EECS120 - Fall 2003
Homework No. 3 Solutions
Questions can be asked in ucb.class.ee120 or in office hours

Problem 3.1 *Book Problems from Lee and Varaiya, chapter 10.*
Problems: 2,3,5
See attached images

Problem 3.2 *Linear Algebra Review*

a. *What is the determinant of aI where I is the $n \times n$ identity matrix? What is the trace?*

The determinant is a^n and the trace is na.

b. *What are the eigenvalues and eigenvectors of the following matrices:*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & 8 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

For the first matrix, we have:

$$\lambda_0 = 0, \vec{v}_0 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The zero eigenvalue and eigenvector should have been able to be found by inspection. The other two involve a little more work.

$$\lambda_1 = \frac{15}{2} + \frac{3\sqrt{33}}{2}, \vec{v}_1 = \begin{bmatrix} \frac{3\sqrt{33}}{22} - \frac{1}{2} \\ \frac{3\sqrt{33}}{44} + \frac{1}{4} \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{15}{2} - \frac{3\sqrt{33}}{2}, \vec{v}_2 = \begin{bmatrix} -\frac{3\sqrt{33}}{22} - \frac{1}{2} \\ -\frac{3\sqrt{33}}{44} + \frac{1}{4} \\ 1 \end{bmatrix}$$

For the second matrix, we immediately see that we have this pair:

$$\lambda_0 = 1, \vec{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The others must be from the two dimensional subspace orthogonal to this one and we get:

$$\lambda_1 = 7 + 2\sqrt{13}, \vec{v}_1 = \begin{bmatrix} 0 \\ \frac{\sqrt{13}-1}{4} \\ 1 \end{bmatrix}$$

$$\lambda_2 = 7 - 2\sqrt{13}, \vec{v}_1 = \begin{bmatrix} 0 \\ \frac{-\sqrt{13}-1}{4} \\ 1 \end{bmatrix}$$

In both of the above two matrices, we avoided having to deal with any third degree equations by noticing one eigenvalue/eigenvector pair immediately.

The third matrix is circulant and so the eigenvectors are the complex exponentials and we get:

$$\lambda_0 = 6, \vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = -\frac{3}{2} + \frac{\sqrt{3}}{2}j, \vec{v}_1 = \begin{bmatrix} 1 \\ e^{j\frac{2\pi}{3}} \\ e^{-j\frac{2\pi}{3}} \end{bmatrix}$$

$$\lambda_2 = -\frac{3}{2} - \frac{\sqrt{3}}{2}j, \vec{v}_2 = \begin{bmatrix} 1 \\ e^{-j\frac{2\pi}{3}} \\ e^{j\frac{2\pi}{3}} \end{bmatrix}$$

c. *Consider vectors in three-dimensional space. Let H be a system that takes an incoming vector, projects it into the plane that is perpendicular to the direction $[111]^T$, and proceeds to rotate the resulting vector 90 degrees around the Z axis. Is H linear? Invertible? Representable as a matrix? (If so, what is the matrix representation?)*

Projections and rotations are both linear and hence their composition is linear as well. This operation can not be invertible since the projection loses information along the direction $[1,1,1]^T$. Finally, all linear operations on finite dimensional vector spaces are representable by matrices and so is this one. To find the matrix, we just take the three standard basis vectors manually through the transformation to get the columns of the matrix. Alternatively, we can write matrices for the operations of projection and rotation and just multiply them together.

A projection involves subtracting out any component that is orthogonal to the subspace being projected to. As such, we need to project the vector onto the unit vector in the direction $[1,1,1]^T$ which means acting on it be the row vector $\vec{p}^T = \frac{1}{\sqrt{3}}[1,1,1]$ and then multiplying the result by the unit vector $\vec{p} = \frac{1}{\sqrt{3}}[1,1,1]^T$. Multiplying things together and writing the entire operation as a matrix gives us:

$$P = I - \vec{p}\vec{p}^T = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Notice that rotation 90 degrees around the Z axis just replaces Y coordinate with the X one, while replacing the X coordinate with the negative of the Y one, and leaves the Z coordinate unchanged. So we can immediately write the rotation matrix as:

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the total transformation:

$$H = RP = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Since P is singular, we can see that H is as well.

d. *Let the $n \times n$ real matrix M be such that it has n distinct eigenvalues λ_i. Show that there exists a coordinate system in which the operation of M can be represented by a diagonal matrix.*

Just use the coordinate system that has the eigenvectors of M as its basis. Since there are n distinct eigenvalues λ_i, there must also exist n distinct unit eigenvectors $\vec{v}_i$ such that $M\vec{v}_i = \lambda_i \vec{v}_i$. Furthermore, all of these eigenvectors are linearly independent.

