EECS120 - Fall 2003
Homework No. 5 Solutions

Problem 5.1 *BIG BONUS:*

Go to www.scitoys.com or other similar sites to look up how to make simple crystal radios. Build one and report on how it works. Can you give an explanation in terms of the concepts taught in this class?

Can you simulate this radio in MATLAB using idealized circuit components and illustrate the operation?

Extended to the end of the term.

Problem 5.2 *Book Problems from Lee and Varaiya. Problem 10.16, 11.4, 11.6*

See attached images

Problem 5.3 *Book Problems from Oppenheim, Willsky, and Nawab Problem 8.35, 8.37, 8.41*

See attached images

Problem 5.4 *What is wrong with* $\sec(t)$

This problem is designed to explore the idea of trying to demodulate a signal by multiplying by $\sec(t)$. *This is meant to fill in and complement the discussion in class.*

Consider a bandlimited signal $x(t)$ *that has no power outside of* $\omega \in [-\frac{1}{4}, +\frac{1}{4}]$.

The modulated signal is given by:

$$y(t) = x(t)\cos(t)$$

We wish to demodulate this signal by multiplication by $\sec(t)$ *or some suitable approximation thereof.*

a. *First, we want to verify that without noise, this makes some sense. Suppose that* $x(t) = \cos(\frac{1}{4}t + \frac{\pi}{3})$. *Let* $z(t) = y(t)\sec(t)$ *for all* t *for which* $\cos(t) \neq 0$. *Show that the right and left limits of* $z(t)$ *at* $t = \frac{\pi}{2}$ *exist and that both equal* $x(\frac{\pi}{2})$.

$$\begin{aligned}
\lim_{t \to \frac{\pi}{2}^+} z(t) &= \lim_{t \to \frac{\pi}{2}^+} \cos(\frac{1}{4}t + \frac{\pi}{3})\cos(t)\sec(t) \\
&= \lim_{t \to \frac{\pi}{2}^+} \cos(\frac{1}{4}t + \frac{\pi}{3})\frac{\cos(t)}{\cos(t)} \\
&= \lim_{t \to \frac{\pi}{2}^+} \cos(\frac{1}{4}t + \frac{\pi}{3}) \\
&= \cos(\frac{\pi}{8} + \frac{\pi}{3}) = x(\frac{\pi}{2})
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to\frac{\pi}{2}^-} z(t) &= \lim_{t\to\frac{\pi}{2}^-} \cos(\frac{1}{4}t+\frac{\pi}{3})\cos(t)\sec(t) \\
&= \lim_{t\to\frac{\pi}{2}^-} \cos(\frac{1}{4}t+\frac{\pi}{3})\frac{\cos(t)}{\cos(t)} \\
&= \lim_{t\to\frac{\pi}{2}^-} \cos(\frac{1}{4}t+\frac{\pi}{3}) \\
&= \cos(\frac{\pi}{8}+\frac{\pi}{3}) = x(\frac{\pi}{2})
\end{aligned}$$

b. *Verify that the average power in the signal* $\sec(t)$ *is infinite.*

To find the average power, we have to integrate $\sec^2(t)$ over one period from 0 to 2π. Clearly since $\sec^2(t) \geq 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \sec^2(t)dt \geq \frac{1}{2\pi}\int_0^{\frac{\pi}{2}-\epsilon} \sec^2(t)dt = \frac{\tan(\frac{\pi}{2}-\epsilon)}{2\pi}$$

which goes to infinity as $\epsilon \to 0$.

c. *To see what might be happening in the frequency domain, we will attempt to calculate a formal Fourier Series for* $\sec(t)$. *Consider the period to be from* $[-\pi,\pi]$ *and do the integrals over the domain* $[-\pi,-\frac{\pi}{2}-\epsilon]\cup[-\frac{\pi}{2}+\epsilon,\frac{\pi}{2}-\epsilon]\cup[\frac{\pi}{2}+\epsilon,\pi]$ *and then let* $\epsilon \to 0$ *to get the integral we are interested in.*

Begin with the DC term. Show that it is zero.

