Discussion 11. EE119 Xi Luo 4/19/2010

I. Diffraction – general expression

$$u(P,t) = Re[U(P)e^{-j\omega t}]$$

$U(P)$: represents the complex field amplitude

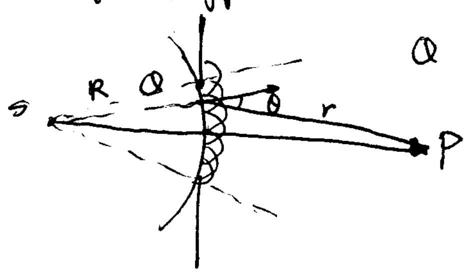
The Huygens-Fresnel Principle

$\tilde{E}_0(\xi,\eta)$ $\tilde{E}(\xi,\eta)$

screen with aperture

$t(\xi,\eta)$

Complex amplitude transmission coefficient.



$$d\tilde{E}(P) = C K(\theta) \frac{A\cdot \exp(ikR)}{R} \frac{\exp(ikr)}{r} d\sigma$$

$\theta = 0$, K max Obliquity factor

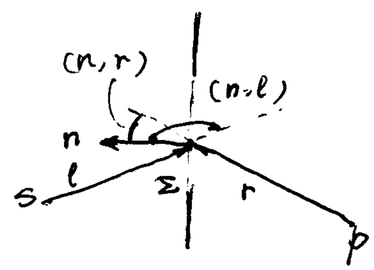
$$\tilde{E}(P) = \frac{CA\exp(ikR)}{R} \iint_{\Sigma} \frac{\exp(ikr)}{r} K(\theta) d\sigma.$$

diffraction essentially is interference.

Kirchhoff found the expression of $K(\theta)$

$$K(\theta) = \frac{\cos(n,r) - \cos(n,l)}{2}$$

$$C = \frac{1}{i\lambda}$$



Fresnel-Kirchhoff diffraction formula

$$\tilde{E}(P) = \frac{A}{i\lambda} \iint_{\Sigma} \underbrace{\frac{\exp(ikl)}{l}}_{\tilde{E}(Q)} \frac{\exp(ikr)}{r} \left[\frac{\cos(n,r) - \cos(n,l)}{2}\right] d\sigma.$$

If S is very far from the aperture, incident light can be treated as plane wave perpendicularly incident on the aperture.

$$K(\theta) = \frac{1+\cos\theta}{2}.$$

① Initial approximation — normally the size of the aperture and the area range on the display screen that we care about are much much smaller than the distance between the aperture and the display screen.

(i) $\cos(n, r) = \cos\theta \approx 1 \Rightarrow K(\theta) = 1$

we do not consider the effect of θ.

(ii) r does not change much with the position of the arbitrary point Q on the aperture.

$r \approx z_1$, neglegible effect on amplitude $\frac{1}{r} \approx \frac{1}{z_1}$

but non-neglegible on phase $\exp(ikr) \not\approx \exp(ikz_1)$

$$\tilde{E}(P) = \frac{1}{i\lambda z} \iint_{\Sigma} \tilde{E}(Q) \exp(ikr)\, d\sigma.$$

$\tilde{E}(Q)$ Complex field amplitude of ~~point~~ Q on the aperture

$$\tilde{E}(Q) = \frac{A\exp(ikl)}{l}$$

② Fresnel Approximation — Fresnel diffraction; near-field region

$$r = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2} = z\left[1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2\right]^{1/2}$$

binomial series

$$r = z\left(1 + \frac{1}{2}\left[\frac{(x-\xi)^2 + (y-\eta)^2}{z^2}\right] - \frac{1}{8}\left[\frac{(x-\xi)^2 + (y-\eta)^2}{z_1^2}\right]^2 + \cdots\right]$$

$$z\left[\cdot \frac{1}{8}\left(\frac{(x-\xi)^2 + (y-\eta)^2}{z^2}\right)^2\right]_{max} \ll \frac{\lambda}{2}$$

$$z_1^3 \gg \frac{1}{4\lambda}\left[(x-x_1)^2 + (y-y_1)^2\right]_{max}$$

$$\Rightarrow r = z\left\{1 + \frac{1}{2}\left[\frac{(x-x_1)^2 + (y-y_1)^2}{z^2}\right]\right\} = z + \frac{x^2+y^2}{2z} - \frac{x\xi + y\eta}{z} + \frac{\xi^2 + \eta^2}{2z} \quad (*)$$

$$\tilde{E}(x,y) = \frac{\exp(ikz)}{i\lambda z} \iint_{\Sigma} \tilde{E}(\xi,\eta) \exp\left\{\frac{ik}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]\right\} d\xi\, d\eta.$$

outside Σ, $\tilde{E}(\xi,\eta) \approx 0$ $\Downarrow$

$$\tilde{E}(x,y) = \text{〃} \iint_{-\infty}^{\infty} \text{〃}$$

$$= \frac{\exp(ikz)}{i\lambda z} \tilde{E}(\xi,\eta) \otimes \exp\left[\frac{ik}{2z}(\xi^2 + \eta^2)\right].$$

③ Fraunhofer Approximation — Fraunhofer diffraction.