We can show linear independence by using a proof by contradiction. Suppose that they were linearly dependent, then we would have a k for which $\vec{v}_k = \sum_{i \neq k} \alpha_i \vec{v}_i$ where all the α_i that are not zero correspond to a linearly independent set of vectors. If $\lambda_k = 0$, then we can solve the equation above to get another vector on the left hand side that corresponds to a nonzero λ_k. Assume now that we have done so and $\lambda_k \neq 0$. Then, $\lambda_k \vec{v}_k = M\vec{v}_k = \sum_{i \neq k} \alpha_i M \vec{v}_i = \sum_{i \neq k} \alpha_i \lambda_i \vec{v}_i$ implies $\vec{v}_k = \sum_{i \neq k} \alpha_i \frac{\lambda_i}{\lambda_k} \vec{v}_i$. Thus $\sum_{i \neq k} \alpha_i \frac{\lambda_i}{\lambda_k} \vec{v}_i = \sum_{i \neq k} \alpha_i \vec{v}_i$. This implies that $0 = \sum_{i \neq k} \alpha_i (1 - \frac{\lambda_i}{\lambda_k}) \vec{v}_i$ and if we focus on the non-zero α_i, then we have shown that the corresponding $\vec{v}_i$ are linearly dependent which contradicts the initial assumption that the nonzero α_i match up to a linearly independent set of vectors.

Since the eigenvectors are linearly independent and there are n of them, they form a basis and hence a coordinate system for the entire n-dimensional vector space. In this coordinate system, the operation of M leaves the coordinate vector $\vec{v}_i$ unchanged except for a scaling by λ_i. Thus it is represented by the diagonal matrix $\mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$.

Problem 3.3 *Finite domains without wrap-around*

Consider signals on the set $\{0, 1, 2, \ldots, n-1\}$. Suppose that we interpret delay/shift to mean that if a signal is shifted by $+i$, then the first i values will be set to zero, while the first $n-i$ values will become the last $n-i$ values. Similarly, if a signal is shifted by $-i$, then the last i values will be set to zero while the last $n-i$ values will become the first $n-i$ values.

a. *Prove that if L is an LTI system, then L is just a scalar gain.*

We already know that as a linear system operating on a finite dimensional vector space, L is representable by a matrix (see the next problem for a full discussion of this) where the columns represent the responses of L to impulses at various times.

Let e_i represent the impulse signal with the impulse at time i and l_i represent the response of of L to that signal. Then $D_{n-1}e_0 = e_{n-1}$ implies by time invariance that $l_{n-1} = Le_{n-1} = LD_{n-1}e_0 = D_{n-1}Le_0 = D_{n-1}l_0$. But by the property of the shift without wrap around $[D_{n-1}l_0](t) = 0$ for every $t < n-1$. Thus, the final column in L must have zeros in all but the last position.

Now, notice that $l_i = D_{-(n-1-i)}l_{n-1}$ from time invariance and thus $l_i(t) = 0$ for $t < i$ since l_{n-1} has zeros in all but the last position. And also $l_i(t) = 0$ for $t > i$ since these are zero from the shift without wrap around. Finally $l_i(i) = l_{n-1}(n-1)$ and thus the matrix L is just $l_{n-1}(n-1)I$ and thus the system is just a scalar gain.

b. *(Bonus) Characterize the entire class of time-invariant systems.*

All time invariant systems must be memoryless — they are just the same function applied to the signal value at each time i.

Problem 3.4 *Finite domains with wrap-around*

Consider the domain Z_n (the integers mod $n > 0$). Here we interpret delay to mean that $[D_\tau x](t) = x(t - \tau \bmod n)$.

