Lecture 20

DIFFRACTION THEORY OF A LENS

We have previously seen that light passing through a lens experiences a phase delay given by:

$$\varphi(\xi, \eta) = \exp\left[-jk(n-1)\left(\frac{\xi^2+\eta^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right] \qquad \text{(neglecting the constant phase)}$$

The focal length, f is given by:

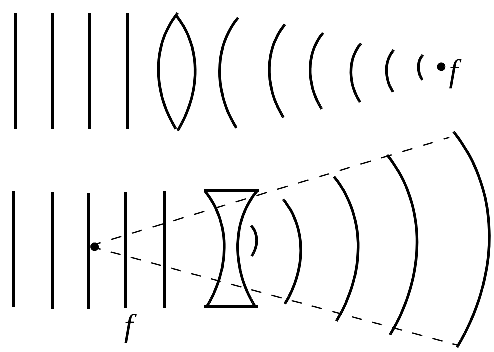
$$\boxed{\frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \qquad \text{The “lens makers formula”}$$

The transmission function is now:

$$\boxed{\varphi(\xi, \eta) = \exp\left[-j\frac{k}{2f}(\xi^2+\eta^2)\right]}$$

This is the paraxial approximation to the spherical phase

Note: the incident plane-wave is converted to a spherical wave converging to a point at f behind the lens (f positive) or diverging from the point at f in front of lens (f negative).



Diffraction from the lens pupil

Suppose the lens is illuminated by a plane wave, amplitude A. The lens “pupil function” is $P(\xi, \eta)$.

The full effect of the lens is $U_l'(\xi, \eta) = \varphi(\xi, \eta)P(\xi, \eta)$

$$U_l'(\xi, \eta) = P(\xi, \eta)\exp\left[-j\frac{k}{2f}(\xi^2+\eta^2)\right]$$

We now use the Fresnel formula to find the amplitude at the “back focal plane” $z = f$

$$U_f(x, y) = \frac{\exp\left[j\frac{k}{2f}(x^2+y^2)\right]}{j\lambda f} \times e^{jkf}\int\int_{-\infty}^{\infty} d\xi d\eta\, U_l'(\xi, \eta)\exp\left[j\frac{k}{2f}(\xi^2+\eta^2)\right]\exp\left[-j\frac{2\pi}{\lambda f}(\xi x+\eta y)\right]$$

The phase terms that are quadratic in $\xi^2 + \eta^2$ cancel each other.

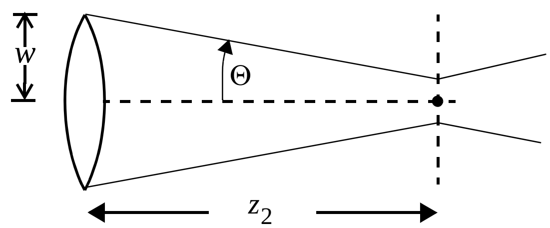
$$U_f(x,y) = \frac{\exp\left[j\frac{k}{2f}(x^2+y^2)\right]}{j\lambda f} e^{jkf} \int\int_{-\infty}^{\infty} d\xi d\eta P(\xi,\eta) \exp\left[-j\frac{2\pi}{\lambda f}(\xi x + \eta y)\right] \tag{A}$$

This is precisely the Fraunhofer diffraction pattern of P ! Note that a large z criterion *does not* apply here.

The focal plane amplitude distribution is a Fourier transform of the lens pupil function $P(\xi,\eta)$, multiplied by a quadratic phase term. However, the intensity distribution is exactly

$$I_f(x,y) = \frac{A^2}{\lambda^2 f^2} |F[P(\xi,\eta)]|^2 \qquad f_x = \frac{x}{\lambda f} \qquad f_y = \frac{y}{\lambda f}$$

