20} = -2.$$

In the seven parts of Fig. 4-8, a number of rays from the head of an arrow representing an object O have been traced through a thin lens of focal length f. Lens aberrations are neglected. The image of the head of the arrow has been located by using two (in some cases, all three) of the rays referred to in Fig. 4-6. In addition, the outermost rays incident on the lens have been drawn. The object distances are respectively, $+3f$, $+2f$, $+\frac{3}{2}f$, $+f$, $+\frac{3}{4}f$, 0, and $-2f$. In parts (1) to (5) inclusive, the object

The *normal magnification* of a telescope is defined as that at which the diameter of its exit pupil is just equal to the pupillary diameter of the eye, usually assumed to be 2 mm.

Answer to Problem 2, Part a

Example. The objective lens of a telescope is 20 mm in diameter and its focal length is 250 mm.

(a) What is the normal magnification of the telescope?

(b) What focal length ocular should be used?

(c) Find the position of the exit pupil.

(d) What would be the diameter of the exit pupil if an ocular were used which gave a magnification 50% in excess of normal?

(e) What would be the diameter of the exit pupil if the magnification were 50% of normal?

Assume all lenses to be thin.

(a) From Eq. (6-4), $\gamma = \dfrac{D}{D'} = \dfrac{20}{2} = 10\times$.

(b) $\gamma = \dfrac{f_1}{f_2}$, $f_2 = \dfrac{f_1}{\gamma} = \dfrac{250}{10} = 25$ mm.

(c) $x' = \dfrac{f_2^2}{x} = \dfrac{f_2^2}{f_1} = \dfrac{f_2}{\gamma} = \dfrac{25}{10} = 2.5$ mm.

That is, the exit pupil is 2.5 mm to the right of the second focal point of the ocular, or 27.5 mm to the right of the ocular itself.

(d) If $\gamma = 15\times$, $D' = \dfrac{D}{\gamma} = \dfrac{20}{15} = 1.33$ mm.

(e) If $\gamma = 5\times$, $D' = \dfrac{20}{5} = 4$ mm.

Answer to Prob. 2 Part b

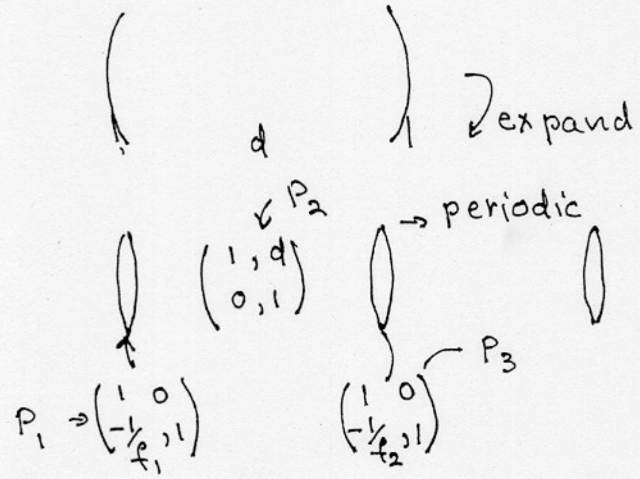
EXAMPLE 5.4. What is the power of a glass marble such as Leeuwenhoek was reputed to have used if the marble is 1 cm in diameter and has a refractive index of 1.50?

SOLUTION: The system has the matrix representation

$$\begin{bmatrix} 1 & -\dfrac{1-1.50}{0.5} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{1.50} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\dfrac{1.50-1}{0.5} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.6667 & -0.66667 \\ 0.6667 & 0.33333 \end{bmatrix}$$

with determinant = 1.00000. The focal length 1/*a* = 1.5 cm, or 0.015 m, and the power is 66.67 D.