Consider $n = 3$.

a. *Show that the set of real-valued signals on this domain is representable by vectors in 3-dimensional space using the standard basis vectors.*

We know that $n = 3, \forall \tau \in Z, x(t - 3\tau \bmod n) = x(t)$, so $x(0)$, $x(1)$, and $x(2)$ completely describe our signal. We can represent this signal with the following vector:

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = x(0)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x(1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x(2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From this, we see that such a signals in this domain can be represented in 3-dimensional space by the standard basis vectors.

b. *Show that all linear systems that map real-valued signals on this domain to real-valued signals on this domain are representable by real 3×3 matrices.*

In part a, we established that we can represent the set of real-valued signals on this domain with the standard basis vectors. That is, all signals on this domain have the following form:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = x_0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where $x_0 = x(0)$, $x_1 = x(1)$, and $x_2 = x(2)$.

Now consider a linear system L that maps real-valued signals on this domain to real-valued signals on this domain. By properties of linearity, we see that:

$$L\left(\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}\right) = x_0 L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + x_1 L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + x_2 L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

Since L maps to real-valued signals on this domain, the output of L can also be represented as a column vector in 3-dimensional space. Using this fact, consider the following 3-by-3 matrix:

$$\begin{bmatrix} L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) & L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) & L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$$

Multiplying our signal by this matrix produces the same result as the linear system L. Thus, we can represent linear systems in the form of L with 3-by-3 matrices.

c. *What is the class of matrices that correspond to LTI systems?*

Time invariance implies that the response of the system to an impulse at time 1 is the same as the impulse response delayed by one unit. Similarly for an impulse at time 2. Thus the class of matrices corresponding to LTI systems are circulant matrices, which have the following form for real-valued signals on this domain:

$$\begin{bmatrix} l_0 & l_2 & l_1 \\ l_1 & l_0 & l_2 \\ l_2 & l_1 & l_0 \end{bmatrix}$$

Here you can see the cyclical shifts across the columns.

d. *Show that there exists a complex coordinate system in which every LTI system is representable by a diagonal matrix.*

This amounts to diagonalizing the matrix found in part d. We do this by finding eigenvectors of the matrix and using those as the basis vectors for the new coordinate system. The first eigenvector is relatively easy to find, which has 1 in each component:

$$\begin{bmatrix} l_0 & l_2 & l_1 \\ l_1 & l_0 & l_2 \\ l_2 & l_1 & l_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (l_0 + l_2 + l_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Recall that the eigenvectors in Z_4 are complex exponentials. So let's try the following vectors:

$$\begin{bmatrix} l_0 & l_2 & l_1 \\ l_1 & l_0 & l_2 \\ l_2 & l_1 & l_0 \end{bmatrix} \begin{bmatrix} 1 \\ e^{j2\pi/3} \\ e^{j4\pi/3} \end{bmatrix} = \begin{bmatrix} l_0 + l_2 e^{j2\pi/3} + l_1 e^{j4\pi/3} \\ l_0 e^{j2\pi/3} + l_2 e^{j4\pi/3} + l_1 \\ l_0 e^{j4\pi/3} + l_2 + l_1 e^{j2\pi/3} \end{bmatrix} = (l_0 + l_2 e^{j2\pi/3} + l_1 e^{j4\pi/3}) \begin{bmatrix} 1 \\ e^{j2\pi/3} \\ e^{j4\pi/3} \end{bmatrix},$$

$$\begin{bmatrix} l_0 & l_2 & l_1 \\ l_1 & l_0 & l_2 \\ l_2 & l_1 & l_0 \end{bmatrix} \begin{bmatrix} 1 \\ e^{j4\pi/3} \\ e^{j8\pi/3} \end{bmatrix} = \begin{bmatrix} l_0 + l_2 e^{j4\pi/3} + l_1 e^{j8\pi/3} \\ l_0 e^{j4\pi/3} + l_2 e^{j8\pi/3} + l_1 \\ l_0 e^{j8\pi/3} + l_2 + l_1 e^{j4\pi/3} \end{bmatrix} = (l_0 + l_2 e^{j4\pi/3} + l_1 e^{j8\pi/3}) \begin{bmatrix} 1 \\ e^{j4\pi/3} \\ e^{j8\pi/3} \end{bmatrix}$$

We now have three eigenvectors and their corresponding eigenvalues, so we can diagonalize the matrix.

