EECS120: Signals and Systems Final
Please put your name and student ID number on the top of every sheet.
Work individually. No collaboration, notes, or books allowed.
"Proofs by Pictures" are worth substantial partial credit if essentially correct, but give some equations alongside for full credit.
There are 250 points possible on this exam. 200 is considered a perfect score.

Trig	$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$	$\sin(\theta+\phi)=\cos\theta\sin\phi+\sin\theta\cos\phi$
CTFT	$X(\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)e^{+j\omega t}d\omega$
CTFS	$X_k=\frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi k}{T}t}dt$	$x(t)=\sum_{k=-\infty}^{+\infty}X_k e^{+j\frac{2\pi k}{T}t}$
DTFT	$X(\omega)=\sum_{t=-\infty}^{+\infty}x[t]e^{-j\omega t}$	$x[t]=\frac{1}{2\pi}\int_0^{2\pi}X(\omega)e^{+j\omega t}d\omega$
DFT	$X_k=\sum_{k=0}^{T-1}x[t]e^{-j\frac{2\pi k}{T}t}$	$x[t]=\frac{1}{T}\sum_{k=0}^{T-1}X_k e^{+j\frac{2\pi k}{T}t}$

$$\text{Z-Transform } X(z)=\sum_{t=-\infty}^{+\infty}x[t]z^{-t}$$

$$\text{Laplace-Transform } X(s)=\int_{-\infty}^{+\infty}x(t)e^{-st}dt$$

$$\text{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$$

If $X(\omega)=1$ for $\omega\in[-\frac{\pi}{T},+\frac{\pi}{T}]$ and 0 otherwise, then $x(t)=\frac{1}{T}\text{sinc}(\frac{t}{T})$

If $x(t)=1$ for $t\in[-\frac{T}{2},+\frac{T}{2}]$ and 0 otherwise, then $X(\omega)=T\text{sinc}(\omega\frac{T}{2\pi})$.

If $y(t)=e^{j\omega_0 t}x(t)$, then $Y(\omega)=X(\omega-\omega_0)$.

If $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$, then $X(\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{T})$.

If $y(t)=e^{-at}x(t)$, then $Y(s)=X(s+a)$ with the RoC of Y given by translating the RoC of X by a in the complex plane.

If $X(s)=\frac{1}{s}$, then either $x(t)=u(t)$ with RoC $\{s|\text{Re}(s)>0\}$ or $x(t)=-u(-t)$ with RoC $\{s|\text{Re}(s)<0\}$.

For $n>1$, if $X(s)=\frac{1}{s^n}$, then either $x(t)=\frac{t^{n-1}}{(n-1)!}u(t)$ with RoC $\{s|\text{Re}(s)>0\}$ or $x(t)=\frac{-t^{n-1}}{(n-1)!}u(-t)$ with RoC $\{s|\text{Re}(s)<0\}$.

If $y[t]=a^{-t}x[t]$, then $Y(z)=X(az)$ with the RoC of Y given by scaling the RoC of X by a in the complex plane.

If $X(z)=\frac{1}{z-a}$, then either $x[t]=a^{t-1}u[t-1]$ with RoC $\{z||z|>|a|\}$ or $x[t]=-a^{t-1}u[-t]$ with RoC $\{z||z|<|a|\}$.

For $n>1$, if $X(z)=\frac{1}{(z-a)^n}$, then either $x[t]=\frac{a^{t-n}u[t-n]}{(n-1)!}\prod_{k=1}^{n-1}(t-k)$ with RoC $\{z||z|>|a|\}$ or $x[t]=\frac{(-1)^n a^{t-n}u[-t]}{(n-1)!}\prod_{k=1}^{n-1}(k-t)$ with RoC $\{z||z|<|a|\}$.

