EE 119 — HOMEWORK 4

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Note Title 3/13/2008

1) REFLECTANCE AT NEAR NORMAL INCIDENCE

$$r_{\parallel} = + \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$= \frac{\sin(\theta_i - \theta_t)}{\cos(\theta_i - \theta_t)} \cdot \frac{\cos(\theta_i + \theta_t)}{\sin(\theta_i + \theta_t)}$$

NOTE FOR SMALL VALUES OF θ_i, $\theta_t = \theta_i / n$

$$\Rightarrow r_{\parallel} = \frac{\sin(\theta_i - \theta_i/n)}{\cos(\theta_i - \theta_i/n)} \cdot \frac{\cos(\theta_i + \theta_i/n)}{\sin(\theta_i + \theta_i/n)}$$

NOTE $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

$\cos x \simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

$\longrightarrow$ OR EVEN SIMPLER, WE CAN USE A TANGENT POWER SERIES

$$\tan x \simeq x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$$

NOTICE $\sin x \simeq \tan x$ UP TO x^3 TERM

THUS IF

$$r_{\perp} = - \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$r_{\parallel} = + \frac{\tan(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$r_{\perp} \approx r_{\parallel}$ EXCEPT OR THE MINUS SIGN

AND THUS $[r_{\parallel}]_{\theta_i \approx 0} = \left(\frac{n-1}{n+1}\right)\left(1 + \frac{\theta_i^2}{n}\right)$

2) THE OSCILLATOR MODEL FOR DIELECTRICS

a) $$m_e\ddot{x} + \underbrace{m_e\gamma\dot{x}}_{\text{DAMPING TERM}} + m_e\omega_0^2 x = \underbrace{q_e E(t)}_{\text{DRIVING TERM}}$$

TOGETHER CREATE OSCILLATIONS

b) $$m_e(i\omega)^2 x_0 e^{i(\omega t-\alpha)} + m_e\gamma(i\omega)x_0 e^{i(\omega t-\alpha)}$$

$$+ m_e\omega_0^2 x_0 e^{i(\omega t-\alpha)} = q_e E_0 e^{i\omega t}$$

$$\Rightarrow m_e x_0\left[(i\omega)^2 + \gamma i\omega + \omega_0^2\right] = q_e E_0 e^{i\alpha}$$

$$x_0\left[-\omega^2 + \omega_0^2 + \gamma i\omega\right] = \frac{q_e E_0}{m_e} e^{i\alpha}$$

$$x_0 = \frac{q_e E_0}{m_e} e^{i\alpha} \frac{1}{\left[(\omega_0^2-\omega^2) + i(\gamma\omega)\right]}$$

NOTE

$$x_0 = \sqrt{|x_0 \cdot x_0^*|}$$

$$= \frac{q_e E_0}{m_e}\left(\frac{1}{(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}\right)^{1/2}$$

c) NOTE $e^{i\alpha} = \cos\alpha + i\sin\alpha$

THUS WE CAN PLUG THIS IN AND SET REAL AND IMAGINARY PARTS EQUAL

① $x_0 \gamma \omega = \frac{q_e E_0}{m_e} \sin\alpha$

② $(\omega_0^2 - \omega^2) x_0 = \frac{q_e E_0}{m_e} \cos\alpha$

$\Rightarrow \tan\alpha = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$

$\alpha = \tan^{-1} \frac{\gamma\omega}{\omega_0^2 - \omega^2}$

IF $\omega \ll \omega_0$

$\alpha = \tan^{-1}(0)$

$= 0$

IF $\omega \approx \omega_0$

$\alpha = \tan^{-1}(\infty)$

$= \pi/2$

IF $\omega \gg \omega_0$

$\alpha = \tan^{-1}(0)$

$= \pi$

b/c) NOTE $D = \epsilon E$

$= \varepsilon_0 E + P$

$= \varepsilon_0 E + \varepsilon_0 \chi E$

$= \varepsilon_0 (1 + \chi) E$

WHERE $\chi = \frac{\text{DIPOLE MOMENT}}{\text{UNIT VOLUME}}$

$= \underbrace{x_0 \cdot q}_{\frac{\text{DIPOLE MOMENT}}{\text{OSCILLATOR}}} \underbrace{N}_{\frac{\text{\# OSCILLATORS}}{\text{VOLUME}}}$

