EE119 Homework 3

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Due Monday, February 16, 2009

1. In class we have discussed that the behavior of an optical system changes when immersed in a liquid. Show that the longitudinal image magnification for an optical system not immersed in air is $(n_2/n_1)m^2$, where the object is immersed in a material of index n_1 and the image is immersed in a material of object n_2. You can also refer to the diagram on the bottom of p. 17 of the notes for the labeling of n_1 and n_2.

   Solution:

   On p. 18 of the notes we are given that the lens law is

$$\frac{n_2}{d_2} = \frac{n_1}{f_1} + \frac{n_1}{d_1}$$

$$d_2 = \frac{n_2}{\frac{n_1}{f_1} + \frac{n_1}{d_1}} = \frac{n_2}{n_1}\frac{1}{\frac{1}{f_1} + \frac{1}{d_1}} = \frac{n_2}{n_1} d_{2\text{original}}$$

   and the longitudinal magnification now is

$$\frac{\partial d_2}{\partial d_1} = \frac{n_2}{n_1}\frac{\partial d_{2\text{original}}}{\partial d_1} = \frac{n_2}{n_1} m^2$$

   for the front and back focal length of a lens.

2. You will explore some of the differences between real and paraxial rays in this problem. For each part below, trace the specified ray and determine where it crosses the optical axis. Show all calculations and include a diagram. Report your answers to 4 decimal places.

   (a) Find L when $\theta = 5°$with real ray (no paraxial approximation).
   Solution:
   First we need to find the incident angle I. From the law of sines, we know that

$$\frac{R}{\sin(\theta)} = \frac{P+R}{\sin(180-I)} = \frac{P+R}{\sin(I)}$$

   So, rearranging, we get that

$$I = \sin^{-1}\left(\frac{P+R}{R}\sin(\theta)\right) \tag{1}$$

   We can use Snell's law to solve for I'

$$n_1 \sin(I) = n_2 \sin(I'). \tag{2}$$

Since $L = R + OP'$, we need to solve for OP'. By the law of sines,

$$\frac{R}{\sin(\phi)} = \frac{OP'}{\sin(I')}$$

From geometry we see that $I = \theta + I' + \phi$, and we can solve for ϕ since we already know θ, I, and I', so

$$OP' = \sin(I')\frac{R}{\sin(I - \theta - I')}.$$

And we get that

$$L = R + \sin(I')\frac{R}{\sin(I - \theta - I')} = R(1 + \frac{\sin(I')}{\sin(I - \theta - I')}) \qquad (3)$$

