EE 119 Homework 3 Solution

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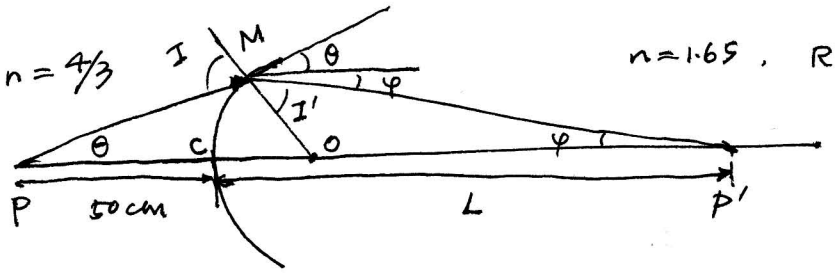
1. (a) $n = 4/3$ $n = 1.65$, $R = 10\,cm$



First we need to find the incident angle I. From the Law of sines, we know that

$$\frac{\overline{MO}}{\sin\theta} = \frac{\overline{PO}}{\sin(180^\circ - I)} \Rightarrow \frac{R}{\sin\theta} = \frac{50\,cm + R}{\sin(180^\circ - I)}$$

$$\Rightarrow \sin(180^\circ - I) = \frac{50+10}{10}\cdot\sin 5^\circ = 0.5229$$

$$\Rightarrow 180^\circ - I = 148.47^\circ \Rightarrow I = 31.5293^\circ$$

By Snell's Law $n_1 \sin I = n_2 \sin I'$

$$\Rightarrow I' \Rightarrow \sin^{-1}\left(\frac{\sin I}{\frac{n_2}{n_1}}\right) = 24.9972^\circ$$

Since $L = R + \overline{OP'}$, we need to solve for $\overline{OP'}$. Again by the law of sines

$$\frac{R}{\sin\varphi} = \frac{\overline{OP'}}{\sin I'}, \quad \varphi = I - I' - \theta = 1.5321^\circ$$

$$\Rightarrow \overline{OP'} = R\,\frac{\sin I'}{\sin\varphi} = 158.0484\,cm$$

$$\Rightarrow L = R + \overline{OP'} = \underline{168.0484\,cm}$$

(b) Similarly, for $\theta = 0.5^\circ$,

$$I = \sin^{-1}\left(\frac{R + 50\,cm}{R}\sin\theta\right) = 3.0013^\circ$$

$$I' = \sin^{-1}\left(\frac{n_1 \sin I}{n_2}\right) = 2.4249^\circ$$

$$\varphi = I - I' - \theta = 0.0764^\circ$$

$$\Rightarrow \overline{OP'} = R\,\frac{\sin I'}{\sin\varphi} = 317.4483\,cm$$

$$\Rightarrow L = R + \overline{OP'} = \underline{327.4483\,cm}$$

(c) Repeat (a) with the paraxial approximation.

We have

$$\frac{n_2}{L} - \frac{n_1}{-pc} = \frac{(n_2 - n_1)}{R}$$

$$\Rightarrow \quad L = \frac{n_2}{\frac{n_1}{-pc} + \frac{(n_2 - n_1)}{R}} = \frac{1.65}{\frac{4/3}{-50} + \frac{(1.65 - 4/3)}{10}} = \underline{330.00\,cm}$$

(d) Repeat (b) with the paraxial approximation. We get the same answer as that is in (c) $L = \underline{330.00\,cm}$.

(e) There's a big difference between the answers of (a) and (b), but no difference between that of (c) and (d).

(f) The answers in (c) and (d) are similar to (b) because $0.5°$ is a small angle and the paraxial approximation is pretty valid. However, even though $5°$ seems pretty small (so you might assume that it's good enough to use the paraxial approximation), the angle of incidence $I = 31.53°$ due to the curvature of the lens, which is not small at all!

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2. (a) We know that $M = d_2/d_1 = -1000$ (Please note that the image is invert from the logo)

As $-d_1 + d_2 \approx d_2 = 1000m$, $d_1 = -1m$.

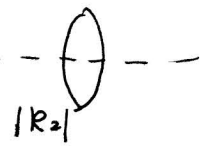
According to the Gaussian Lens Law,

$$\frac{1}{d_2} = \frac{1}{f} + \frac{1}{d_1}$$

$$\Rightarrow \quad f = \frac{1}{\frac{1}{d_2} - \frac{1}{d_1}} = \underline{0.999\,m}$$

Notice that since the clouds are very far away (close to the mathematical infinity) that the object needs to be placed very close to the focal length.