Periodic Lens System



$$\begin{matrix}A & B\\ C & D\end{matrix} = P_3 P_2 P_1 = \begin{pmatrix}1, & 0\\ -\frac{1}{f_2}, & 1\end{pmatrix}\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\begin{pmatrix}1, & 0\\ -\frac{1}{f_1}, & 1\end{pmatrix}$$

$$= \begin{pmatrix}1, & d\\ -\frac{1}{f_2}, & -\frac{d}{f_2}+1\end{pmatrix}\begin{pmatrix}1, & 0\\ -\frac{1}{f_1}, & 1\end{pmatrix}$$

$$P = \begin{pmatrix}1-\frac{d}{f_1} & , d\\ -\frac{1}{f_2}+\frac{d}{f_1 f_2}-\frac{1}{f_1} & , -\frac{d}{f_2}+1\end{pmatrix}$$

Note det $P = 1$ $AD - BC = 1$

$$\begin{pmatrix}r_2\\ r_2'\end{pmatrix} = P \begin{pmatrix}r_1\\ r_1\end{pmatrix}$$

periodic systen

$$\left.\begin{matrix}r_n = r_0 e^{iknd}\\ r_n' = r_0' e^{iknd}\end{matrix}\right\} \text{use phasors.}$$

$$\cancel{e^{iknd}}\, r_0 = A r_0 e^{ik(n-1)d} + B r_0' e^{ik(n-1)d}$$

$$\cancel{e^{iknd}}\, r_0' = C r_0 e^{ik(n-1)d} + D r_0' e^{ik(n-1)d}$$

Same equations apply for e^{-iknd} since A, B, C, D real. . General solution

$r_n = r_n \cos kdn + r_0' \sin kdn \to$ Initial slope and

$$\det\begin{pmatrix} 1-Ae^{ikd}, & Be^{ikd} \\ Ce^{ikd}, & 1-De^{ikd}\end{pmatrix}=0$$

$$1-(A+D)e^{ikd}+ADe^{i2kd}$$

$$-BCe^{i2kd}=0$$

$$1-(A+D)e^{ikd}+(e^{ikd})^2=0$$

$$e^{ikd}=\frac{A+D}{2}\pm\left(\left(\frac{A+D}{2}\right)^2-1\right)^{1/2}$$

$$=\frac{A+D}{2}\pm i\left(1-\left(\frac{A+D}{2}\right)^2\right)^{1/2}$$

$$\therefore\ \boxed{\cos kd=\frac{A+D}{2}}\longrightarrow$$ Note analogy to Kronecg-Penney model is solid state : ⇒ periodic system

$$\sin kd=\left(1-\left(\frac{A+D}{2}\right)^2\right)^{1/2}$$

stability criteria $-1<\frac{A+D}{2}<1$

Full peridicity

$$\begin{pmatrix}1, & d\\ 0, & 1\end{pmatrix}\begin{pmatrix}1-d/f_1 & , & d\\ -1/f_2+d/f_1f_2-1/f_1 & , & -d/f_2+1\end{pmatrix}$$

$$A = 1-d/f_1-d/f_2+d^2/f_1f_2-d/f_1$$

$$D = 1-d/f_2$$

$$\frac{A+D}{2} = 1-d/f_1-d/f_2+\frac{1}{2}d^2/f_1f_2$$

Add 1 and divide by 2

$$0<\left(1-d/2f_1\right)\left(1-d/2f_2\right)<1$$

Note : Confocal cavitity $d=f$ for both - just on edge of stability

(13a)

The re-entrant cavity

$r_1 = r_2 = R \qquad d = R \qquad f_1 = f_2 = R/2$

$\therefore \quad A = 1 - 2 - 2 + 4 - 2 = -1$

$D = 1 - 2 = -1$

$C = -2/R + 4/R - 2/R = 0$

$B = d + d - d^3/f_2 = 0$

$\therefore \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ← this is $\begin{matrix} A & B \\ C & D \end{matrix}$ for 1 round trip

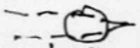
for two round trips it is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

so beam returns exactly

Problem No. 4 - Schaefer

Spectacle lenses, or eyeglasses, are an additional optical element added to the eye system. In contrast, contact lenses provide an artificial surface to the cornea and correct vision by changing the corneal power. In either case, the correction usually returns vision to its near-optimal state with a full range of accommodation.

The 10 D range of accommodation mentioned is that for a typical person of college age. As one ages, the lens of the eye continues to grow and fills the sack in which it sits. The result of this growth, known as *presbyopia*, aging eye, is that the range of accommodation is reduced from 10 D to about 1 D at age 65. At 45 years of age the range on average is about 3 D, so that the nearest clearly imaged object is at 33 cm. Since optimal reading and viewing distance is usually taken at 25 cm, some visual correction is necessary. This is usually in the form of reading glasses or bifocal spectacles. Bifocal correction is increased regularly until it reaches 3 to 3.5 D at about 65 years of age, after which no further change is usually required.

