Notes 05 largely plagiarized by %khc

1 System Representations

As touched upon in notes04, there are four major ways of representing an LTI system: linear differential equation with constant coefficients (LDE), impulse response, frequency response, and magnitude/phase response. Each of these four system representations captures the behavior of a given system, albeit in a slightly different manner.

An LDE with constant coefficients is just what you've seen in math5?, a differential equation of the form:

$$\sum_{i_0}^{N} a_i \frac{d^i}{dt^i} y(t) = \sum_{j_0}^{N} b_j \frac{d^j}{dt^j} x(t)$$

where the a_i and b_j are constants. As we will find out later, because the scaling terms are constants and the differentials of $x(t)$ and $y(t)$ are only of unity power, any system described by an LDE of such a form is both linear and time invariant. If we had the LDE for a system and the input $x(t)$, the output $y(t)$ would be kind of painful to find though.

The impulse response is another way of respresenting a system. By saying that you have an impulse response, you are actually implying that:

1. that the input is $\delta(t)$ [if it weren't, then the term "impulse response" makes no sense].
2. that the system is linear [you can try to say that a nonlinear system has an "impulse response", but knowing that response tells you close to nothing useful about the system], following the development in ps2, problem 8a.
3. that the system is time invariant, especially if you write the impulse response as a function of a single variable, as in ps2, problem 8b.
4. that the initial conditions are zero at time t_0, because the system is implied to be linear [for $t_0 < t < 0$, the input is identically zero and the system is linear, so the output had better be identically zero over that same interval].
5. that $t_0 = -\infty$, because the initial conditions are zero and the system is linear [for $t < 0$, the input is identically zero and the system is linear, so the output had better be identically zero over that same interval; since the interval starts at $t = -\infty$, t_0 had better be $-\infty$.

If we have a step response, the input $x(t)$ is a unit step, but all the other points above still hold. To get from a step response to an impulse response, take the derivative of the step response. To see this, note that the system is LTI and write the output $y(t)$ in terms of the input $x(t)$ and the impulse response $h(t)$.

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
y_{step}(t) &= u(t) * h(t) \\
&= \int_{-\infty}^{t} h(\tau) d\tau \\
\frac{d}{dt} y_{step}(t) &= h(t)
\end{aligned}$$

An alternative proof of this is:

$$\begin{aligned}
y_{step}(t) &= u(t) * h(t) \\
\dot{\delta}(t) * y_{step}(t) &= \dot{\delta}(t) * u(t) * h(t) \\
\frac{d}{dt} y_{step}(t) &= \delta(t) * h(t) \\
&= h(t)
\end{aligned}$$

Either way though, if we knew what $x(t)$ and $h(t)$ were, instead of solving an LDE, we would have to do an equally character-building convolution to find the output $y(t)$.[1]

[1] Whoever dies with the most character wins.

At this time, it is difficult to see that having a frequency response $H(\omega)$ is useful; we only know how to determine the output $y(t)$ for inputs of the form $e^{j\omega t}$, $\cos \omega t$, or $\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$. If the input is of any of these forms and the frequency response is known, you can immediately determine the output without either having to solve an LDE or performing a convolution. However, within the next two weeks, we will see that the frequency response is extremely handy in determining the output for inputs of other forms.

Since the magnitude and phase responses are just the magnitude and phase of the frequency response, respectively, there really isn't anything too conceptually deep here.

2 Putting It All Together

There are a number of different ways to get between the four system representations mentioned in the previous section.

To get the impulse response from the LDE, we can

1. substitute $\delta(t)$ for the input and solve for the output $y(t)$
2. substitute $u(t)$ for the input, solve for the step response, and differentiate the step response to get the impulse response.