d) FOR METALS, ELECTRONS ARE DELOCALIZED AND ACT AS FREE, THUS $\omega_0 = 0$ (RESONANT FREQUENCY). LIGHT BELOW ω_p IS REFLECTED BECAUSE THE ELECTRONS SCREEN THE ELECTRIC FIELD OF THE LIGHT. ABOVE ω_p, ELECTRONS CANNOT ACT FAST ENOUGH TO SCREEN IT AND LIGHT IS TRANSMITTED

e) $$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon}}$$

$$= \sqrt{\frac{N(1.60\times10^{-19})^2}{(9.11\times10^{-31})(8.85\times10^{-12})}}$$

$$= 56.4\sqrt{N}$$

$$= 56.4\sqrt{3\times10^{28}}$$

$$\approx 10^{16} \text{ RAD/SEC}$$

f) IN THE HAGEN RUBENS REGION CONDUCTIVITY DOMINATES AND $\omega \ll \gamma$

NOTE - HAGEN RUBENS, ϵ_i DOMINATES

$$\boxed{n_i^2 = \frac{\epsilon_i}{2\epsilon_0}}$$

$$n_i^2 \simeq n_r^2 = \frac{\epsilon_i}{2\epsilon_0}$$

REFLECTIVITY

$$|r|^2 = R = \frac{(1-\overset{n_r + in_i}{N})(1-N^*)}{(1+N)(1+N^*)}$$

$$= \frac{((1-n_r)^2 + (n_i^2))}{((1+n_r)^2 + (n_i)^2)}$$

$$\simeq \frac{[(1-n)^2+n^2]}{[(1+n)^2+n^2]}$$

$$= \frac{n^2}{n^2}\left(\frac{(1/n-1)^2+1}{(1/n+1)^2+1}\right)$$