Plugging in numbers, $I = 31.5293, I' = 24.9972$ and $L = 168.0442$. I used the following matlab code for this:

```
theta=0.5*pi/180
P=50
R=10
n1=4/3
n2=1.65

I=asin((P+R)*sin(theta)/R)
Iprime=asin(n1*sin(I)/n2)

L=R*(1+sin(Iprime)/sin(I-theta-Iprime))
sols=[I*180/pi, Iprime*180/pi, L]

Lparaxial=n2/(-n1/P+(n2-n1)/R)
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(b) Find L when $\theta = 0.5°$with real ray (no paraxial approximation). Using the above matlab code gives $I = 3.0013, I' = 2.4249, L = 327.3102$cm

(c) Repeat part (a) with the paraxial ray (paraxial approximation).
In the paraxial approximation, we have the relationship

$$\frac{n_2}{L} - \frac{n_1}{-P} = \frac{(n_2 - n_1)}{R}$$

So, with numbers,

$$L = \frac{n_2}{\frac{n_1}{-P} + \frac{(n_2 - n_1)}{R}} = \frac{1.65}{\frac{4/3}{-50} + \frac{(1.65 - 4/3)}{10}} = 330.00\text{cm}$$

(d) Repeat part (b) with the paraxial ray (paraxial approximation). In the paraxial approximation, L is the same, and is again 330.00

(e) Is there a difference between your answers in (a) and (b)? Is there any difference between your answers in (c) and (d)? There is a difference between the answers of a and b, but not between answers of c and d.

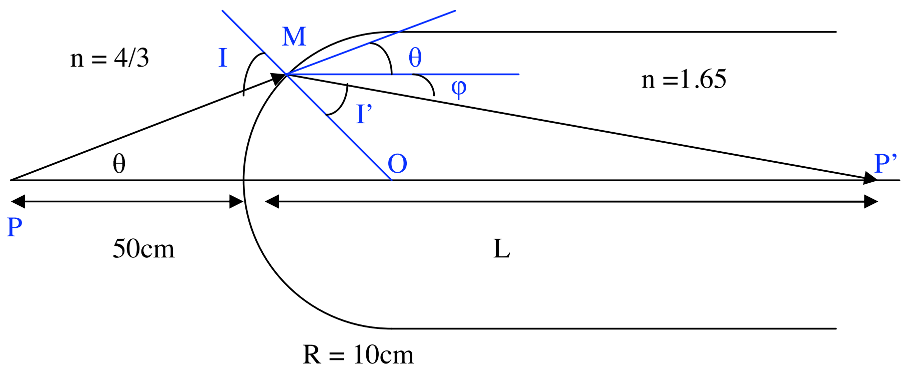


Figure 1: diagram for problem 2.

(f) Now compare your answers in (b) and either (c) or (d). Why are they so similar? The answers in c and d are similar to b because 0.5 is a small angle and the paraxial approximation is pretty valid. However, even though 5°is pretty small, the incident angle of 30 is not small at all!

3. Consider a thin, spherical plano-convex lens having a radius of curvature of 50.0mm and a refractive index of 1.5.

(a) Determine the focal length in air.

Solution:

$$\frac{1}{f} = (n_l - 1)(\frac{1}{R_1} - \frac{1}{R_2}) = 0.5 \times (\frac{1}{50} - 0) = \frac{1}{100}$$

So the focal length is 100 mm.

(b) Suppose this lens is placed right on the surface of a tank of water. At what depth below the surface would a collimated light beam from above come to a focus? (The refractive index of water is 1.3.)

Solution:

We use $\frac{n_1}{f_2} = \frac{n_2}{f_2}$ to get that the focal length in water is 130 mm=1.3 cm. If you use the lensmaker's equation and assumed that the lens is covered in water, you would get

$$\frac{n_{\text{medium}}}{f} = (n_l - n_{\text{medium}})(\frac{1}{R_1} - \frac{1}{R_2}) = 0.2 \times (\frac{1}{50} - 0) = \frac{2}{500}$$

so the focal length would be $1.3 \times 250 = 325$ mm in water. This isn't a completely accurate interpretation of the problem, but if I took off points for this answer I'll give you the points back.

4. A spherical wave can be mathematically described by:

$$E = \frac{E_0}{r}\text{Re}\left(e^{-ik\cdot r+\phi}\right) = \frac{E_0}{r}\cos(-kr + \phi)$$

Where r is the distance of the wave front from the source in 3D space, k is the wave vector (the amplitude of k is given by $\frac{2\pi}{\lambda}$, and ϕ is a constant phase. In this problem, the optic axis is on the z-axis.

