10.

1. The signal $x(t)$ is first transmitted through the channel. Assuming that $x(t)$ has a FT $X(j\omega)$; then the output of the channel $y(t)$ has the FT:

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= X(j\omega)e^{j\phi(\omega)} \end{aligned}$$

2. $y(t)$ is recorded and flown back to the site of the original transmission. $y(t)$ is then played backwards though the channel. That means that $y(-t)$ is transmitted through the channel. Since $y(t)$ has FT $X(j\omega)e^{j\phi(\omega)}$, $y(-t)$ has FT $Y(-j\omega)$ by time reversal. Transmission through the channel gives output $z(t)$ with FT:

$$\begin{aligned} Z(j\omega) &= Y(-j\omega)H(j\omega) \\ &= X(-j\omega)e^{j\phi(-\omega)}e^{j\phi(\omega)} \\ &= X(-j\omega)e^{-j\phi(\omega)}e^{j\phi(\omega)} \\ &= X(-j\omega) \end{aligned}$$

since $\phi(-\omega) = -\phi(\omega)$.

3. $z(t)$ is then played backwards. $z(-t)$ has FT $Z(-j\omega) = X(j\omega)$, which is the FT of the original transmitted signal $x(t)$. So $x(t)$ is recovered.

9.

(a)

$$\begin{aligned} H_x(\omega) &= \int_{-\infty}^{\infty} x(t)(\cos\omega t + \sin\omega t)dt \\ &= \int_{-\infty}^{\infty} x(t)[\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} + \frac{1}{2j}e^{j\omega t} - \frac{1}{2j}e^{-j\omega t}]dt \\ &= \frac{1}{2}[\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt]^* + \frac{1}{2}[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt] \\ &\quad + \frac{1}{2j}[\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt]^* - \frac{1}{2j}[\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt] \\ &= \mathcal{R}\text{e}[X(j\omega)] - \mathcal{I}\text{m}[X(j\omega)] \end{aligned}$$

where $\mathcal{R}\text{e}[X(j\omega)] = \frac{1}{2}[X(j\omega) + X^*(\omega)]$ and $\mathcal{I}\text{m}[X(j\omega)] = \frac{1}{2j}[X(j\omega) - X^*(\omega)]$.

Now, we plug the above into the inverse Hartley transform, and perform some algebra:

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} H_x(\omega)(\cos\omega t + \sin\omega t)d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathcal{R}\text{e}[X(j\omega)] - \mathcal{I}\text{m}[X(j\omega)])(\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} + \frac{1}{2j}e^{j\omega t} - \frac{1}{2j}e^{-j\omega t})d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\mathcal{R}\text{e}[X(j\omega)]e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\mathcal{R}\text{e}[X(j\omega)]e^{-j\omega t}d\omega \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{j}{2}\mathcal{R}\text{e}[X(j\omega)]e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{j}{2}\mathcal{R}\text{e}[X(j\omega)]e^{-j\omega t}d\omega \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\mathcal{I}\text{m}[X(j\omega)]e^{j\omega t}d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\mathcal{I}\text{m}[X(j\omega)]e^{-j\omega t}d\omega \end{aligned}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}\mathcal{I}\mathrm{m}[X(j\omega)]e^{j\omega t}d\omega-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}\mathcal{I}\mathrm{m}[X(j\omega)]e^{-j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}(\mathcal{R}\mathrm{e}[X(j\omega)]+j\mathcal{I}\mathrm{m}[X(j\omega)])e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}(\mathcal{R}\mathrm{e}[X(j\omega)]-j\mathcal{I}\mathrm{m}[X(j\omega)])e^{-j\omega t}d\omega$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}(\mathcal{R}\mathrm{e}[X(j\omega)]-j\mathcal{I}\mathrm{m}[X(j\omega)])e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}(\mathcal{R}\mathrm{e}[X(j\omega)]+j\mathcal{I}\mathrm{m}[X(j\omega)])e^{-j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}X(j\omega)e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}X^*(\omega)e^{-j\omega t}d\omega$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}X^*(\omega)e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}X(j\omega)e^{-j\omega t}d\omega$$

We now are in a position to apply the formula for the inverse Fourier transform.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}X(j\omega)e^{j\omega t}d\omega &= \frac{1}{2}x(t)\\
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}X^*(\omega)e^{-j\omega t}d\omega &= \frac{1}{2}[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega]^*\\
&= \frac{1}{2}x^*(t)\\
-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}X^*(\omega)e^{j\omega t}d\omega &= -\frac{j}{2}[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega(-t)}d\omega]^*\\
&= -\frac{j}{2}x^*(-t)\\
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{j}{2}X(j\omega)e^{-j\omega t}d\omega &= \frac{j}{2}[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega(-t)}d\omega]\\
&= \frac{j}{2}x(-t)
\end{aligned}$$

Combining the above with the assumption that $x(t)$ is real, we have:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}H_x(\omega)(\cos\omega t+\sin\omega t)d\omega\\
&\quad=\frac{1}{2}x(t)+\frac{1}{2}x^*(t)-\frac{j}{2}x^*(-t)+\frac{j}{2}x(-t)\\
&\quad=\frac{1}{2}x(t)+\frac{1}{2}x(t)-\frac{j}{2}x(-t)+\frac{j}{2}x(-t)\\
&\quad=x(t)
\end{aligned}$$

So the inverse Hartley transform of the Hartley transform of $x(t)$ is just $x(t)$, as desired.