$$= \frac{1-\frac{2}{n}+\cdots+1}{1+2/n+\cdots+1}$$

$$\simeq (1-1/n)(1-1/n)$$

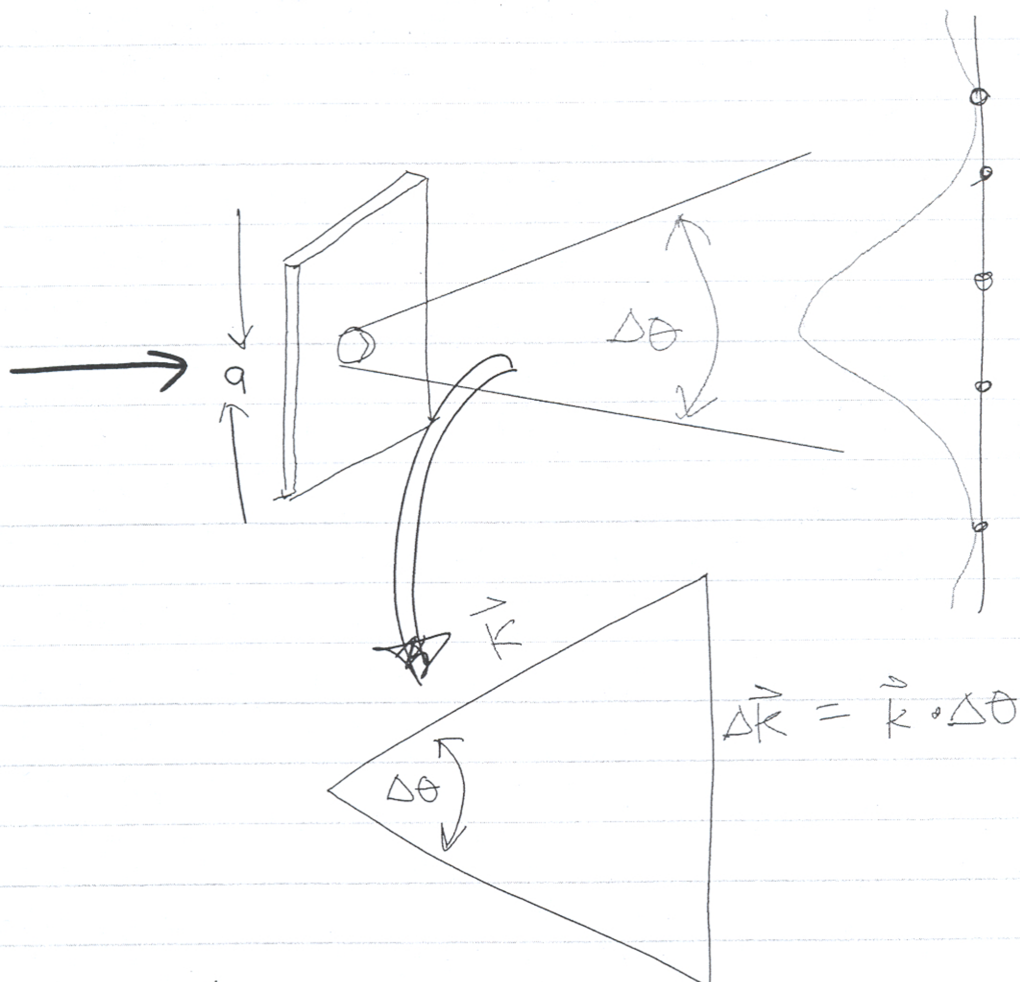
$$= (1-2/n)$$

$$\simeq \left[1-2\left(\frac{2\epsilon_0}{\epsilon_i}\right)\right]$$

$$\simeq 1-2\left(\frac{2\omega\varepsilon}{\omega_p^2}\right)^{1/2}$$

3) BASIC RESOLUTION AND UNCERTAINTY

a)



$$\Delta x \Delta p \sim \frac{\hbar}{2}$$

$$\Delta x(\hbar k) \sim \frac{\hbar}{2}$$

$$\Delta x \Delta k \sim \frac{1}{2}$$

$$\Delta x \, k\cdot\Delta\theta \sim \frac{1}{2}$$

$$\Delta\theta \sim \frac{1}{2k\Delta x}$$

$$\Delta\theta \sim \frac{\lambda}{2(2\pi)(\Delta x)}$$

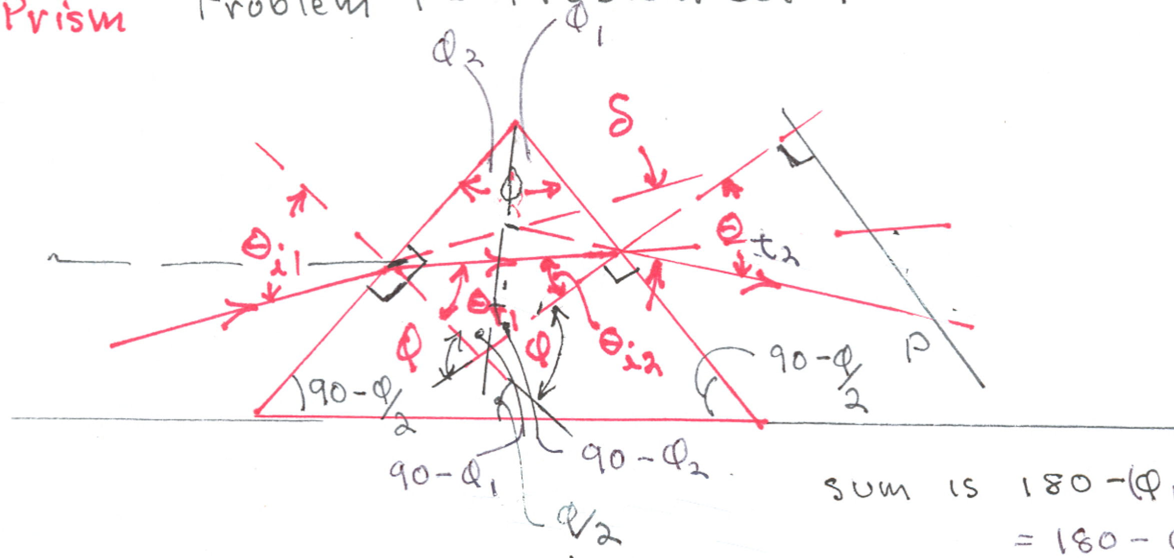
$$\Delta\theta \sim \frac{\lambda}{4\pi a} \approx \frac{\lambda}{a}$$

b) IN THIS CASE, THE ENTRANCE PUPIL IS ALSO IMAGED BY THE OBJECTIVE LENS. THUS IT ACTS AS THE CIRCULAR BOUNDARY FOR THE OBJECTIVE LENS. THIS EXPLAINS WHY ONE HAS TO LOOK AT JUST THE RIGHT ANGLE TO SEE THROUGH A TELESCOPE.