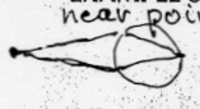


far sighted

need + lens.

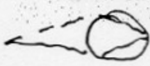
EXAMPLE 5.1 A myopic individual has his far-point of best vision at 16.6 cm and his near-point at about 6.5 cm. What is his range of accommodation? What spectacle correction will restore his far-point to infinity? What will then be his new near-point?

near point



SOLUTION: The far-point of 0.166 m implies a power of 1/0.166, or 6.0 D, and the near-point of 0.0625 implies a power of 16.0 D, thus giving an accommodative range of 10 D. The eye is 6.0 D too strong, so the spectacle correction should be −6.0 D. The range of accommodation will remain the same at 10 D. The new near-point will be 1/10 D or 0.1 m (10 cm).

near sighted



far point

$\frac{1}{S_o} + \frac{1}{S_1} = \frac{1}{f_1}$

count

$\frac{1}{16.6} = -\frac{1}{S_1} + \frac{1}{f_1}$

$\frac{1}{\infty} = \underbrace{-\frac{1}{S_1} + \frac{1}{f_1}}_{\frac{1}{16}} + \frac{1}{f}$

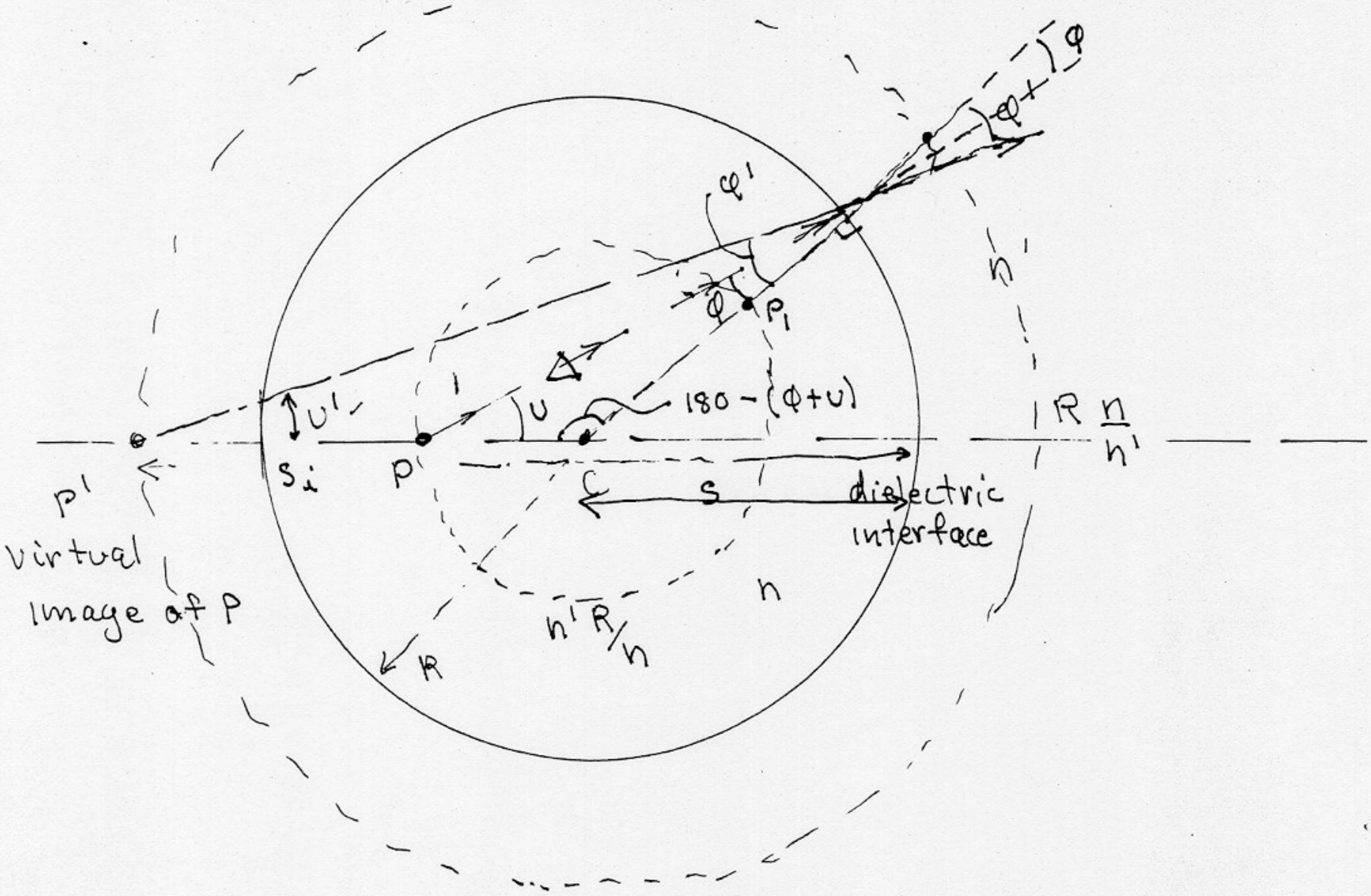
EXAMPLE 5.2 A hyperopic individual has a near-point of 1.5 m. What correction is necessary to move the near-point to 25 cm?