(b) With $H_x(\omega)$ and $H_y(\omega)$ the Hartley transforms for $x(t)$ and $y(t)$, and $z(t)=x(t)*y(t)$:

$$\begin{aligned}
Z(j\omega) &= X(j\omega)Y(j\omega)\\
H_z(\omega) &= \mathcal{R}\mathrm{e}[Z(j\omega)]-\mathcal{I}\mathrm{m}[Z(j\omega)]\\
&= \mathcal{R}\mathrm{e}[X(j\omega)Y(j\omega)]-\mathcal{I}\mathrm{m}[X(j\omega)Y(j\omega)]\\
&= \mathcal{R}\mathrm{e}[(\mathcal{R}\mathrm{e}[X(j\omega)]+j\mathcal{I}\mathrm{m}[X(j\omega)])(\mathcal{R}\mathrm{e}[Y(j\omega)]+j\mathcal{I}\mathrm{m}[Y(j\omega)])]\\
&\quad-\mathcal{I}\mathrm{m}[(\mathcal{R}\mathrm{e}[X(j\omega)]+j\mathcal{I}\mathrm{m}[X(j\omega)])(\mathcal{R}\mathrm{e}[Y(j\omega)]+j\mathcal{I}\mathrm{m}[Y(j\omega)])]\\
&= \mathcal{R}\mathrm{e}\{\mathcal{R}\mathrm{e}[X(j\omega)]\mathcal{R}\mathrm{e}[Y(j\omega)]+j\mathcal{R}\mathrm{e}[X(j\omega)]\mathcal{I}\mathrm{m}[Y(j\omega)]+j\mathcal{I}\mathrm{m}[X(j\omega)]\mathcal{R}\mathrm{e}[Y(j\omega)]-\mathcal{I}\mathrm{m}[X(j\omega)]\mathcal{I}\mathrm{m}[Y(j\omega)]\}\\
&\quad-\mathcal{I}\mathrm{m}\{\mathcal{R}\mathrm{e}[X(j\omega)]\mathcal{R}\mathrm{e}[Y(j\omega)]+j\mathcal{R}\mathrm{e}[X(j\omega)]\mathcal{I}\mathrm{m}[Y(j\omega)]+j\mathcal{I}\mathrm{m}[X(j\omega)]\mathcal{R}\mathrm{e}[Y(j\omega)]-\mathcal{I}\mathrm{m}[X(j\omega)]\mathcal{I}\mathrm{m}[Y(j\omega\\
&= \mathcal{R}\mathrm{e}[X(j\omega)]\mathcal{R}\mathrm{e}[Y(j\omega)]-\mathcal{I}\mathrm{m}[X(j\omega)]\mathcal{I}\mathrm{m}[Y(j\omega)]-\mathcal{R}\mathrm{e}[X(j\omega)]\mathcal{I}\mathrm{m}[Y(j\omega)]-\mathcal{I}\mathrm{m}[X(j\omega)]\mathcal{R}\mathrm{e}[Y(j\omega)]
\end{aligned}$$

For any real signal w,

$$\begin{aligned} H_w(\omega) &= \mathcal{R}\mathrm{e}[W(j\omega)] - \mathcal{I}\mathrm{m}[W(j\omega)] \\ H_w(-\omega) &= \mathcal{R}\mathrm{e}[W(-j\omega)] - \mathcal{I}\mathrm{m}[W(-j\omega)] \end{aligned}$$

Because w is real, $W(j\omega)$ exhibits conjugate symmetry: $W(j\omega) = W^*(-\omega)$. This implies that the real part of $W(j\omega)$ is even and the imaginary part of $W(j\omega)$ is odd:

$$\begin{aligned} \mathcal{R}\mathrm{e}[W(j\omega)] &= \mathcal{R}\mathrm{e}[W(-j\omega)] \\ \mathcal{I}\mathrm{m}[W(j\omega)] &= -\mathcal{I}\mathrm{m}[W(-j\omega)] \end{aligned}$$

We then have:

$$\begin{aligned} H_w(\omega) &= \mathcal{R}\mathrm{e}[W(j\omega)] - \mathcal{I}\mathrm{m}[W(j\omega)] \\ H_w(-\omega) &= \mathcal{R}\mathrm{e}[W(j\omega)] + \mathcal{I}\mathrm{m}[W(j\omega)] \end{aligned}$$

Adding and subtracting the above, we obtain:

$$\begin{aligned} \mathcal{R}\mathrm{e}[W(j\omega)] &= \frac{1}{2}[H_w(\omega) + H_w(-\omega)] \\ \mathcal{I}\mathrm{m}[W(j\omega)] &= -\frac{1}{2}[H_w(\omega) - H_w(-\omega)] \end{aligned}$$

Plugging this into our formula for $H_z(\omega)$ gives us:

$$\begin{aligned} H_z(\omega) &= \frac{1}{2}[H_x(\omega) + H_x(-\omega)]\frac{1}{2}[H_y(\omega) + H_y(-\omega)] - \frac{1}{2}[H_x(\omega) - H_x(-\omega)]\frac{1}{2}[H_y(\omega) - H_y(-\omega)] \\ &\quad + \frac{1}{2}[H_x(\omega) + H_x(-\omega)]\frac{1}{2}[H_y(\omega) - H_y(-\omega)] + \frac{1}{2}[H_x(\omega) - H_x(-\omega)]\frac{1}{2}[H_y(\omega) + H_y(-\omega)] \\ &= \frac{1}{2}H_x(\omega)H_y(\omega) + \frac{1}{2}H_x(\omega)H_y(-\omega) + \frac{1}{2}H_x(-\omega)H_y(\omega) - \frac{1}{2}H_x(-\omega)H_y(-\omega) \end{aligned}$$

(c) The advantage of the Hartley transform is that we don't have to deal with complex valued transforms. However, convolution does not turn into a simple product in the frequency domain, as it does with Fourier transforms. This is why we use Fourier transforms instead.