c) FOR A TELESCOPE

$$R = \frac{\lambda}{D}$$

$$= \frac{\lambda}{2a}$$



sum is $180-(\varphi_1+\varphi_2)$
$= 180-\varphi$

① $n_0 \sin\theta_{i_1} = n \sin\theta_{t_1}$

gives θ_{t_1}

② $\theta_{i_2} = 180-(\theta_{t_1}+180-\varphi)$

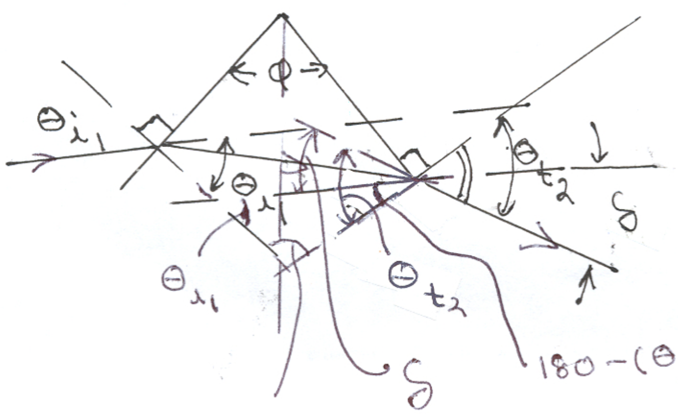
$= (\varphi-\theta_{t_1})$

③ $n \sin(\varphi-\theta_{t_1}) = n_0 \sin\theta_{t_2}$

$n(\sin\varphi \cos\theta_{t_1} - \cos\varphi \sin\theta_{t_1}) = n_0 \sin\theta_{t_2}$

$$n\left(\sin\varphi\sqrt{1-\left(\frac{n_0\sin\theta_{i_1}}{n}\right)^2} - \cos\varphi\,\frac{n_0}{n}\sin\theta_{i_1}\right) = n_0\sin\theta_{t_2}$$

④ The deviation $= \delta$



$180-(\theta_{i_1}+(180-\varphi))$

$180-\varphi$ $\quad = \varphi-\theta_{i_1}$

Thus $\boxed{\delta = \theta_{t_2}-(\varphi-\theta_{i_1})}$ (Eq. (5.53) of text)

Thus

$$\delta = \theta_{i_1} - \overset{\text{Apex angle}}{\varphi} + \sin^{-1}\left[\frac{n}{n_0}\left(\sin\varphi\sqrt{1-\left(\frac{n_0 \sin\theta_{i1}}{n}\right)^2} - \cos\varphi \frac{n_0}{n}\sin\theta_{i1}\right]\right.$$

To determine the minimum deviation we take $\frac{d\delta}{d\theta_{i_1}} = 0$. Letting $\left(\frac{n}{n_0}\right)^2 - \sin^2\theta_{i1} = y$

and $\sin\varphi\, y^{1/2} - \cos\varphi \sin\theta_{i1} = x$

$$\frac{d\delta}{d\theta_{i1}} = 1 + \frac{d}{d\theta_{i1}}\sin^{-1}(x(y))$$

$$= 1 + \frac{1}{\sqrt{1-x^2}}\frac{dx}{d\theta_{i1}}$$

$$= 1 + \frac{1}{\sqrt{1-x^2}}\left[\frac{\sin\varphi}{2}\frac{1}{y^{1/2}}\frac{dy}{d\theta_{i1}} - \cos\varphi\cos\theta_{i_1}\right]$$

$$= 1 + \frac{1}{\sqrt{1-x^2}}\left[\sin\varphi\frac{1}{y^{1/2}}(-\sin\theta_{i1}\cos\theta_{i1}) - \cos\varphi\cos\theta_{i1}\right]$$

$$= 0$$

so $-y^{1/2}\sqrt{1-x^2} = \sin\varphi(-\sin\theta_{i1}\cos\theta_{i_1}) - y^{1/2}\cos\varphi\cos\theta_{i_1}$

$$y = \left(\frac{n}{n_0}\right)^2 - \sin^2\theta_{i_1} = \left(n/n_0\right)^2 - \left(\frac{n}{n_0}\right)^2\sin^2\theta_{t_1} = \left(n/n_0\right)^2\cos^2\theta_{t_1}$$