$$\begin{bmatrix} l_0 & l_2 & l_1 \\ l_1 & l_0 & l_2 \\ l_2 & l_1 & l_0 \end{bmatrix} = VHV^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{j2\pi/3} & e^{j4\pi/3} \\ 1 & e^{j4\pi/3} & e^{j8\pi/3} \end{bmatrix} \begin{bmatrix} f_0 & 0 & 0 \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & e^{-j2\pi/3}/3 & e^{-j4\pi/3}/3 \\ 1/3 & e^{-j4\pi/3}/3 & e^{-j8\pi/3}/3 \end{bmatrix},$$

where $f_0 = (l_0 + l_2 + l_1)$, $f_1 = (l_0 + l_2 e^{j2\pi/3} + l_1 e^{j4\pi/3})$, and $f_2 = (l_0 + l_2 e^{j4\pi/3} + l_1 e^{j8\pi/3})$. Here, V^{-1} transforms vectors to the new coordinate system, H is the resulting diagonal matrix corresponding to the LTI system, and the columns of V are the basis vectors used to represent this system.

e. *Represent D_1 and D_2 in both the coordinate system of (c) and (d).*

D_1 and D_2 are both delays mod 3, so we can represent them as follows:

$$D_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

In the diagonal coordinate system, they are as follows:

$$H_{D_1} = V^{-1} D_1 V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-j2\pi/3} & 0 \\ 0 & 0 & e^{-j4\pi/3} \end{bmatrix},$$

$$H_{D_2} = V^{-1} D_2 V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j2\pi/3} & 0 \\ 0 & 0 & e^{j4\pi/3} \end{bmatrix}$$

f. *Show explicitly that the coordinate system of part (d) is orthogonal. (i.e. the basis vectors are all orthogonal to each other using the regular Euclidean inner product on complex spaces.)*

The regular Eucliden inner product on complex spaces of two vectors $\mathbf{u}$ and $\mathbf{v}$ is:

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u}^* \ \mathbf{v} \ ,$$

where * denotes the complex conjugate of the transpose of $\mathbf{u}$. The basis vectors of the new coordinate system amount to the columns of matrix V, defined in part d. They are:

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{u}_1 = \begin{bmatrix} 1 \\ e^{j2\pi/3} \\ e^{j4\pi/3} \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ e^{j4\pi/3} \\ e^{j8\pi/3} \end{bmatrix}$$

Checking the following inner products confirms all the basis vectors are orthogonal:

$$< \mathbf{u}_0, \mathbf{u}_1 >= (1 + e^{j2\pi/3} + e^{j4\pi/3}) = (1 + \frac{-1}{2} + j\frac{\sqrt{3}}{2} + \frac{-1}{2} - j\frac{\sqrt{3}}{2}) = 0$$

$$< \mathbf{u}_0, \mathbf{u}_2 >= (1 + e^{j4\pi/3} + e^{j8\pi/3}) = (1 + \frac{-1}{2} - j\frac{\sqrt{3}}{2} + \frac{-1}{2} + j\frac{\sqrt{3}}{2}) = 0$$

$$< \mathbf{u}_1, \mathbf{u}_2 >= (1 + e^{j2\pi/3} + e^{j4\pi/3}) = 0$$

g. *Do all complex diagonal matrices in the coordinate system of (d) correspond to real LTI systems? If not, which subset of the complex diagonal matrices correspond to real LTI systems?*

Not all complex diagonal matrices in the coordinate system of (d) correspond to real LTI systems. After all, even a complex circulant matrix would share the same eigenvectors and hence would give rise to a diagonal matrix in the transformed coordinates.

Recall the following definitions from part (d): $f_0 = (l_0 + l_2 + l_1)$, $f_1 = (l_0 + l_2 e^{j2\pi/3} + l_1 e^{j4\pi/3})$, and $f_2 = (l_0 + l_2 e^{j4\pi/3} + l_1 e^{j8\pi/3})$. Because l_0,l_2, and l_1 are real-valued in a real LTI system, the following must be real-valued:

$$\begin{gathered} f_0 + f_1 + f_2 \\ f_0 + e^{j2\pi/3} f_1 + e^{-j2\pi/3} f_2 \\ f_0 + e^{-j2\pi/3} f_1 + e^{j2\pi/3} f_2 \end{gathered}$$

For this to happen, f_0 must be real valued and f_1 and f_2 must be complex conjugates of each other. To see this, write $f_k = r_k + j i_k$ and notice that we have the following constraints:

$$\begin{aligned} i_0 + i_1 + i_2 &= 0 \\ i_0 + \sin(2\pi/3) r_1 + \cos(2\pi/3) i_1 - \sin(2\pi/3) r_2 + \cos(2\pi/3) i_2 &= 0 \\ i_0 - \sin(2\pi/3) r_1 + \cos(2\pi/3) i_1 + \sin(2\pi/3) r_2 + \cos(2\pi/3) i_2 &= 0 \end{aligned}$$