In $(*)$, when z is large enough so that

$$\frac{\xi^2 + \eta^2}{2z} \ll \frac{\lambda}{2} \Rightarrow z \gg \frac{(\xi^2+\eta^2)_{max}}{\lambda}$$

$$z \gg \frac{k(\xi^2+\eta^2)_{max}}{2}$$

"far-field" or Fraunhofer region.

Example: $\lambda = 500nm$ monochromatic light normally incident square aperture of $3cm \times 3cm$.

The approximate range for Fraunhofer diffraction

$$\frac{k \; (\xi^2+\eta^2)_{max}}{2z} << \pi$$

$$z >> \frac{(\xi^2+\eta^2)}{\lambda} = \frac{(\frac{3}{2})^2 \times 2 \times 10^{-4}}{500 \times 10^{-9}} = \underline{900m}.$$

II. Fraunhofer diffraction and Fourier transform

$$\hat{E}(x,y) = \underbrace{\frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{ik}{2z}(x^2+y^2)\right]}_{C} \iint_{\Sigma} \hat{E}(\xi,\eta)\exp\left[-\frac{ik}{z}(x\xi+y\eta)\right]d\xi d\eta.$$

$$\iint_{\Sigma} \Downarrow \iint_{-\infty}^{\infty}$$

$$\iint_{-\infty}^{\infty} \hat{E}(\xi,\eta)\exp[-i2\pi(u\xi+v\eta)]d\xi d\eta \Leftarrow \iint_{-\infty}^{\infty} \hat{E}(\xi,\eta)\exp\left[-i\frac{2\pi}{\lambda z}(x\xi+y\eta)\right]d\xi d\eta$$

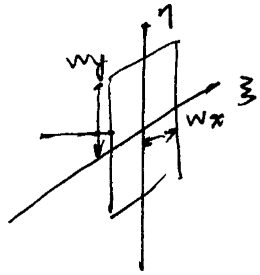
$$u = f_x = \frac{x}{\lambda z}, \quad v = f_y = \frac{y}{\lambda z}.$$

spatial frequency of $\hat{E}(\xi,\eta)$ on x and y.

$$\boxed{\begin{aligned} f(x,y) &= \iint F(u,v)\exp[i2\pi(ux+vy)]\,du\,dv. \\ F(u,v) &= \iint f(x,y)\exp[-i2\pi(ux+vy)]\,dx\,dy \end{aligned}} \leftarrow \text{Fourier Transform}$$

$$\hat{E}(x,y) = C \cdot \mathcal{F}[\hat{E}(\xi,\eta)]\Big|_{f_x=\frac{x}{\lambda z},\, f_y=\frac{y}{\lambda z}}.$$

Example in the Lecture Note.



$$t_A = \mathrm{rect}\left(\frac{\xi}{2W_x}\right)\mathrm{rect}\left(\frac{\eta}{2W_y}\right)$$

$\hat{E}(\xi,\eta) = t_A(\xi,\eta)$ with plane-wave illumination.

$$\hat{E}(x,y) = C \cdot \mathcal{F}[\hat{E}(\xi,\eta)]\Big|_{f_x,f_y}$$

$$= \frac{\exp ikz}{i\lambda z}\exp\left[\frac{ik}{2z}(x^2+y^2)\right] A\,\mathrm{sinc}\left(\frac{2W_x x}{\lambda z}\right)\mathrm{sinc}\left(\frac{2W_y y}{\lambda z}\right)$$

$$A = 4W_xW_y$$

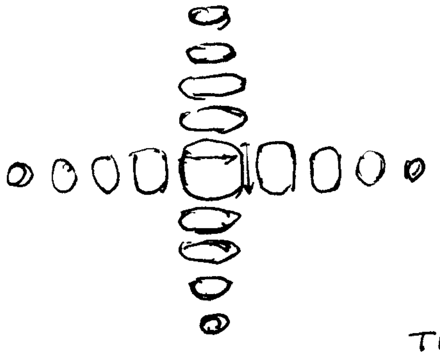
$$I = |\hat{E}(x,y)|^2 = \frac{A^2}{\lambda^2 z^2}\mathrm{sinc}^2\left(\frac{2W_x x}{\lambda z}\right)\mathrm{sinc}^2\left(\frac{2W_y y}{\lambda z}\right)$$

$x=0, y=0$ $\quad I_{max}$.

when $\frac{2W_x x}{\lambda z} = 1$, $\quad I=0 \quad x = \frac{\lambda z}{2W_x}$ or $y = \frac{\lambda z}{2W_y}$.