Solution is $\theta_{t_1} = \varphi/2 = \theta_{i2}$ (the transmission thru the prism is horizontal: check

$$x = \sin\varphi\, n/n_0 \overset{\text{used Snell}}{\cos\theta_{t_1}} - \cos\varphi\, n/n_0 \sin\theta_{t_1} =$$

$$= n/n_0\left(\sin(\varphi - \theta_{t_1})\right) \quad (1-x^2)^{1/2}$$

Thus

$$n/n_0 \cos\theta_{t_1}\left(1 - \left(n/n_0 \sin(\varphi-\theta_{t_1})\right)^2\right)^{1/2} = \sin\varphi\,(+\underbrace{\sin\theta_{i1}}_{n/n_0 \sin\theta_{t_1}}\cos\theta_{i1})$$

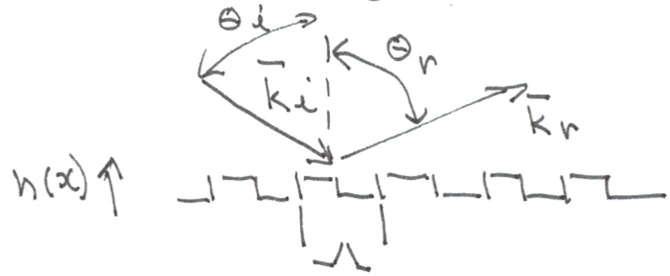
$$+ n/n_0\cos\theta_{t_1}\cos\varphi\cos\theta_{i1}$$

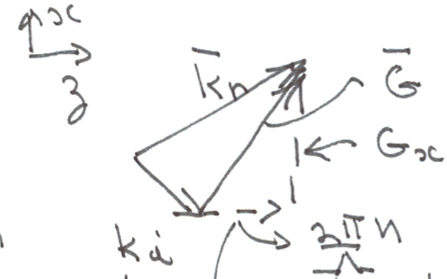
$$n/n_0\cos\theta_{t_1}\left(1-\left(n/n_0\right)^2\sin^2\underbrace{(\varphi-\theta_{t_1})}_{\theta_{i2}}\right)^{1/2} = +n/n_0\cos\theta_{i1}\cos(\varphi-\theta_{t_1})$$

so $\cos\theta_{t_1}\cos\theta_{t2} = +\cos\theta_{i_1}\cos(\varphi - \theta_{t_1})$

$\therefore$ $\theta_{t2} = \theta_{i_1}$ and $\theta_{t_1} = \theta_{i2} = \varphi/2$

Problem No. 5 Part a





Simple Approach

Conservation of momentum

(2) $\bar{k}_r = \bar{k}_i + \bar{G} \leftarrow$ momentum suppled (or absorbed)
Grating
by grating

This is basically a phase argument

The grating can be expanded as a Fourier Series

$$h(x) = \sum F_n e^{i\left(\frac{2\pi}{\Lambda} n\right) z} \quad (1)$$

This is reflected in such properties as the dielectric coefficient

But $D = \epsilon_0 E + \epsilon_0 \chi E$

so if χ is of the form of (1) (Fourier series) and $\bar{E}$ is the incident field then $\bar{D}$ has terms with $k_{iz} \mp n\frac{2\pi}{\Lambda}$ which generates the radiated field thus $k_{rz} = k_{iz} + n\frac{2\pi}{\Lambda}$

or $k \sin\theta_r = k \sin\theta_i + n\frac{2\pi}{\Lambda}$; $k = \omega/c$

This is the z-component of (2).