Problem 4.1 *True/False.*

If the bold statement is true, give a proof for it. If the statement is false, show a counterexample or proof that it is false.

a. *20 pts* *Let* L *be a system that acts on continuous time signals as follows:* $[Lx](t) = x(t)\cos(t)$. **Then,** L **is L.T.I.**

b. 20pts **All continuous-time LTI systems having a real-valued impulse response $h(t)$ also have a well defined CTFT $H(\omega)$.**

c. *20pts* *Let* $x(t)$ *be a periodic continuous time signal that has period* $1ms$ *and is also bandlimited so that its CTFT* $X(\omega) = 0$ *for* $|\omega| \geq 4000\pi$. **It is possible to reconstruct all of $x(t)$ from a *finite* number of samples.**

Problem 4.2 *Echoes and removing them*

This problem is about what happens when a wireless signal is received along with an echo of it. For convenience, we will use microseconds as our units of time to avoid carrying 10^6 *around everywhere.*

The transmitted signal is generated from a real discrete time signal $d[n]$ *(with 1 microsecond between values) as follows:*

1. *The discrete time signal* $d[k]$ *is run through a zero-order hold DAC to give rise to continuous time* $c(t) = d[k]$ *for* $t \in [k - \frac{1}{2}, k + \frac{1}{2})$.

2. *The continuous time signal* $c(t)$ *is modulated by a carrier at 900.125 MHz to give us* $x(t) = c(t)\cos(2\pi 900.125 t)$.

The $x(t)$ *is the signal that is actually transmitted.*

The signal received by the receiver is $y(t)$ *and is processed as follows:*

1. *The received signal is multiplied by cosine and sine carriers to give us the in-phase and quadrature signals.*

$$\begin{aligned} y_i(t) &= y(t)\cos(2\pi 900.125 t) \\ y_q(t) &= y(t)\sin(2\pi 900.125 t) \end{aligned}$$

2. *These signals are perfectly low-pass filtered to eliminate any images that might have appeared around 1800.25 MHz. For our purposes, you can consider the signal* $c(t)$ *to be essentially bandlimited to be less than 100MHz wide. Ignore any aliasing distortion.*

$$\begin{aligned} i(t) &= LPF(y_i)(t) \\ q(t) &= LPF(y_q)(t) \end{aligned}$$

3. *The signals are sampled every microsecond and interpreted as the real and imaginary parts of a single signal.*

$$r[k] = i(k) + jq(k)$$

The complex signal $r[k]$ *is then to be processed further to recover the original* $d[k]$.

a. *10pts* *Suppose that in addition to the original transmitted signal, the receiver picked up a single echo that arrived with half the amplitude exactly 1 microsecond later.*

Show that the system that takes x to y is LTI and find its impulse response.

b. *10pts* **Give the Laplace transform and RoC for the system in part (a).**

*(*HINT: It is not a rational function of s. Think about what the RoC means.*)*

c. *20pts* *Assume the model from part (a).* **Give a difference equation relating the complex discrete-time signal $r[k]$ at the receiver to the original signal $d[k]$ at the transmitter.**

*(*HINT: You do not need part (b) to do this. Work in time-domain where it makes sense and remember that you can treat $c(t)$ as though it was band-limited and can pass through the low-pass-filter without any distortion. But do be careful with the delayed cosine.*)*

d. 20pts **Give the Z-transform for the discrete-time system determined in part (c), the RoC that makes physical sense, and draw a plot showing the locations of any poles and zeros.**

*(*Escape Route: If at this point, you do not have confidence in your answer from part (c), you can hedge your bets for $\frac{2}{3}$ credit by proceeding as though the answer in (c) was $r[k] = \frac{15j}{2}d[k] + (5+3j)d[k-1] + d[k-2]$. This is not the right answer.*)*

e. *20pts* *You decide to post-process (equalization) the $r[k]$ from part (d) using a discrete-time LTI filter H to recover $d[k]$ from it.* **Please give the Z-transform representation for H, the RoC that makes physical sense, and draw a plot showing the locations of any poles and zeros.**

*(*Escape Route: You can continue along the escape route from part d.*)*

f. 20pts **Give the discrete-time impulse response for H in part (e) as well as a compact difference equation to realize the filter for processing purposes.**

*(*Escape Route: You can continue along the escape route from part e.*)*

Problem 4.3 *Sampling and modulation*

Justify your answers for full credit: use any combination of time-domain and transform-domain reasoning to show that your answer is correct.

Depending on the part, you are allowed to use some of the following ingredients in making your system. If you are allowed to use it, you can use as many of that item as you would like, including none at all. If it is parametric, you can set the parameters for it as you like and those parameters can vary among instantiations used. Unless specified otherwise, the continuous or discrete-time versions of the ingredients can be used as appropriate.