(b) For a double-convex lens, $R_1 > 0$, $R_2 < 0$



There are two possibilities, $|R_1| = 4|R_2|$ or $4|R_1| = |R_2|$

1° $|R_1| = 4|R_2|$ $\quad \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ (lensmaker's equation)

$$\frac{1}{f} = (1.5-1)\left(\frac{1}{R_1} - \frac{1}{-\frac{R_1}{4}}\right)$$

$$\Rightarrow \underline{R_1 = 2.4975\,m,}$$

$$\underline{R_2 = -0.6244\,m}$$

2° $4|R_1| = |R_2|$, $\quad \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$$= (n-1)\left(-\frac{1}{4R_1} + \frac{1}{R_1}\right)$$

$$\Rightarrow \underline{R_1 = 0.3746\,m}$$

$$\underline{R_2 = 1.4985\,m}$$

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3. (a) Going from spherical to cartesian coordinates gives $r = \sqrt{x^2+y^2+z^2}$. When $z \gg (x, y)$, this expression can be rewritten and Taylor expanded

$$r = z\sqrt{1 + \frac{x^2+y^2}{z^2}} \approx z\left(1 + \frac{x^2+y^2}{2z^2}\right)$$

We also need the approximation that $\frac{1}{1+x} \approx 1$ for $x \ll 1$. Substituting the approximations into the mathematical expression for a spherical wave,

$$E = \frac{E_0}{z\left(1+\frac{x^2+y^2}{2z}\right)} \cos\left(-kz\left(1+\frac{x^2+y^2}{2z^2}\right) + \phi\right) \approx \frac{E_0}{z}\cos\left(-kz - k\frac{x^2+y^2}{2z} + \phi\right) \quad \text{—(eqn. *)}$$

At constant z, the $-kz$ term can be incorporated into the constant phase ϕ, leaving us with

$$E = \frac{E_0}{z}\cos\left(-\frac{k(x^2+y^2)}{2z} + \phi\right).$$

(b) Assume that the lens is at $z=0$. A spherical wave emerging from an object point located at d_1 has the form

$$E = \frac{E_0}{|d_1|}\cos\left[-\frac{k(x^2+y^2)}{2d_1}\right]$$

The phase shift that this light acquires when it passes through a thin lens is $\Delta\phi = -\frac{k(x^2+y^2)}{2f}$

So the form of the resulting wave after passing through the lens will be

$$E = \frac{E_0}{d_1}\cos\left[-\frac{k(x^2+y^2)}{2d_1} - \frac{k(x^2+y^2)}{2f}\right]$$

$$= \frac{E_0}{d_1}\cos\left[-\frac{k(x^2+y^2)}{2}\left(\frac{1}{d_1}+\frac{1}{f}\right)\right]$$

$$= \frac{E_0}{d_1}\cos\left[-\frac{k(x^2+y^2)}{2d_2}\right]$$

Here we use the thin lens approximation, as $\Delta z \ll d_1$, the amplitude of the wave traversing the thin-lens is treated as constant $\left(\frac{E_0}{d_1}\right)$.

Therefore the emerging wave has the form of a spherical wave emanating from a point d_2.

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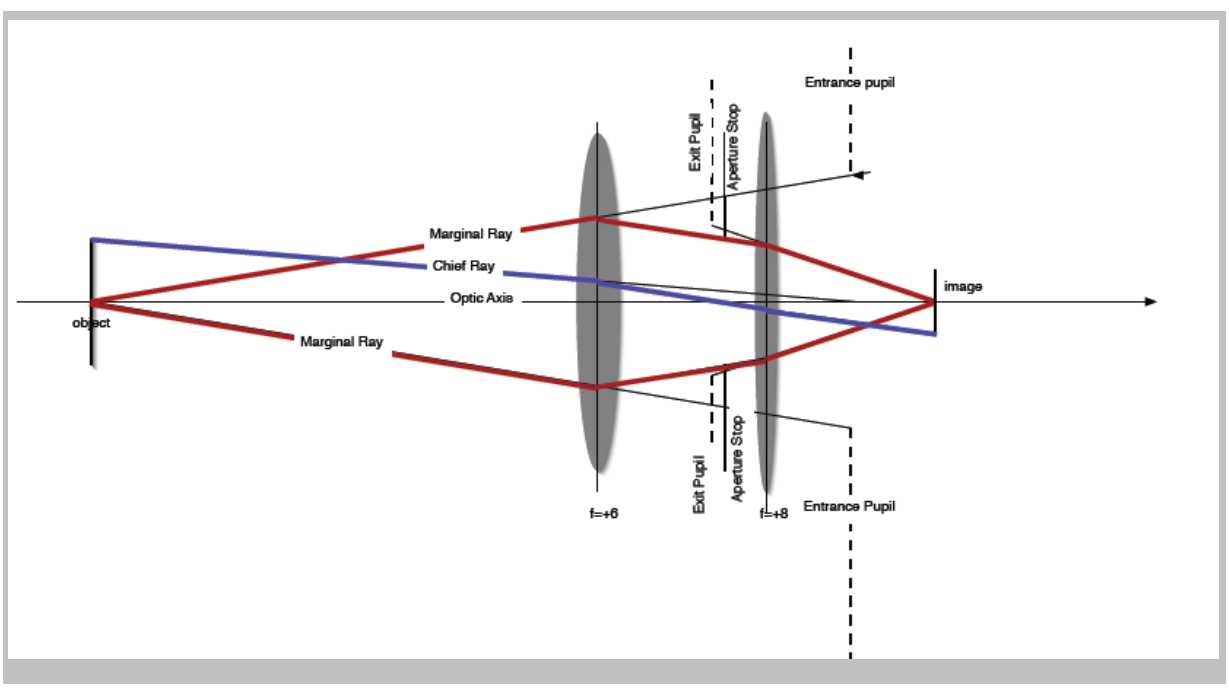
4. (a)