Example: a circular lens, with radius w



$$P = \text{circ}\left(\frac{q}{w}\right) \qquad (q^2 = \xi^2 + \eta^2)$$

$$\text{let} \quad h(r) = F[P(\lambda z_2 q)] = F\left[\text{circ}\left(\frac{\lambda z_2 q}{w}\right)\right] \qquad (r^2 = x^2 + y^2)$$

$$= \frac{A}{\lambda z_2}\left[2\frac{J_1(2\pi wr/\lambda z_2)}{2\pi wr/\lambda z_2}\right]$$

$$|h(r)|^2 = \frac{A^2}{\lambda^2 z_2^2}\left[2\frac{J_1(2\pi wr/\lambda z_2)}{2\pi wr/\lambda z_2}\right]^2$$

The spot diameter is $\boxed{d = 1.22\frac{\lambda f}{w}}$

$= 1.22\frac{\lambda}{\Theta}$ paraxial approximation

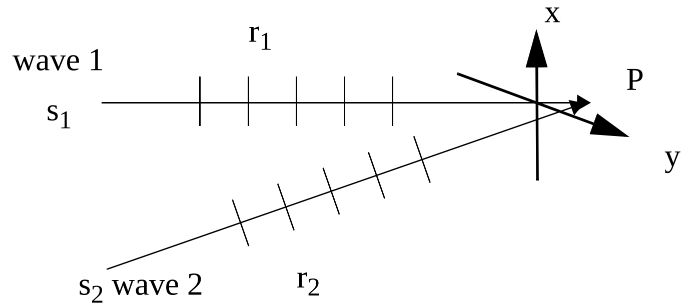
The “Rayleigh” resolution of the lens is $d/2 = 0.66\lambda/\Theta$.

For a large Θ, $\boxed{d/2 = 0.61\frac{\lambda}{\sin\Theta} = 0.61\frac{\lambda}{NA}}$

INTERFERENCE

[Reading assignment: Hecht 9.1, 9.3 (to p. 396 only), 9.4, 9.7.2, 9.8.2, 9.8.3]

Interference occurs when light from different sources or different paths are superimposed. As an electromagnetic wave, when two waves superimpose, it is the electric field amplitudes that add.



Let the two sources radiate plane waves so that

$$E_1 = A_1 \cos\left(\omega_1 t - \frac{2\pi r_1}{\lambda_1} + \phi_1\right) = A_1 \cos(\omega_1 t + \alpha_1)$$

$$E_2 = A_2 \cos\left(\omega_2 t - \frac{2\pi r_2}{\lambda_2} + \phi_2\right) = A_2 \cos(\omega_2 t + \alpha_2)$$

At P, we add the fields

$$\boxed{E(P) = E_1 + E_2}$$

The intensity is generally what is detected

$$\boxed{I(P) = \langle |E_1 + E_2|^2 \rangle}$$

where the average is a time average over detector response time,

$$= \langle A_1^2 \cos^2(\omega_1 t + \alpha_1) + A_2^2 \cos^2(\omega_2 t + \alpha_2)$$
$$+ A_1 A_2 \cos[(\omega_1 + \omega_2)t + (\alpha_1 + \alpha_2)] + A_1 A_2 \cos[(\omega_1 - \omega_2)t + (\alpha_1 - \alpha_2)] \rangle$$

When $\omega_1 \neq \omega_2$, the $\cos(\omega_1 \pm \omega_2)t$ terms average out

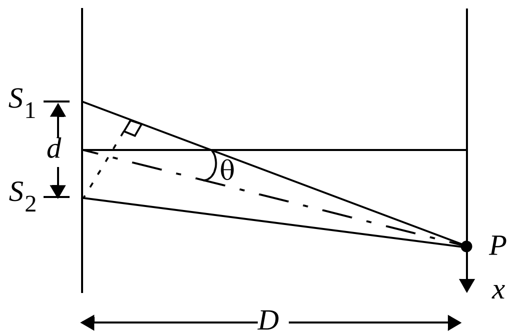
$$\boxed{I(P) = I_1 + I_2}$$

If $\boxed{\omega_1 = \omega_2 = \omega}$, then we get an interference:

$$\begin{aligned} I(P) &= I_1 + I_2 + \langle A_1 A_2 \cos[2\omega t + (\alpha_1 + \alpha_2)] + A_1 A_2 \cos(\alpha_1 - \alpha_2)\rangle \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\alpha_1 - \alpha_2) \\ &= I_1 + I_2 + \underbrace{2\sqrt{I_1 I_2}\cos\left[\left(\frac{2\pi}{\lambda}\right)((r_1 - r_2) + (\phi_1 - \phi_2))\right]}_{\text{interference term}} \end{aligned}$$