For the x-component a sharp boundary has a continuous k_x spectrum. It can thus supply (or absorb) any G_x necessary to guarantee that $k_r = (\omega/c)$ (from the wave equation)

b) Resolution

$$\overbrace{L = N\Lambda}^{\text{Number of grating periods}}$$

$$\longleftrightarrow \quad \vec{z}$$

$$\Delta k_z \Delta z \simeq 2\pi$$

$$\Delta k_z \simeq \pi/\Delta z \simeq 2\pi/L$$

$$\therefore \; \Delta(k \sin\theta_r) \simeq 2\pi/L$$

$$\text{or } (\Delta \sin\theta_r) \simeq \frac{2\pi}{kL} \simeq \frac{\lambda}{L}$$

Scattering from a Grating: - Using an effective susceptibility (1)

. Prob 5 Problem Set 4 ↘ More rigorous approach.

part b) $\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t}$

$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

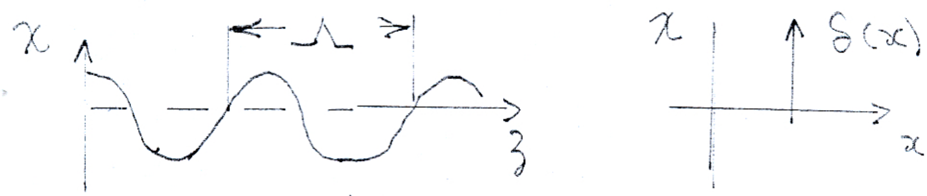
$\frac{\partial \bar{P}}{\partial t}$ is the current excited by the incident radiation field

$\bar{B} = \nabla \times A$

$$\nabla \times \frac{(\nabla \times \bar{A})}{\mu} = \frac{\partial \epsilon_0 \bar{E}}{\partial t} + \frac{\partial \bar{P}}{\partial t}$$

$$\bar{E} = -\nabla \varphi - \frac{\partial \bar{A}}{\partial t} \qquad \nabla \cdot \bar{A} + \mu \epsilon_0 \frac{\partial \varphi}{\partial t} = 0$$

$$\therefore -\nabla^2 \bar{A} + \mu \epsilon_0 \frac{\partial^2 \bar{A}}{\partial t^2} = \boxed{\mu_0 \frac{\partial \bar{P}}{\partial t}}$$ Driving term

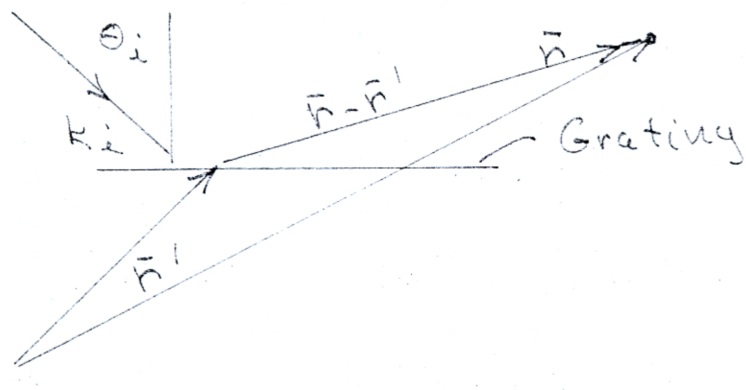


$$\chi(z,x) = \int e^{i k_x x} \sum_m e^{-i \frac{2\pi}{\Lambda} m z} A_m \, dk_x$$

Driving term is $i\omega\, \chi(z,x)\, E_i\, e^{-i k_{ix} x - i k_{iz} z + i\omega}$ which causes radiation

Solution for $\bar{A}$

$$\bar{A} = \frac{\mu \epsilon_0}{4\pi} \int \frac{i\omega\, \chi(z', x')}{|\bar{r} - \bar{r}'|} e^{i\omega t - i \bar{k}_r \cdot (\bar{r} - \bar{r}')} E_i$$



$$\tilde{A} \doteq \frac{\mu}{4\pi}\frac{i\omega}{r}\iint \overbrace{A_m e^{+ik_x x'} e^{-i\frac{2\pi}{\Lambda} m z'}}^{\text{Grating}} \cdot \overbrace{E_i\, e^{+ik_{ix}x' - ik_{iz}z'}}^{\text{Inc Field}}$$