(b) The exit pupil is the image of the aperture stop through the lenses between the aperture stop and the image.

Here the aperture stop is located 1cm to the left of the lens, its image will be at

$$\frac{1}{d_2} = \frac{1}{f} + \frac{1}{d_1} = \frac{1}{8} + \frac{1}{-1} = -\frac{7}{8}$$

So the exit pupil is $\underline{\frac{8}{7}\text{ cm}}$ to the left of the final lens.

$m = \frac{d_2}{d_1} = \frac{8}{7}$, so the diameter of the exit pupil is $\frac{24}{7}$ $\underline{= 3.4286\text{ cm}}$.

The position and size of the entrance pupil can be found in the same way.

$$\frac{1}{d_2} = \frac{1}{f} + \frac{1}{d_1} = \frac{1}{6} + \frac{1}{-3} = -\frac{1}{6}$$

So the entrance pupil is $\underline{6\text{ cm}}$ to the right of the first lens (as now the right side is the object side), and its diameter will be

$$3 \times \frac{-6}{-3} = \underline{6\text{ cm}}$$

(c) The image formed by the first lens is located at

$$\frac{1}{d_2} = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$

$$\Rightarrow d_2 = 12\text{ cm}$$

So the first image is located 12 cm to the right of the first lens.

The magnification of the first lens is -1.

This image is the object for the second lens, so d_1 for the second lens is $12 - 4 = 8$cm. Then the final image is at

$$\frac{1}{d_2} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$\Rightarrow d_2 = 4\text{ cm}$$

So the final image is $\underline{4\text{ cm}}$ to the right of the final lens.

$m = \frac{4}{8} = 0.5$. The total magnification of the lens system is -0.5.

The image will be $3 \times 0.5 = \underline{1.5\text{ cm high}}$ and it's inverted.

(e) See diagram in (a).

5. (a) For convex-plano lens,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5-1)\left(\frac{1}{20} - \frac{1}{\infty}\right) \Rightarrow f = 40\,\text{cm}$$

for plano-convex lens,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5-1)\left(\frac{1}{\infty} - \frac{1}{-20}\right) \Rightarrow f = 40\,\text{cm}.$$

(b) For convex-plano lens,

$$\sin I_1 = \frac{h}{R_1} = \frac{10}{20} \Rightarrow I_1 = 30^\circ$$

Using Snell's Law,

$$1.5 \sin I_1' = \sin I_1 \Rightarrow I_1' = \sin^{-1}\left(\frac{\sin I_1}{1.5}\right) = 19.47^\circ$$

$$\Rightarrow I_2 = I_1 - I_1' = 10.53^\circ$$

Using Snell's Law again

$$1.5 \sin I_2 = \sin I_2' \Rightarrow I_2' = 15.91^\circ$$

$$L' = \frac{h}{\tan I_2'} = 35.09\,\text{cm}$$

$$\Delta = f - L' = \underline{4.91\,\text{cm}}$$

For plano-convex lens, $I_1 = \sin^{-1}\left(\frac{h}{R_1}\right) = 30^\circ$

Using Snell's Law, $n \sin I_1 = \sin I_1' \Rightarrow I_1' = \sin^{-1}(n \sin 30^\circ)$
$= 48.56^\circ$

$$\Rightarrow I_2 = I_1' - I_1 = 18.56^\circ$$

$$L' = \frac{h}{\tan I_2} = 29.73\,\text{cm}$$

$$\Delta = f - L' = \underline{10.27\,\text{cm}}$$

(c) Convex-plano lens configuration is better since $\Delta = f - L'$ is much smaller, meaning less spherical aberration — the position where the ray intersects the optical axis is closer to the focal point.

To understand this conclusion, just recall the fact that the incident ray will undergo a minimum deviation when it makes, more or less, the same angle as does the emerging ray. (Homework #2 Prob 1)