$$\begin{aligned}
2\pi S_{0,\epsilon} &= \int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \sec(t)dt + \int_{-\frac{\pi}{2}-\epsilon}^{0} \sec(t)dt + \int_0^{\frac{\pi}{2}-\epsilon} \sec(t)dt + \int_{\frac{\pi}{2}-\epsilon}^{\pi} \sec(t)dt \\
&= \int_0^{\frac{\pi}{2}-\epsilon} \sec(t-\pi)dt + \int_{-\frac{\pi}{2}-\epsilon}^{0} \sec(t)dt + \int_0^{\frac{\pi}{2}-\epsilon} \sec(t)dt + \int_{-\frac{\pi}{2}-\epsilon}^{0} \sec(t-\pi)dt \\
&= \int_0^{\frac{\pi}{2}-\epsilon} \sec(t-\pi)+\sec(t)dt + \int_{-\frac{\pi}{2}-\epsilon}^{0} \sec(t)+\sec(t-\pi)dt \\
&= \int_0^{\frac{\pi}{2}-\epsilon} 0dt + \int_{-\frac{\pi}{2}-\epsilon}^{0} 0dt \\
&= 0
\end{aligned}$$

And so the limit as $\epsilon \to 0$ must also be zero.

d. *Continue part (c) and calculate the Fourier Series terms at* $\omega = \pm 1$.

We follow the hint and write $e^{jt} = \cos(t) + j\sin(t)$ and then do separate integrals for the real and imaginary parts.

$$\begin{aligned}
2\pi S_{\pm 1,\epsilon} &= \int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \sec(t)(\cos(t) \pm j\sin(t))dt + \int_{-\frac{\pi}{2}+\epsilon}^{0} \sec(t)(\cos(t) \pm j\sin(t))dt \\
&\quad + \int_{0}^{\frac{\pi}{2}-\epsilon} \sec(t)(\cos(t) \pm j\sin(t))dt + \int_{\frac{\pi}{2}+\epsilon}^{\pi} \sec(t)(\cos(t) \pm j\sin(t))dt \\
&= \int_{-\pi}^{-\frac{\pi}{2}-\epsilon} dt + \int_{-\frac{\pi}{2}+\epsilon}^{0} dt + \int_{0}^{\frac{\pi}{2}-\epsilon} dt + \int_{\frac{\pi}{2}+\epsilon}^{\pi} dt \\
&\quad \pm j\left(\int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \tan(t) + \tan(-t)dt + \int_{-\frac{\pi}{2}-\epsilon}^{0} \tan(t) + \tan(-t)dt\right) \\
&= 2\pi - 4\epsilon
\end{aligned}$$

Take the limit to see that $S_{\pm 1} = 1$.

e. *To continue part (c), write a recursion that relates the k-th Fourier Series coefficient to the $k-2$ one.*

We first prove a little lemma that $\cos(kt) = 2\cos(t)\cos((k-1)t) - \cos((k-2)t)$ since

$$\begin{aligned}
\cos(kt) &= \frac{1}{2}(e^{jkt} + e^{-jkt}) \\
&= \frac{1}{2}(e^{j(k-1)t}e^{jt} + e^{-j(k-1)t}e^{-jt} + e^{j(k-2)t} + e^{-j(k-2)t} - (e^{j(k-2)t} + e^{-j(k-2)t})) \\
&= \frac{1}{2}(e^{j(k-1)t}e^{jt} + e^{-j(k-1)t}e^{-jt} + e^{j(k-1)t}e^{-jt} + e^{-j(k-1)t}e^{jt} - (2\cos((k-2)t))) \\
&= \frac{1}{2}(e^{j(k-1)t} + e^{-j(k-1)t})\frac{2}{2}(e^{jt} + e^{-jt}) - \cos((k-2)t) \\
&= 2\cos(t)\cos((k-1)t) - \cos((k-2)t)
\end{aligned}$$