For $\mathrm{sinc}\,\alpha$, $\quad \alpha = 0$, I_{max}; $\quad \alpha = \pm n\pi$, $I = 0$

diffraction pattern of a ~~square~~ rectangular aperture.



$\Delta x_0 = \frac{\lambda z}{W_x}$ $\quad \Delta\theta_{x/2} = \frac{\lambda}{2W_x}$

$\Delta y_0 = \frac{\lambda z}{W_y}$ $\quad \Delta\theta_{y/2} = \frac{\lambda}{2W_y}$

$\Delta x \propto \frac{1}{W_x}$

The more confined the incident,
The more divergent the diffraction field.

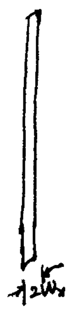
$\Delta x \propto \lambda$

large λ, the more pronounced the phenomenon of diffraction.

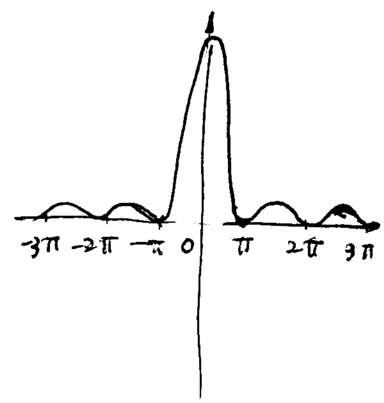
Example: a slit of width $2W_x$.

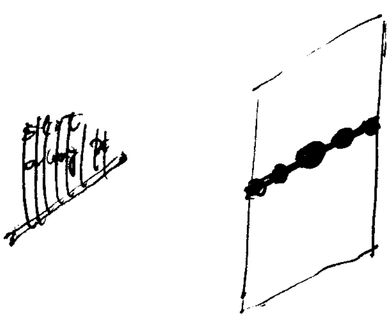
$2W_y >> W_x$.

one-dimensional.



$$I = I_0 \, \mathrm{sinc}^2\left(\frac{2W_x x}{\lambda z}\right)$$





Example

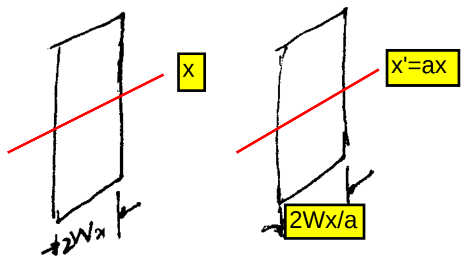
Fraunhofer diffraction of a circular aperture will be covered in the next lecture. (see lecture note)

Application of Fourier Transform properties on diffraction.

① $\mathcal{F}[f(ax)] = \frac{1}{|a|}\mathcal{F}\left(\frac{u}{a}\right)$

coordinates shrink to $\frac{1}{a}$ of the original;
spatial frequency increases to a of the original.

⇒ more confined the incident light;
more pronounced the diffraction.



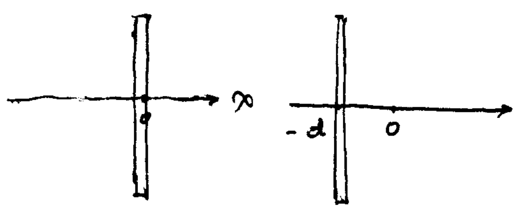
$$I = I_0 \,\mathrm{sinc}^2\left(\frac{2W_x x}{\lambda z}\right)$$

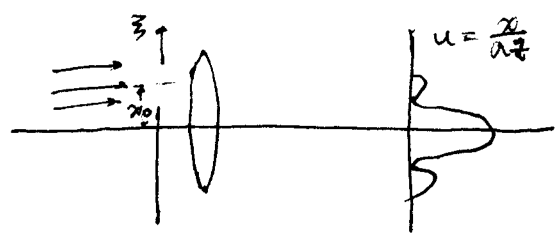
$$u = \frac{x}{\lambda z} \Rightarrow u' = \frac{x}{\lambda z a}$$

$$I' = \frac{I_0}{a^2}\,\mathrm{sinc}^2\left(\frac{2W_x x}{\lambda z a}\right)$$

② $\mathcal{F}[f(x-x_0)] = F(u)\exp[-i2\pi u x_0]$

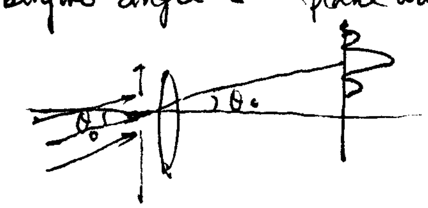
The lateral translation of the aperture does not affect the amplitude distribution of the diffracted field; only caused a linear shift in phase.