SOLUTION: Using a spectacle lens we need to map objects at 0.25 m onto a plane at a distance of 1.5 m. Using a thin lens and equation (4-1), we obtain

$$-\frac{1}{-0.25} + \frac{1}{-1.5} = P$$

and $P = 3.33$ D. Spectacle lenses are usually available in 0.25 D steps, so that a +3.25 D lens would be used.

Aplanatic Points **Problem No 5** Reference **Sears** (15)



$$s = -\left(R + \frac{n'R}{n}\right) \quad (0) \quad R < 0$$

or $$\frac{(R+S)}{R} = +\frac{n'}{n} \quad (1)$$

(2)

Snell $n \sin\varphi = n' \sin\varphi'$ $\varphi' = U$

Law of sins $$\frac{n'}{n}\frac{\sin\varphi}{\cancel{R}} = \frac{\sin U}{\cancel{R}} = \frac{\sin\varphi'\, n'/n}{n'/n} = \sin\varphi' \quad (3)$$

$$\frac{\sin U'}{R} = \frac{\sin\varphi'}{(S_i - R)} = \frac{n}{n'}\frac{\sin\varphi}{(S_i - R)} \quad (4)$$

Sum of angles of $\triangle$ = 180

$U' + \varphi' + (180) - (\varphi + U) = 180 \Rightarrow U' + \varphi' = \varphi + U$

But $\varphi' = U$ thus $U' = \varphi$.

From 4 then $\frac{1}{R} = \frac{n}{n'}\frac{1}{S_i - R}$ or

$$S_i = R(1 + n/n')$$

Now
$$\frac{n'}{+S_\ell} + \frac{n}{s} = \frac{n'}{+(R\frac{n}{n'}+R)} + \frac{n}{-(R+n'/_n R)}$$

$$= -\left(\frac{-n'R - (n')^2/_n R + n^2/_{n'} R + nR}{R^2(1+n/_{n'})(1+n'/_n)}\right)$$

$$= (\frac{-(n')^2+n^2)/n' + (n^2-n'^2)/n}{R(1+n/_{n'})(1+n'/_n)})$$

$$= -\frac{(n^2-(n')^2)(+1/_{n'}+1/_n)}{R(1+n/_{n'})(1+n'/_n)} = \frac{n'-n}{R}$$

Thus for this case we obtain the paraxial lens equation without the assumption that φ is small!