Adding the second and third equations together cancels the real parts and gives us:

$$i_0 + \cos(2\pi/3)(i_1 + i_2) = 0$$

combining with the first equation immediately gives us:

$$i_0 = 0, (i_1 + i_2) = 0$$

Substituting into the second equation immediately gives us $r_1 = r_2$ which proves that f_0 is real and $f_1 = f_2^*$.

h. *Give an LTI system that removes the DC offset of a signal, but otherwise leaves it unchanged. Express it both in terms of the impulse response and the "frequency response."*

Removing the DC offset in a signal eliminates its zero-frequency component. This amounts to having an eigenvalue of 0 for $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Since the remaining frequency components remain unchanged, they have eigenvalues of 1. Thus, the matrix in (d)'s coordinate system looks like this:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This means that an LTI system, expressed with our standard basis vectors, is as follows:

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

From these matrices, we can get our impulse response and frequency response:

$$h(n) = \begin{cases} 2/3 & n = 0 \\ -1/3 & n = 1, 2 \end{cases}$$

$$H(k) = \begin{cases} 0 & k = 0 \\ 1 & k = 1 \\ 1 & k = 2 \end{cases}$$

i. *Repeat parts d,e,f,g,h for $n = 5, 6$*

For $n = 5$, we can use an approach similar to the $n = 3$ case to get the following:

$$\begin{bmatrix} l_0 & l_4 & l_3 & l_2 & l_1 \\ l_1 & l_0 & l_4 & l_3 & l_2 \\ l_2 & l_1 & l_0 & l_4 & l_3 \\ l_3 & l_2 & l_1 & l_0 & l_4 \\ l_4 & l_3 & l_2 & l_1 & l_0 \end{bmatrix} = V_5 H V_5^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{j2\pi/5} & e^{j4\pi/5} & e^{j6\pi/5} & e^{j8\pi/5} \\ 1 & e^{j4\pi/5} & e^{j8\pi/5} & e^{j12\pi/5} & e^{j16\pi/5} \\ 1 & e^{j6\pi/5} & e^{j12\pi/5} & e^{j18\pi/5} & e^{j24\pi/5} \\ 1 & e^{j8\pi/5} & e^{j16\pi/5} & e^{j24\pi/5} & e^{j32\pi/5} \end{bmatrix}$$

$$
\begin{bmatrix} f_0 & 0 & 0 & 0 & 0 \\ 0 & f_1 & 0 & 0 & 0 \\ 0 & 0 & f_2 & 0 & 0 \\ 0 & 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & 0 & f_4 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j2\pi/5} & e^{-j4\pi/5} & e^{-j6\pi/5} & e^{-j8\pi/5} \\ 1 & e^{-j4\pi/5} & e^{-j8\pi/5} & e^{-j12\pi/5} & e^{-j16\pi/5} \\ 1 & e^{-j6\pi/5} & e^{-j12\pi/5} & e^{-j18\pi/5} & e^{-j24\pi/5} \\ 1 & e^{-j8\pi/5} & e^{-j16\pi/5} & e^{-j24\pi/5} & e^{-j32\pi/5} \end{bmatrix} (d),
$$

$$
D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}
$$

$$
H_{D_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{-j2\pi/5} & 0 & 0 & 0 \\ 0 & 0 & e^{-j4\pi/5} & 0 & 0 \\ 0 & 0 & 0 & e^{-j6\pi/5} & 0 \\ 0 & 0 & 0 & 0 & e^{-j8\pi/5} \end{bmatrix},
$$

$$
H_{D_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{-j4\pi/5} & 0 & 0 & 0 \\ 0 & 0 & e^{-j8\pi/5} & 0 & 0 \\ 0 & 0 & 0 & e^{-j12\pi/5} & 0 \\ 0 & 0 & 0 & 0 & e^{-j16\pi/5} \end{bmatrix} (e),
$$

$$
(f) :< \mathbf{u}_0, \mathbf{u}_1 >=< \mathbf{u}_0, \mathbf{u}_2 >=< \mathbf{u}_0, \mathbf{u}_3 >=< \mathbf{u}_0, \mathbf{u}_4 >= 0
$$