③ $\mathcal{F}[f(x)\exp[i2\pi u_0 x]] = \mathcal{F}(u-u_0)$

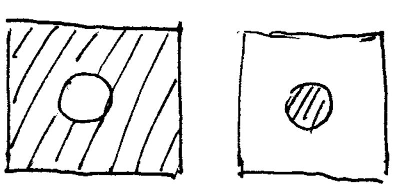
At Oblique angle ← plane wave illumination



shape of (pattern) diffraction pattern same;
shift in lateral position.

$$u_0 = \frac{x_0}{\lambda z} = \frac{\sin\theta_0}{\lambda}$$

④



complementary apertures.

$T_2(\xi,\eta) = 1 - t_1(\xi,\eta)$

$T_2(u,v) = \delta(u,v) - T_1(u,v)$ $\longrightarrow$ complex amplitude.

If we do not consider the central point,

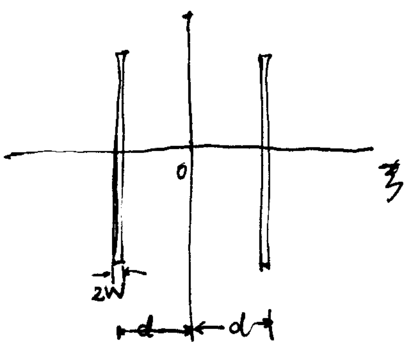
$$T_2(u,v) = -T_1(u,v)$$

$$\Rightarrow I_1(u,v) = I_2(u,v)$$

Except for the central point, the diffraction patterns for complementary apertures are the same.

⑤ Others: Linear superposition; $\mathcal{F}(f \otimes g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$.

Example: Method I. For a single slit at $\xi = 0$. $\hat{E}(\xi) = t_A(\xi) = \mathrm{rect}\left(\frac{\xi}{2W}\right)$



$$\hat{E}_0(x) = \frac{e^{jkz}}{j\lambda z} \exp j\frac{k}{z}x^2\, \mathcal{F}[\hat{E}(\xi)]\Big|_{f_x = \frac{x}{\lambda z}}$$

$$= \frac{e^{jkz}}{j\lambda z} \cdot e^{j\frac{k}{z}x^2} A\,\mathrm{sinc}\left(\frac{2Wx}{\lambda z}\right)$$

$$A = 2W$$

$$I_0(x) = \frac{4W^2}{\lambda^2 z^2}\,\mathrm{sinc}^2\left(\frac{2Wx}{\lambda z}\right)$$

For the single slit at $\xi = -d$,

~~$\hat{E}(\xi) = t$~~

applying rule ②. $\hat{E}'(x) = \hat{E}(x) \exp[-i2\pi u(-d)]$

$$= \hat{E}(x) \exp\left[i2\pi \frac{x}{\lambda z} d\right]$$

For the single slit at $\xi = d$,

$$\hat{E}''(x) = \hat{E}(x) \exp\left[-i2\pi \frac{x}{\lambda z} d\right]$$

$$\hat{E}_{tot}(x) = \hat{E}'(x) + \hat{E}''(x) = \hat{E}(x)\left(\exp\left[i2\pi\frac{x}{\lambda z}d\right] + \exp\left[-i2\pi\frac{x}{\lambda z}d\right]\right)$$

$$= \hat{E}(x) \cdot 2\cos\left(2\pi\frac{x}{\lambda z}d\right)$$

$$I_{tot}(x) = I_0(x) \cdot 4\cos^2\left(2\pi\frac{x}{\lambda z}d\right).$$

Method II, $\hat{E}(\xi) = t_A(\xi) = \mathrm{rect}\left(\frac{\xi}{2W}\right) \otimes [\delta(\xi+d) + \delta(\xi-d)]$

$$\hat{E}(x) = C\,\mathcal{F}(\hat{E}(\xi)) = C\,\mathcal{F}\left(\mathrm{rect}\left(\frac{\xi}{2W}\right)\right) \times \mathcal{F}[\delta(\xi-d) + \delta(\xi+d)]$$

$$= C \cdot A\,\mathrm{sin}\left(\frac{2Wx}{\lambda z}\right) \times [\exp(i2\pi u d) + \exp(i2\pi u(-d))]$$

$$= C \cdot A\,\mathrm{sin}\left(\frac{2Wx}{\lambda z}\right) \cdot 2\cos(2\pi u d)$$

$$\Rightarrow I_{tot}(x) = I_0(x) \cdot 4\cos^2\left(2\pi\frac{x}{\lambda z}d\right)$$