Next, we notice that $\sin(kt)$ is an odd function while $\sec(t)$ is an even function. As a result $\sec(t)\sin(kt)$ is also an odd function of t and so:

$$
\begin{aligned}
2\pi S_{k,\epsilon} &= \int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \sec(t)(\cos(kt)+j\sin(kt))dt + \int_{-\frac{\pi}{2}+\epsilon}^{0} \sec(t)(\cos(kt)+j\sin(kt))dt + \\
&\int_{0}^{\frac{\pi}{2}-\epsilon} \sec(t)(\cos(kt)+j\sin(kt))dt + \int_{\frac{\pi}{2}+\epsilon}^{\pi} \sec(t)(\cos(kt)+j\sin(kt))dt \\
&= \int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \sec(t)(2\cos(t)\cos((k-1)t)-\cos((k-2)t))dt \\
&+ \int_{-\frac{\pi}{2}+\epsilon}^{0} \sec(t)(2\cos(t)\cos((k-1)t)-\cos((k-2)t))dt \\
&+ \int_{0}^{\frac{\pi}{2}-\epsilon} \sec(t)(2\cos(t)\cos((k-1)t)-\cos((k-2)t))dt \\
&+ \int_{\frac{\pi}{2}+\epsilon}^{\pi} \sec(t)(2\cos(t)\cos((k-1)t)-\cos((k-2)t))dt \\
&+j\int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \sec(t)\sin(kt)-\sec(t)\sin(kt)dt + j\int_{-\frac{\pi}{2}+\epsilon}^{0} \sec(t)\sin(kt)-\sec(t)\sin(kt)dt) \\
&= -S_{(k-2),\epsilon} \\
&+2\int_{-\pi}^{-\frac{\pi}{2}-\epsilon} \cos((k-1)t)dt + 2\int_{-\frac{\pi}{2}+\epsilon}^{0} \cos((k-1)t)dt \\
&+2\int_{0}^{\frac{\pi}{2}-\epsilon} \cos((k-1)t)dt + 2\int_{\frac{\pi}{2}+\epsilon}^{\pi} \cos((k-1)t)) \\
&= -S_{(k-2),\epsilon} \\
&+\frac{2(\sin((k-1)(\frac{\pi}{2}-\epsilon))-\sin((k-1)(\frac{\pi}{2}+\epsilon)))}{k-1} \\
&+\frac{2(\sin((k-1)(-\frac{\pi}{2}-\epsilon))-\sin((k-1)(-\frac{\pi}{2}+\epsilon)))}{k-1}
\end{aligned}
$$

It is clear from the above that $S_k = \lim_{\epsilon\to 0} S_{k,\epsilon} = -S_{(k-2)}$ since the other terms will all cancel out.

f. *Apply (c,d,e) to write the formal Fourier Series for* $\sec(t)$. *Use the relation between Fourier Series and the CTFT to write out and plot the Fourier Transform of* $\sec(t)$ *using Dirac delta functions.*

First, we notice that $S_k = 0$ for even k since by subtracting or adding 2 we can always get to 0 and that by part (c) is 0. For odd k, we notice that we must alternate between ± 1 with every subsequent odd number starting at $+1$ for $k = \pm 1$. This gives us:

$$
\sec(t) = \sum_{k \text{ odd}} (-1)^{\frac{|k|-1}{2}} e^{jkt}
$$

and so the Fourier Transform $S(\omega)$ has a $2\pi\delta$ at each of the odd ks corresponding

$\omega = k$ giving us:

$$S(\omega) = \sum_{k \text{ odd}} 2\pi(-1)^{\frac{|k|-1}{2}}\delta(\omega - k)$$

This has a plot with a Dirac delta with magnitude $\pm 2\pi$ alternating at odd $\omega = k$ and zero everywhere else.

g. *Now, suppose we choose to approximate* $\sec(t)$ *by only the* $-2m+1 \leq k \leq 2m-1$ *terms of the Fourier Series when doing the multiplication for* $z(t)$*. Simplify the resulting expression for* $z(t)$ *and show what it looks like in the Frequency Domain. What is happening as we increase the order of approximation* m *?*