$$
< \mathbf{u}_1, \mathbf{u}_2 >=< \mathbf{u}_1, \mathbf{u}_3 >=< \mathbf{u}_1, \mathbf{u}_4 >= 0
$$

$$
< \mathbf{u}_2, \mathbf{u}_3 >=< \mathbf{u}_2, \mathbf{u}_4 >= 0
$$

$$
< \mathbf{u}_3, \mathbf{u}_4 >= 0
$$

$(g) : f_0+f_1+f_2+f_3+f_4, f_0+f_1e^{j2\pi/5}+f_2e^{j4\pi/5}+f_3e^{j6\pi/5}+f_4e^{j8\pi/5}, f_0+f_1e^{j4\pi/5}+f_2e^{j8\pi/5}+f_3e^{j12\pi/5}+f_4e^{j}$
$f_0+f_1e^{j6\pi/5}+f_2e^{j12\pi/5}+f_3e^{j18\pi/5}+f_4e^{j24\pi/5}, f_0+f_1e^{j8\pi/5}+f_2e^{j16\pi/5}+f_3e^{j24\pi/5}+f_4e^{j32\pi/5} \in \Re$,
which implies that f_0 is real and that the others are in complex conjugate pairs using the identical arguments we used earlier.

$$
(h) :
$$

$$
h(n) = \begin{cases} 4/5 & n = 0 \\ -1/5 & n = 1, 2, 3, 4 \end{cases}
$$

$$
H(k) = \begin{cases} 0 & k = 0 \\ 1 & k = 1 \\ 1 & k = 2 \\ 1 & k = 3 \\ 1 & k = 4 \end{cases}
$$

Likewise, for the $n = 6$ case:

$$\begin{bmatrix} l_0 & l_5 & l_4 & l_3 & l_2 & l_1 \\ l_1 & l_0 & l_5 & l_4 & l_3 & l_2 \\ l_2 & l_1 & l_0 & l_5 & l_4 & l_3 \\ l_3 & l_2 & l_1 & l_0 & l_5 & l_4 \\ l_4 & l_3 & l_2 & l_1 & l_0 & l_5 \\ l_5 & l_4 & l_3 & l_2 & l_1 & l_0 \end{bmatrix} = V_6 H V_6^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{j2\pi/6} & e^{j4\pi/6} & e^{j6\pi/6} & e^{j8\pi/6} & e^{j10\pi/6} \\ 1 & e^{j4\pi/6} & e^{j8\pi/6} & e^{j12\pi/6} & e^{j16\pi/6} & e^{j20\pi/6} \\ 1 & e^{j6\pi/6} & e^{j12\pi/6} & e^{j18\pi/6} & e^{j24\pi/6} & e^{j30\pi/6} \\ 1 & e^{j8\pi/6} & e^{j16\pi/6} & e^{j24\pi/6} & e^{j32\pi/6} & e^{j40\pi/6} \\ 1 & e^{j10\pi/6} & e^{j20\pi/6} & e^{j30\pi/6} & e^{j40\pi/6} & e^{j50\pi/6} \end{bmatrix}$$

$$\begin{bmatrix} f_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_5 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j2\pi/6} & e^{-j4\pi/6} & e^{-j6\pi/6} & e^{-j8\pi/6} & e^{-j10\pi/6} \\ 1 & e^{-j4\pi/6} & e^{-j8\pi/6} & e^{-j12\pi/6} & e^{-j16\pi/6} & e^{-j20\pi/6} \\ 1 & e^{-j6\pi/6} & e^{-j12\pi/6} & e^{-j18\pi/6} & e^{-j24\pi/6} & e^{-j30\pi/6} \\ 1 & e^{-j8\pi/6} & e^{-j16\pi/6} & e^{-j24\pi/6} & e^{-j32\pi/6} & e^{-j40\pi/6} \\ 1 & e^{-j10\pi/6} & e^{-j20\pi/6} & e^{-j30\pi/6} & e^{-j40\pi/6} & e^{-j50\pi/6} \end{bmatrix} (d),$$