$$dk_x\, e^{-ik_{rx}(x-x')}\, e^{-ik_{rz}(z-z')}\, dx'dz'$$

Integrate over $\underline{dx'}$ gives $\underline{\delta(k_{rx} + k_x + k_{ix})}$ $\int dk_x$ is cleared with $\underline{k_{rx} = -(k_x + k_{ix})}$. Left with

$$\tilde{A} = \tilde{E}_i \frac{\mu\epsilon_0}{4\pi}\frac{i\omega}{r} A_m \int_{-L/2}^{+L/2} e^{-i\frac{2\pi}{\Lambda} m z'}\, e^{-ik_{iz}z'}\, e^{+ik_{rz}z'}\, dz'$$

$$\times\, e^{-ik_{rz}z - ik_{rx}x}$$

$$= \underline{\text{Const}}\; e^{-ik_{rz}z - ik_{rx}x}\, \frac{L}{2}\left(\frac{e^{-i(\Delta k)L/2} - e^{+i\Delta k \frac{L}{2}}}{i\,\Delta k\, L/2}\right)$$

$$\text{Const} = \tilde{E}_i\, \mu/4\pi\, \epsilon_0\, i\omega/r\, A_m$$

$$\Delta k = \left(k_{iz} + m\frac{2\pi}{\Lambda} - k_{rz}\right)$$

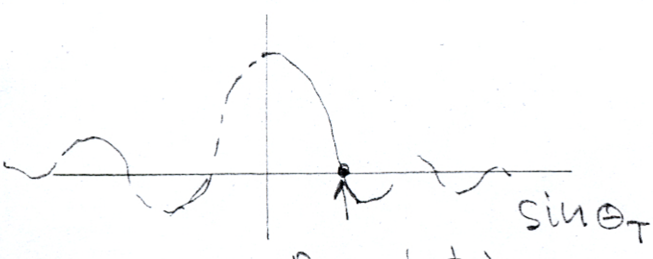
$$= L\ \text{Const}\; e^{-ik_{rz}z - ik_{rx}x}\, \frac{\sin \Delta k\, L/2}{(\Delta k\, L/2)}$$

$$k_{iz} = \omega/c\, \sin\theta_i \;;\; k_{rz} = \omega/c\, \sin\theta_r$$

$\therefore$ The linewidth behavior is

$$\frac{\sin\left(\frac{\omega}{c}\frac{L}{2}\right)\left[\sin\theta_i - \sin\theta_T - m\frac{\lambda}{\Lambda}\right]}{\frac{\omega}{c}\frac{L}{2}\left[\sin\theta_i - \sin\theta_T - m\frac{\lambda}{\Lambda}\right]}$$

$$\frac{\omega}{c}\frac{L}{2}\left[\Delta(\sin\theta) - m\frac{\lambda}{\Lambda}\right] = \pi$$



Resolution Condition

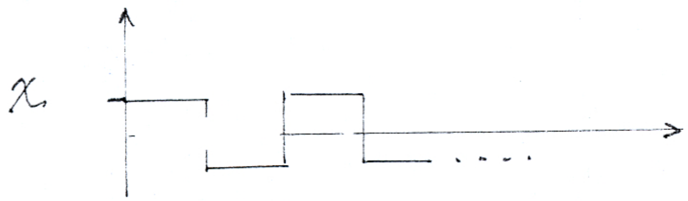
Let $\sin\theta_r = \sin\theta_i - m\lambda/\Lambda + \delta$ then

$\delta \frac{\omega}{c} \frac{L}{2} = \pi$ or $\delta \simeq (\lambda/L)$

Thus the angular resolution is $\underline{\text{wavelength}}$ / L

For a metal

$$p(t) = \left(\frac{\omega_p^2}{-\omega^2 + i\omega\delta}\right) E_i e^{i\omega t} \simeq -\frac{\omega_p^2}{\omega^2} E_i e^{i\omega t}$$