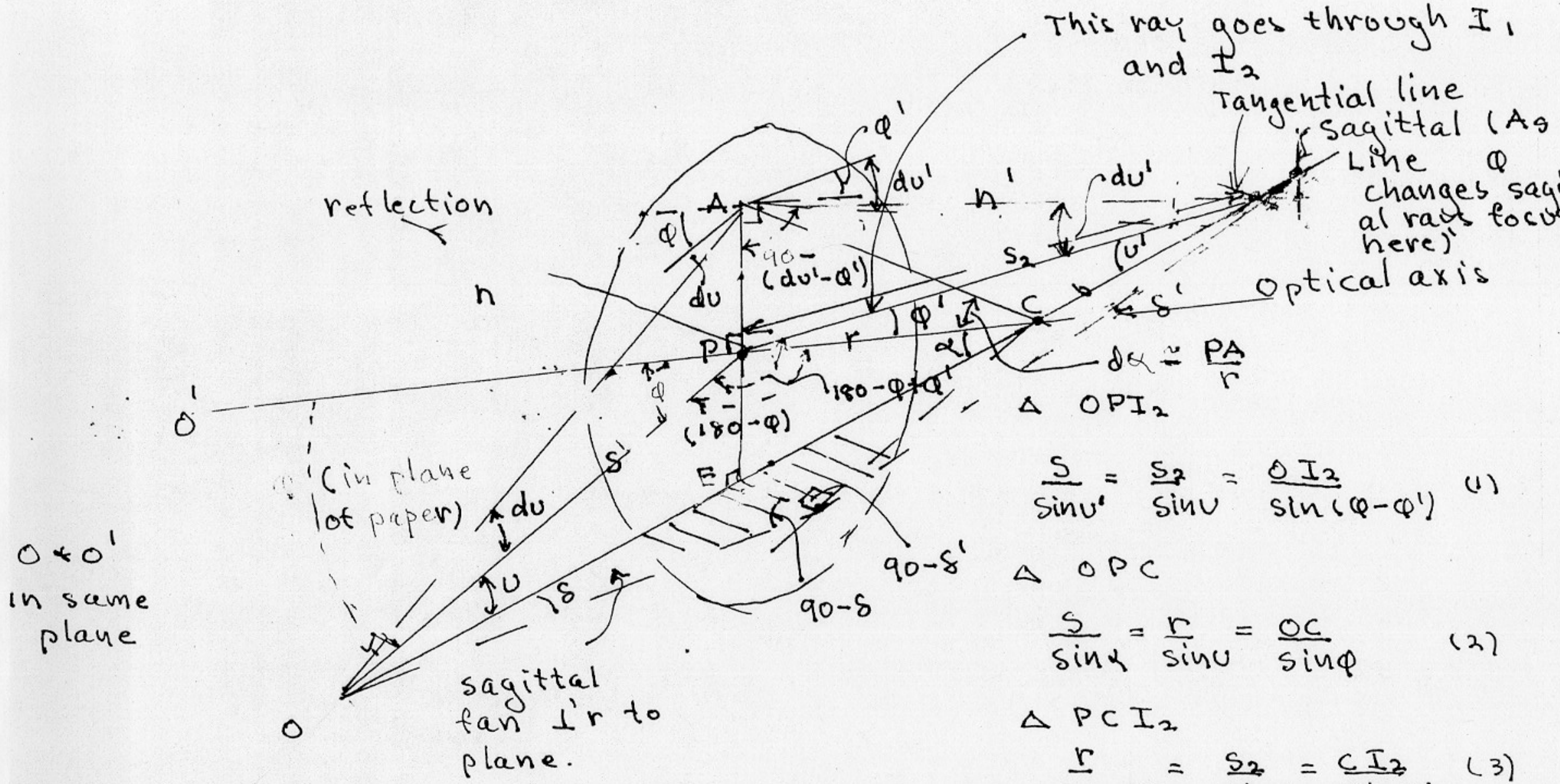
The Abbe sine condition for perfect imaging

$$\sin u = \sin\varphi' = \frac{n}{n'}\sin\varphi = \frac{n}{n'}\sin u'$$

$$\therefore \underrightarrow{n'\sin u = n\sin u'}$$

The outer and inner spheres are perfect images of one another and p and p' are called aplanatic points.

Problem No. six - Astigmatism



$\Delta\ OPI_2$

$$\frac{S}{\sin u'} = \frac{S_2}{\sin u} = \frac{OI_2}{\sin(\varphi-\varphi')} \quad (1)$$

$\Delta\ OPC$

$$\frac{S}{\sin\alpha} = \frac{r}{\sin u} = \frac{OC}{\sin\varphi} \quad (2)$$

$\Delta\ PCI_2$

$$\frac{r}{\sin u'} = \frac{S_2}{\sin\alpha} = \frac{CI_2}{\sin\varphi'} \quad (3)$$

From (2) $OC = \frac{r\sin\varphi}{\sin u}$ - (4) From (3) $CI_2 = \frac{r\sin\varphi'}{\sin u'}$ -(5)