(a) Show that at a large distance from the point source, close to the optic axis, at constant z, the wave can be approximated by

$$E = \frac{E_0}{z}\cos[-\frac{k(x^2+y^2)}{2z} + \phi]$$

In this approximation, z ≫(x,y). Hint: the taylor expansion of $\sqrt{1+x}$ for $x \ll 1$ is $\sqrt{1+x} \approx 1 + \frac{x}{2}$.

Solution:

Going from spherical to cartesian coordinates gives $r = \sqrt{x^2+y^2+z^2}$. When $z \gg (x, y)$, this expression can be rewritten and Taylor expanded

$$r = z\sqrt{1 + \frac{x^2+y^2}{z^2}} \approx z(1 + \frac{x^2+y^2}{2z^2})$$

We also need the approximation that $\frac{1}{1+x} \approx 1$ for $x \ll 1$. Substituting the approximation for r into the original expression,

$$E = \frac{E_0}{z(1+\frac{x^2+y^2}{2z})}\cos(-kz(1+\frac{x^2+y^2}{2z^2}) + \phi) \approx \frac{E_0}{z}\cos(-kz - k\frac{x^2+y^2}{2z} + \phi) \quad (4)$$

At constant z, the amplitude is constant, and the $-kz$ term can be incorporated into the constant phase ϕ, leaving us with

$$E \approx \frac{E_0}{z}\cos(-k\frac{x^2+y^2}{2z} + \phi)$$

(b) Show that when this wave is propagating in the z-direction, it can be approximated by a plane wave of constant amplitude. Assume that the distance it travels, Δz, is much smaller than the distance from the point source. A plane wave can be mathematically described by

$$E = E_0\cos(kz + \phi)$$

Where E_0 is the amplitude, k is the wave vector and ϕ is a constant phase.

Solution:

We use equation 4 and set $x = y = 0$ on the z-axis. this gives

$$E \approx \frac{E_0}{z}\cos(-kz + \phi)$$

Since the range over which the wave travels is much smaller than the distance from the point source, we can assume that the magnitude of z is constant (As opposed to the cosine of the magnitude, which oscillates). So we can absorb the z in the magnitude into the constant magnitude, getting

$$E \approx E\cos(-kz + \phi)$$

(c) Show that a spherical wave emerging from a given object point will be converted into a spherical wave converging to the image point given by the usual Gaussian lens law, by using the thin-lens-phase shift.

Solution:

Assume that the lens is at the origin. A spherical wave emerging from an object point located at d_1 has the form

$$E = \frac{E_0}{d_1}\cos[-\frac{k(x^2+y^2)}{2d_1}]$$

The phase shift that this light acquires when it passes through a thin lens is

$$\Delta\phi = -\frac{k(x^2+y^2)}{2f}$$

so the form of the resulting wave after passing through the lens will be

$$E = \frac{E_0}{d_1}\cos[-\frac{k(x^2+y^2)}{2d_1} - \frac{k(x^2+y^2)}{2f}]$$

$$= \frac{E_0}{d_1}\cos[-\frac{k(x^2+y^2)}{2}(\frac{1}{d_1}+\frac{1}{f})] = \frac{E_0}{d_1}\cos[-\frac{k(x^2+y^2)}{2d_2})]$$

So the resulting wave has the form of a spherical wave emanating from a point d_2.

5. Two thin lenses with focal lengths of + 6.0 cm and + 8.0 cm and apertures of 8.0 cm and 9.0 cm, respectively, are located 4 cm apart. An aperture stop (3 cm in diameter) is located between the two lenses, 3 cm from the first lens. An object 3 cm high is located with its center 12 cm in front of the first lens.