Master Ingredients List:

1. *Adder: Takes as many inputs as you like and returns the sum of them on a time-by-time basis. e.g.* $x(t) = a(t) + b(t)$

2. *Scalar Multiply: Is parametrized by the (possibly complex) scalar gain* α*. e.g.* $y(t) = \alpha c(t)$

3. *Function Generator: Is parameterized by the (possibly complex) function of time* $f(t)$ *and just returns the signal* $f(t)$*. e.g. Use* $f(t) = \cos(t)$ *to get a signal* $z(t) = \cos(t)$*. NOTE: You are allowed to use Dirac delta functions within* $f(t)$ *if you would like.*

4. *Memoryless Function: Is parameterized by the (possibly complex) function* $g(w)$ *of a single complex variable* w *and returns the input signal having been run through* $g(w)$ *on a time-by-time basis. e.g. Use* $g(w) = |w|$ *and then get* $y(t) = g(d(t))$*.*

5. *Signal Multiplier: Returns the product of its input signals on a time-by-time basis. e.g.* $y(t) = a(t)b(t)$

6. *LTI filter: Parametrized by either the impulse response or the frequency response. Runs the input signal through an LTI system and outputs the result. e.g. Use* $h(t) = sinc(t)$ *and get* $y(t) = x(t) * h(t)$ *where* $*$ *denotes convolution.*

7. *Sampler: Parametrized by the inter-sample time* T_s*. Given input signal* $i(t)$*, it has as output the discrete-time signal* $i_d[k] = i(kT_s)$*.*

8. *Weighted Summer: Parametrized by a (possibly complex) pulse-shape* $p(t)$ *and an inter-pulse time* T_p*. Given a discrete-time signal* $i_d[k]$*, it has as output the continuous-time signal* $l(t) = \sum_{k=-\infty}^{+\infty} i_d[k]p(t - kT_p)$

In this problem, you will build a transmitter that takes a discrete time signal and generates a continuous time signal $x(t)$ at the appropriate higher frequency. You will also build a receiver that recovers the original discrete-time signals from the transmissions.

a. *30pts* *Given a real-valued discrete-time signal $d[n]$ which generates a new value every millisecond ($T_s = \frac{1}{1000}$),* **please construct a continuous-time real-valued signal $x(t)$ for which $X(\omega) = 0$ unless $|\omega| \in [200000\pi, 201000\pi]$.** *You can use any of the ingredients listed.*

Show how to recover $d[n]$ from $x(t)$.

b. 30pts **Repeat part (a), but now (8) Weighted Summer is broken. You are free to choose T_p, but can only use a first-order-hold**

$$p(t) = \begin{cases} 0 & \textbf{if } t < -T_p \\ \frac{t}{T_p} + 1 & \textbf{if } -T_p \le t \le 0 \\ 1 - \frac{t}{T_p} & \textbf{if } 0 < t < T_p \\ 0 & \textbf{if } t \ge T_p \end{cases}$$

In addition, your function generators can only generate pure sines and cosines.

For this problem, it suffices to just show what you would change from your answer in part a. You do not need to repeat the details that are identical.

c. 30pts **Repeat Part (a), except this time you are not allowed to use any LTI filters at the transmitter, and only ideal lowpass filters at the receiver.** *You do have access to a pair of new parts: a splitter and a combiner that help convert a single signal into a pair of signals at half the rate and vice-versa.*

- *Splitter: takes a discrete time signal $x[k]$ with inter-sample times given by T_s and returns two discrete time signals $x_e[k] = x[2k]$ and $x_o[k] = x[2k+1]$ with inter-sample times given by $2T_s$ each.*
- *Combiner: takes two discrete time signals $x_e[k]$ and $x_o[k]$ with inter-sample times given by $2T_s$ and returns a single discrete time signal*

$$x[k] = \begin{cases} x_e[\frac{k}{2}] & \text{if } k \text{ even} \\ x_o[\frac{k-1}{2}] & \text{if } k \text{ odd} \end{cases}$$

  with inter-sample time given by T_s.

(HINT: Use the splitter at the transmitter side and the combiner at the receiver side. Think about using carriers at 20500π)

Extra workspace.