(4) +(5) $OC + CI_2 = OI_2 = r\left(\frac{\sin\varphi}{\sin u} + \frac{\sin\varphi'}{\sin u'}\right)$ - (6)

Substituting in Eq (1)

$$\frac{s}{\sin u'} = \frac{r}{\sin(\phi-\phi')}\left[\frac{\sin\phi}{\sin u} + \frac{\sin\phi'}{\sin u'}\right] \qquad -(7)$$

Eq. (1) $\sin u = s_2 \sin u'/s$ → Eliminate $\sin u$

$$\frac{s}{\cancel{\sin u'}} = \frac{r}{\sin(\phi-\phi')}\left[\frac{s\sin\phi}{s_2\cancel{\sin u'}} + \frac{\sin\phi'}{\cancel{\sin u'}}\right]$$

$$\sin\phi\cos\phi' - \sin\phi'\cos\phi$$

$$n\sin\phi = n'\sin\phi' \qquad -(9)$$

$$s = \frac{r}{\frac{n'}{n}\cancel{\sin\phi'}\cos\phi' - \cancel{\sin\phi'}\cos\phi}\left[s\,\frac{n'}{n}\frac{\cancel{\sin\phi'}}{s_2} + \cancel{\sin\phi'}\right]$$

or $$\frac{n}{s} = \frac{1}{r}\left[n'\cos\phi' - n\cos\phi\right]\frac{1}{\left(s\frac{n'}{n}\cdot\frac{1}{s_2}+1\right)}$$

or $$\frac{n'}{s_2} + \frac{n}{s} = \frac{n'\cos\phi' - n\cos\phi}{r}$$

In reflection $\phi' \to 180-\phi$ and $n'=n$

$$\frac{n}{s_2} + \frac{n}{s} = -\frac{n\cos\phi}{r} \rightarrow \underline{\text{sagittal focus}}$$

All the rays ⊥'r to the tangential plane will focus similarly!

tangential focus.

$180 - \varphi = 180 - (\alpha + u)$ or $\varphi = \alpha + u$

$\varphi' = \cancel{180} - (u' + \cancel{180} - \alpha) = \alpha - u'$

11) - $d\varphi = d\alpha + du \qquad d\varphi' = d\alpha - du' \quad -(12)$

$du \simeq \sin(du) = \qquad du' \simeq \sin du' \qquad \sin d\alpha \simeq d\alpha$

using similarity then

$$\frac{|du|}{PA} = \frac{\sin(90 - (\varphi + du))}{s} = \frac{\cos(\varphi + du)}{s} \simeq \frac{\cos\varphi}{s} - \frac{\sin\varphi \, du}{s}$$

$$\frac{|du'|}{PA} = \frac{\sin(90 - (du' - \varphi'))}{s_2} = \frac{\cos(\varphi' - du')}{s_2} \simeq \frac{\cos\varphi'}{s_2} + \frac{\sin\varphi' du'}{s_2}$$

↑ for φ reasonably small these can be neglected

$$d\alpha = \frac{PA}{r}$$

From (11)

$$\therefore d\varphi = \frac{PA}{r} + PA\frac{\cos\varphi}{s} \qquad (13)$$

$$d\varphi' = \frac{PA}{r} - PA\frac{\cos\varphi'}{s_2}$$

differentiating Snell! $\quad n\cos\varphi \, d\varphi = n'\cos\varphi' d\varphi' \quad (14)$

using (13) then (PA cancels) in (14)

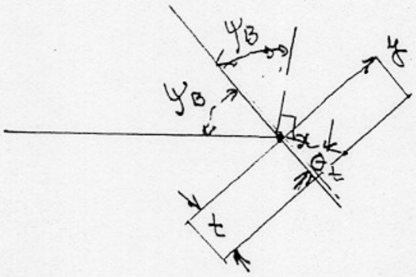
$$n\frac{\cos^2\varphi}{s} + \frac{n\cos\varphi}{r} = n'\left(-\frac{\cos^2\varphi'}{s_2} + \frac{\cos\varphi'}{r}\right)$$

or

$$n\frac{\cos^2\varphi}{s} + n'\frac{\cos^2\varphi'}{s_2} = \frac{n'\cos\varphi' - n\cos\varphi}{r}$$