   (a) Draw a diagram that shows the entire imaging system.

   (b) Find the position and size of the entrance and exit pupils and draw them in the diagram from part (a).

   Solution:

   The exit pupil is the image of the aperture stop through the lenses between the aperture stop and the image. In this case, the exit pupil is the image of the aperture stop through the 8 cm lens. Since the aperture stop is located 1 cm to the left of the lens, its image will be at

$$\frac{1}{d_2} = \frac{1}{8} + \frac{1}{-1} = -\frac{7}{8}$$

   So the exit pupil will be $\frac{8}{7}$ cm to the left of the first lens. The magnification is $\frac{8}{7}$, so the diameter of the exit pupil is $\frac{24}{7} = 3.4286$ We can calculate the entrance pupil in the same way:

$$\frac{1}{d_2} = \frac{1}{6} + \frac{1}{-3} = -\frac{1}{6}$$

   So the entrance pupil will be 6 cm to the right of the first lens, and its diameter will be $3 \times \frac{6}{3} = 6cm$.

   (c) Find the position and size of the final image and draw it in the diagram from part (a).

(d) the image formed by the first lens is located at

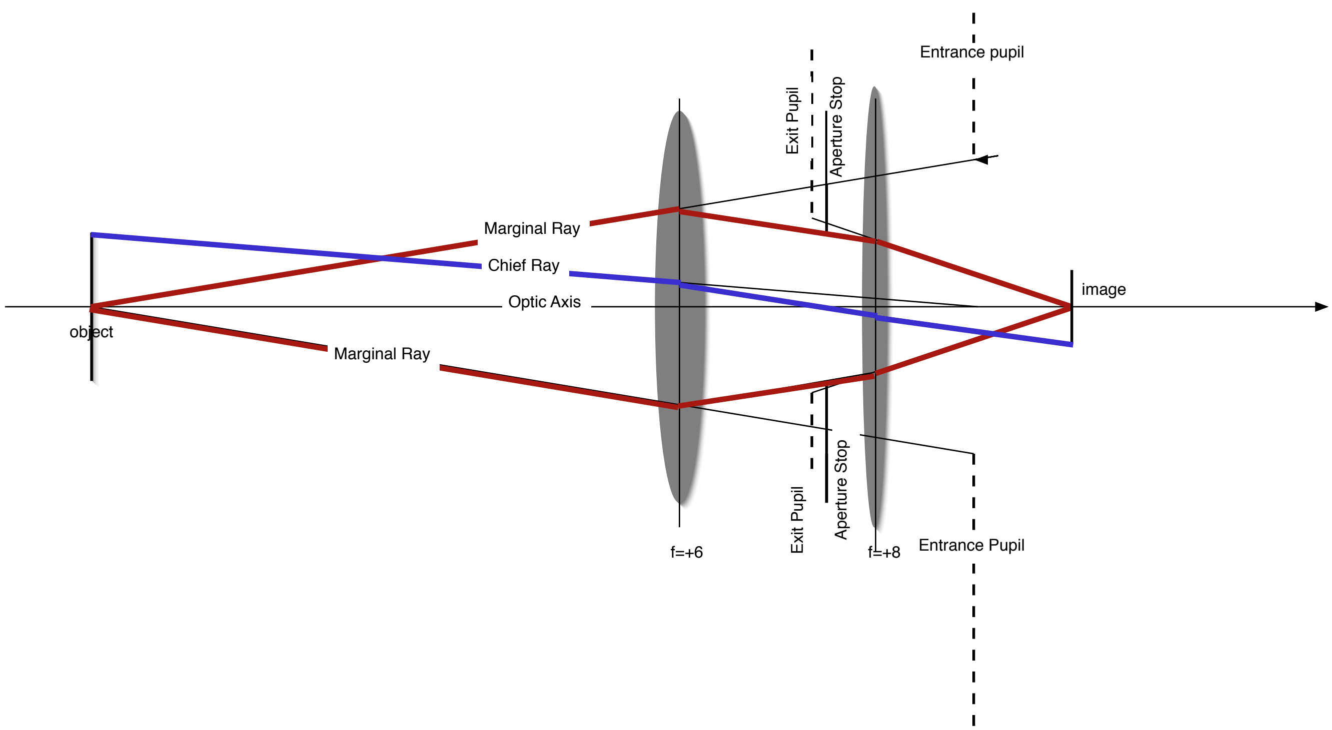
$$\frac{1}{d_2} = \frac{1}{6} - \frac{1}{12} = \frac{1}{12} =$$

so the first image is located 12 cm to the right of the first lens. The magnification of the first lens is -1. This image is the object for the second lens, so d_1 for the second lens is 12-4=8 cm. Then the final image is at

$$\frac{1}{d_2} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

So the final image is 4 cm to the right of the final lens. The magnification for the second lens is $4/8 = 0.5$. So the total magnification of the lens system is -0.5, meaning that the image will be 1.5 cm high.

(e) Draw the two marginal rays and the chief ray from the top end of the object and trace them all the way to the image.



6. [Adapted from Hecht 6.14] A compound lens is composed of two thin lenses separated by 10 cm. The first of these has a focal length of +20cm, and the second a focal length of 30cm. Determine the focal length of the combination and locate the corresponding principal points. Draw a diagram of the system.

Solution:

Equation 6.8 of Hecht tells us that the effective focal length of a compound lens is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \frac{1}{20} + \frac{1}{30} - \frac{10}{600} = \frac{4}{60}$$

So the effective focal length of the combination is 15 cm. The distance from the center of the first lens to the first principal plane ie

$$h_1 = \frac{fd}{f_2} = \frac{15 \times 10}{30} = 5\text{cm}$$

And the distance from the center of the second lens to the second principal plane ie

$$h_2 = \frac{fd}{f_1} = \frac{15 \times 10}{20} = 7.5\text{cm}$$

With these numbers, we can find that the front focal length is 15-5=10 cm and back focal length is 15-7.5=7.5 cm.