This has

$$\begin{aligned}
z(t) &= x(t)\cos(t) \sum_{\substack{-2m+1 \leq k \leq 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} e^{jkt} \\
&= x(t)\frac{1}{2}(e^{jt} + e^{-jt}) \sum_{\substack{-2m+1 \leq k \leq 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} e^{jkt} \\
&= \frac{x(t)}{2} \sum_{\substack{-2m+1 \leq k \leq 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} (e^{j(k+1)t} + e^{j(k-1)t}) \\
&= \frac{x(t)}{2}(\sum_{\substack{-2m+1 \leq k < 0 \\ k \text{ odd}}} (-1)^{\frac{-k-1}{2}} (e^{j(k+1)t} + e^{j(k-1)t}) \\
&\quad + \sum_{\substack{0 < k < 2m-1 \\ k \text{ odd}}} (-1)^{\frac{k-1}{2}} (e^{j(k+1)t} + e^{j(k-1)t})) \\
&= \frac{x(t)}{2}(\sum_{l=1}^{m}(-1)^{l-1}(e^{-j2lt} + e^{-j(2l-2)t}) + \sum_{l=1}^{m}(-1)^{l-1}(e^{j2lt} + e^{j(2l-2)t})) \\
&= \frac{x(t)}{2}(1 + (-1)^{m-1}e^{-2jmt} + 1 + (-1)^{m-1}e^{2jmt}) \\
&= x(t) + x(t)\cos(2mt)
\end{aligned}$$

and so $Z(\omega) = X(\omega) + \frac{1}{2}(X(\omega - 2m) + X(\omega + 2m))$. As we increase the order of approximation for the $\sec(t)$, the extra images in the frequency domain are getting pushed further and further out.

Unfortunately for us, this trick has very little practical use in most analog settings because it brings in noise from the bands around the harmonics of the carrier into our baseband signal. However, in the discrete-time domain, it can sometimes be of use when we are working with severely oversampled signals.

h. *BONUS: Use Matlab or other plotting software to graphically compare the time-domain realization of* $\sec(t)$ *with that of various orders of Fourier Series approximations to it. Does it look like it is converging in the normal sense?*

If you make a plot, you will see that the various approximations do not seem to be converging in the normal sense. Rather, they are just wiggling more and more around the $\sec(t)$ function. The average value is in the right place, but the magnitude of the differences remains high.

i. *What if the original signal was* $y(t) = x(t)\cos(t + \phi)$ *where* ϕ *is small. Repeat part (g) with this mismatch in the phase.*

If ϕ is small, then $\cos(t + \phi) = \cos(t)\cos(\phi) - \sin(t)\sin(\phi) \approx \cos(t) - \phi\sin(t)$. So, we just need to see what happens to the part that corresponds to $-x(t)\phi\sin(t)$ when we multiply it by an approximation to $\sec(t)$.

$$
\begin{aligned}
& \phi x(t)\sin(t) \sum_{\substack{-2m+1 \le k \le 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} e^{jkt} \\
= \; & \phi x(t)\frac{1}{2j}(e^{jt} - e^{-jt}) \sum_{\substack{-2m+1 \le k \le 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} e^{jkt} \\
= \; & \phi x(t)\frac{1}{2j} \sum_{\substack{-2m+1 \le k \le 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} (e^{j(k+1)t} - e^{j(k-1)t}) \\
= \; & \phi x(t)\frac{1}{2j}(\sum_{\substack{-2m+1 \le k \le 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} e^{j(k+1)t} \\
& - \sum_{\substack{-2m+1 \le k \le 2m-1 \\ k \text{ odd}}} (-1)^{\frac{|k|-1}{2}} e^{j(k-1)t})
\end{aligned}
$$

Now, instead of canceling, the terms in the two sums above are adding constructively to be ± 2 for most positions and ± 1 for the extreme ones. So, we get $\pm 2\phi x(t)\sin(kt)$ for even positions $|k| < 2m$ and $\pm\phi x(t)\sin(2mt)$ where the sign depends on whether k or $2m$ is a multiple of 4 or not. Thus, the corruption in the demodulated signal increases with an increased number of terms in the approximation for $\sec(t)$. And so, even a tiny ϕ can cause a big problem.