for reflection $\varphi' = 180 - \varphi$ and $n' = n$

$$n/s + n'/s_2 = (n' - n)/(\cos\varphi \; r)$$



$w_x = w$

$w_y = \frac{w}{\cos\psi_B}$ just outside

Snell $n_0 \sin\psi_B = n \sin\theta_t = n\cos\psi_B$

for $\theta_t = 90 - \psi_B$

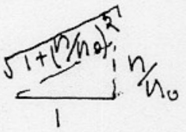
transmitted $w_x = w$

$$w_y = \left(\frac{w}{\cos\psi_B}\right)\cdot\cos\theta_t = \frac{w}{\cos\psi_B}\sin\psi_B$$

$$= w\tan\psi_B = w\left(\frac{n}{n_0}\right)$$

The width traversed by the beam

$$\xi = \frac{t}{\cos\theta_t} = \frac{t}{\sin\psi_B} = \frac{t}{(1-\cos^2\psi_B)^{1/2}} = \frac{t}{\left(1-\frac{1}{(1+(\frac{n}{n_0})^2)}\right)^{1/2}} = t\,\frac{\sqrt{1+(\frac{n}{n_0})^2}}{n/n_0}$$



Now must take diffraction into account

Assume the beam confocal region is at the cell ($z=0$ at cell input)

$w_x = w = w_{0x}$

$w_y = wn = w_{0y}$

The Gaussian beam $w(z)$ is given by

$$w(z) = w_0\left(1+\left(\frac{z}{z_0}\right)^2\right)^{1/2}$$

so $w_x(\xi) = w_{0x}\left(1+\left(\xi/\xi_{0x}\right)^2\right)^{1/2}$ $\quad \xi_{0x} = \frac{\pi w^2}{\lambda} n$

Thus the effective distance is

$$\frac{\xi}{n} = \frac{t}{n^2}\left(1+\left(\frac{n}{n_0}\right)^2\right)^{1/2} = d_x$$

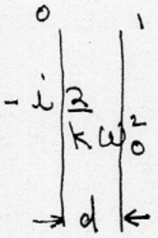
$w_y(\xi) = w_{0y}\left(1+\left(\xi/\xi_{0y}\right)^2\right)^{1/2}$ $\quad \xi_{0y} = \frac{\pi w^2}{\lambda}\left(\frac{n}{n_0}\right)^2 n$

Thus the effective distance is

$$\xi/(n/n_0)^2 \cdot \frac{1}{n} = \frac{t}{n}\left(1+\left(\frac{n}{n_0}\right)^2\right)^{1/2}\Big/\left(\frac{n}{n_0}\right)^3$$

Focussing of a Gaussian beam

$$\frac{1}{q_0} = -i\frac{2}{k\omega_0^2}$$



$$q_1 = \frac{Aq_0+B}{Cq_0+D}$$

$$= \frac{\cos\left(\sqrt{\epsilon_2/\epsilon_0}\,d\right)\left(-i\frac{2}{k\omega_0^2}\right) + \sin\sqrt{\epsilon_2/\epsilon_0}\,d \Big/ \sqrt{\epsilon_2/\epsilon_0}}{\sqrt{\frac{\epsilon_2}{\epsilon_0}}\,\sin\left(\sqrt{\epsilon_2/\epsilon_0}\,d\right)\left(-i\frac{2}{k\omega_0^2}\right) + \cos\sqrt{\frac{\epsilon_2}{\epsilon_0}}\,d}$$

$$\simeq \frac{-i\frac{2}{k\omega_0^2} + d}{\epsilon_2/\epsilon_0\,d\left(-i\frac{2}{k\omega_0^2}\right) + 1}$$

$$\begin{pmatrix} A, & B \\ C & D \end{pmatrix} \simeq \begin{pmatrix} 1 & , & d \\ -\epsilon_2/\epsilon_0\,d & , & 1 \end{pmatrix}$$

$$\therefore\ + f = \frac{\epsilon_0}{\epsilon_2 d} \simeq \left(\frac{1}{500}\right)\frac{1}{d} \simeq$$

$$\sqrt{\frac{\epsilon_2}{\epsilon_0}} \simeq 25\,cm^{-1};\ \sqrt{\epsilon_2/\epsilon_0}\,d \simeq$$