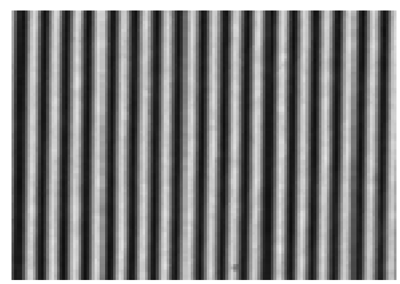
The intensity observed shows maxima + minima as the *path length difference* $(r_1 - r_2)$ varies [assumes phase factors do not vary (ϕ_1, ϕ_2)] This gives rise to constructive and destructive interference. The phase difference is $\boxed{\delta = \frac{2\pi}{\lambda}(r_1 - r_2)}$.

Young's two-slit interference experiment



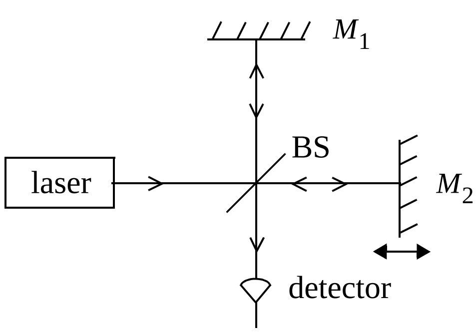
Phase difference

$$\boxed{\begin{aligned} \delta &= \frac{2\pi}{\lambda}(S_1 P - S_2 P) \\ &\cong \frac{2\pi}{\lambda} d \sin\theta = \frac{2\pi}{\lambda} d \frac{x}{D} \end{aligned}}$$



$$I = 2I_1 + 2I_1 \cos\frac{2\pi dx}{\lambda D}$$ "fringe pattern"

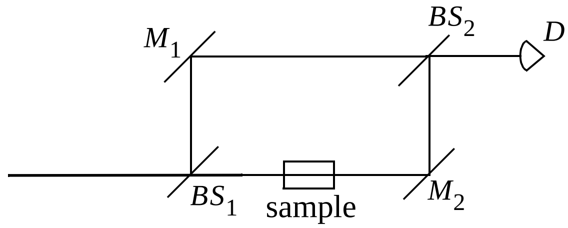
Michelson interferometer



As M_2 moves, the detected intensity changes. The light double passes the M_2 arm, so when M_2 moves by $\lambda/2$, the detected intensity changes sinusoidally from I_{max} to I_{min} to I_{max} again (1 full cycle).

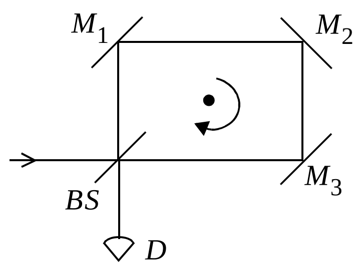
Used for distance measuring. As M_2 moves, we count cycles using the detector. With good S/N ratio and a stable laser, movement as small as $\lambda/1000$ (~ 5A°!) can be measured. Large movement can be measured with this accuracy. Such interferometers are very useful for very precise servo control of positioning systems.

Mach-Zender



This interferometer can be used for measuring material properties. If the index of refraction of the sample varies, then the phase difference varies and the intensity at D varies. As an example, one can determine the temperature dependence of the index of refraction n for air or other gases.

Sagnac interferometer (modified Mach-Zender)



If the interferometer is rotating clockwise, the clockwise light has a longer time-of-flight than the opposite direction.

# of fringes shift $N = \dfrac{4A\Omega}{c\lambda}$ A: area
Ω: rot vel

By using a spool of fiber instead of discrete mirrors, a very stable arrangement can be made and sensitivity is increased by n, the number of turns of fiber on the spool. This is called the “fiber-ring gyro,” very popular in inertial navigation.