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$H_{D_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-j2\pi/6} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-j4\pi/6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-j6\pi/6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-j8\pi/6} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-j10\pi/6} \end{bmatrix},$$

$$H_{D_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-j4\pi/6} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-j8\pi/6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-j12\pi/6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-j16\pi/6} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-j20\pi/6} \end{bmatrix} (e),$$

$$(f) :< \mathbf{u}_0, \mathbf{u}_1 >=< \mathbf{u}_0, \mathbf{u}_2 >=< \mathbf{u}_0, \mathbf{u}_3 >=< \mathbf{u}_0, \mathbf{u}_4 >=< \mathbf{u}_0, \mathbf{u}_5 >= 0$$

$$< \mathbf{u}_1, \mathbf{u}_2 >=< \mathbf{u}_1, \mathbf{u}_3 >=< \mathbf{u}_1, \mathbf{u}_4 >=< \mathbf{u}_1, \mathbf{u}_5 >= 0$$

$$< \mathbf{u}_2, \mathbf{u}_3 >=< \mathbf{u}_2, \mathbf{u}_4 >=< \mathbf{u}_2, \mathbf{u}_5 >= 0$$

$$< \mathbf{u}_3, \mathbf{u}_4 >=< \mathbf{u}_3, \mathbf{u}_5 >= 0$$

$$< \mathbf{u}_4, \mathbf{u}_5 >= 0$$

$$(g) : f_0 + f_1 + f_2 + f_3 + f_4 + f_5, f_0 + f_1 e^{j2\pi/6} + f_2 e^{j4\pi/6} + f_3 e^{j6\pi/6} + f_4 e^{j8\pi/6} + f_5 e^{j10\pi/6},$$

$$f_0 + f_1 e^{j4\pi/6} + f_2 e^{j8\pi/6} + f_3 e^{j12\pi/6} + f_4 e^{j16\pi/6} + f_5 e^{j20\pi/6},$$

$$f_0 + f_1 e^{j6\pi/6} + f_2 e^{j12\pi/6} + f_3 e^{j18\pi/6} + f_4 e^{j24\pi/6} + f_5 e^{j30\pi/6},$$
$$f_0 + f_1 e^{j8\pi/6} + f_2 e^{j16\pi/6} + f_3 e^{j24\pi/6} + f_4 e^{j32\pi/6} + f_5 e^{j40\pi/6},$$
$$f_0 + f_1 e^{j10\pi/6} + f_2 e^{j20\pi/6} + f_3 e^{j30\pi/6} + f_4 e^{j40\pi/6} + f_5 e^{j50\pi/6} \in \Re,$$

which again implies that f_0 and f_3 are real (since they always have real multipliers in the above equations) and the others are complex conjugate pairs.

$$(h):$$

$$h(n) = \begin{cases} 5/6 & n = 0 \\ -1/6 & n = 1, 2, 3, 4, 5 \end{cases}$$

$$H(k) = \begin{cases} 0 & k = 0 \\ 1 & k = 1 \\ 1 & k = 2 \\ 1 & k = 3 \\ 1 & k = 4 \\ 1 & k = 5 \end{cases}$$

So in general, we have seen how things work though the examples above. All the cases of n even are like Z_4 and Z_6 while the odd ones will behave like Z_3 and Z_5. For extra credit, I encourage you to write up the solution for the general case of real signals on Z_n and LTI systems on them. The best writeup will get posted to the web. Use LaTeX if you want it to look nice like this.

Problem 3.5 *Suppose that $x(t)$ is a discrete-time signal that is periodic with period T (positive integer). If $y = Lx$ where L is an LTI system, is y guaranteed to be a periodic signal? Why or why not?*

Yes, it is guaranteed to be periodic. This can be seen in one way since all periodic discrete time signals with period T can be represented by T-dimensional vectors and all LTI systems correspond to $T \times T$ matrices. As a result, $x(t)$ can be represented by $\vec{x}$ and the action of the linear system L can be seen as $\vec{y} = L\vec{x}$. Thus, the output $y(t)$ must be represented by $\vec{y}$ and is thus periodic with period T.

Understandably, the above explanation leaves something to be desired. Here is a way of showing it directly using only the properties of Time Invariant systems. Let D_{kT} be the system that delays by kT.

$$\begin{aligned} y(t + kT) &= [D_{kT} L x](t) \\ &= [L D_{kT} x](t) \\ &= [Lx](t) \\ &= y(t) \end{aligned}$$

and thus y is periodic. The key fact is just that x is periodic with period T and hence $D_{kT} x = x$.