and $\omega_p/\omega \simeq 10$ in the visible

Note that at the peak, the radiated field is

$$L \text{ Const } e^{-ik_{iz}z - ik_{rx}x + im\frac{2\pi}{\Lambda}z}$$

and k_r is given by

$$\left(k_{iz} - m\frac{2\pi}{\Lambda}\right)^2 + (k_{rx})^2 = \omega^2/c^2$$

since the reflected field must satisfy the wave equation

Note the extra phase change $m\frac{2\pi}{\Lambda}z$

due to the grating momentum

c) Dispersion $\sin\theta_r = \sin\theta_i - m\lambda/L$

$$\cos\theta_r \frac{d\theta_r}{d\lambda} = m/L$$

$$\therefore \left(\frac{d\theta_r}{d\lambda}\right) = \frac{m}{L}\frac{1}{\cos\theta_r} = \frac{-\sin\theta_r + \sin\theta_i}{\lambda\cos\theta_r}$$

Thus

$$\delta = \theta_{i_1} - \underbrace{\varphi}_{\text{Apex angle}} + \sin^{-1}\left[\frac{n}{n_0}\left(\sin\varphi\sqrt{1-\left(\frac{n_0 \sin\theta_{i1}}{n}\right)^2} - \cos\varphi\,\frac{n_0}{n}\sin\theta_{i1}\right]\right.$$

To determine the minimum deviation we take $\frac{d\delta}{d\theta_{i_1}} = 0$. Letting $\left(\frac{n}{n_0}\right)^2 - \sin^2\theta_{i1} = y$

and $\sin\varphi\, y^{1/2} - \cos\varphi \sin\theta_{i1} = x$

$$\frac{d\delta}{d\theta_{i1}} = 1 + \frac{d}{d\theta_{i1}}\sin^{-1}(x(y))$$

$$= 1 + \frac{1}{\sqrt{1-x^2}}\frac{dx}{d\theta_{i1}}$$

$$= 1 + \frac{1}{\sqrt{1-x^2}}\left[\frac{\sin\varphi}{2}\frac{1}{y^{1/2}}\frac{dy}{d\theta_{i1}} - \cos\varphi\cos\theta_{i1}\right]$$

$$= 1 + \frac{1}{\sqrt{1-x^2}}\left[\sin\varphi\frac{1}{y^{1/2}}(-\sin\theta_{i1}\cos\theta_{i1}) - \cos\varphi\cos\theta_{i1}\right]$$

$$= 0$$

so $-y^{1/2}\sqrt{1-x^2} = \sin\varphi(-\sin\theta_{i1}\cos\theta_{i1}) - y^{1/2}\cos\varphi\cos\theta_{i1}$

$$y = \left(\frac{n}{n_0}\right)^2 - \sin^2\theta_{i_1} = (n/n_0)^2 - \left(\frac{n}{n_0}\right)^2\sin^2\theta_{t1} = (n/n_0)^2\cos^2\theta_{t1}$$

Solution is $\theta_{t1} = \varphi/2 = \theta_{i2}$ (the transmission thru the prism is horizontal: check

$$x = \sin\varphi\, n/n_0 \cos\theta_{t1} - \cos\varphi\, n/n_0 \sin\theta_{t1} =$$

used Snell

$$= n/n_0(\sin(\varphi - \theta_{t1})) \qquad (1-x^2)^{1/2}$$

Thus

$$n/n_0\cos\theta_{t1}\left(1-\left(n/n_0\sin(\varphi-\theta_{t1})\right)^2\right)^{1/2} = \sin\varphi\,(+\sin\theta_{i1}\cos\theta_{i1})$$

$$\sin\theta_{i1} = n/n_0 \sin\theta_{t1}$$

$$+ n/n_0\cos\theta_{t1}\cos\varphi\cos\theta_{i1}$$

$$n/n_0\cos\theta_{t1}\left(1-\left(\frac{n}{n_0}\right)^2\sin^2(\underbrace{\varphi-\theta_{t1}}_{\theta_{i2}})\right)^{1/2} = +n/n_0\cos\theta_{i1}\cos(\varphi-\theta_{t1})$$

so $\cos\theta_{t1}\cos\theta_{t2} = +\cos\theta_{i1}\cos(\varphi-\theta_{t1})$

$\therefore\ \theta_{t2} = \theta_{i1}$ and $\theta_{t1} = \theta_{i2} = \varphi/2$