The first method was covered in lecture. For the second method, let's consider the system $\dot{y}(t) + \frac{y(t)}{RC} = \frac{x(t)}{RC}$. The step response is $y_{step}(t) = (1 - e^{-t/RC})u(t)$. When we differentiate the step response, we need to be a little careful.

$$\begin{aligned} y_{impulse}(t) &= \dot{y}_{step}(t) \\ &= \frac{1}{RC} e^{-t/RC} u(t) + (1 - e^{-t/RC})\delta(t) \\ &= \frac{1}{RC} e^{-t/RC} u(t) + (1 - e^{-0/RC})\delta(t) \\ &= \frac{1}{RC} e^{-t/RC} u(t) \end{aligned}$$

To get the frequency response from the LDE, let the input $x(t) = e^{j\omega t}$, our friendly neighborhood eigenfunction, and let the output be $y(t) = H(\omega)e^{j\omega t}$. We can then solve for $H(\omega)$. [In fact, this trick is quite similar to the one we will use to convert all LDEs that we see into algebraic equations.]

To get back and forth between the impulse response and the frequency response, we can use the Fourier transform. More on this in a week.

The magnitude and the phase response can be found from the frequency response as follows. If $H(\omega)$ is of the form:

$$H(\omega) = K \frac{\prod_{n=1}^{N}(1 + a_n j\omega)}{\prod_{m=1}^{M}(1 + b_m j\omega)}$$

we can rewrite both sides in polar form to get:

$$\begin{aligned} |H(\omega)|e^{j\angle H(\omega)} &= |K|e^{j\angle K} \frac{\prod_{n=1}^{N}|1 + a_n j\omega|e^{j\angle(1+a_n j\omega)}}{\prod_{m=1}^{M}|1 + b_m j\omega|e^{j\angle(1+b_m j\omega)}} \\ |H(\omega)| &= |K| \frac{\prod_{n=1}^{N}|1 + a_n j\omega|}{\prod_{m=1}^{M}|1 + b_m j\omega|} \\ &= |K| \frac{\prod_{n=1}^{N}\sqrt{1 + (a_n\omega)^2}}{\prod_{m=1}^{M}\sqrt{1 + (b_m\omega)^2}} \\ \angle H(\omega) &= \angle K + \sum_{n=1}^{N}\angle(1 + a_n j\omega) - \sum_{m=1}^{M}\angle(1 + b_m j\omega) \\ &= \angle K + \sum_{n=1}^{N}\arctan(a_n\omega) - \sum_{m=1}^{M}\arctan(b_m\omega) \end{aligned}$$

In English, the magnitude response is the product of the magnitudes in the numerator divided by the product of the magnitudes in the denominator; the phase response is the sum of the angles in the numerator less the sum of the angles in the denominator. But this should not be surprising, since you have already seen the phase response when you make Bode plots. If the magnitude response were expressed in dB, you would also have seen that in Bode plots too.

Anyway, you should be comfortable with jumping back and forth between all these system representations. The relationships are summarized below in Figure **??**. We will add additional relationships later.

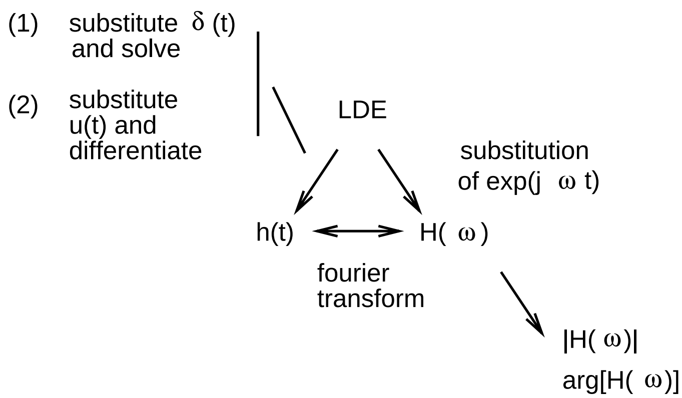

Figure 1: Relating the methods of representing a system.

3 Fourier Series

Fourier series depends on the fact that the family of complex exponentials $\{e^{jm\omega_0 t}\}_{m=-\infty}^{m=\infty}$, m an integer, forms an orthogonal set. As you may or may not have seen in either linear algebra or calculus (yes, ancient history to some), if two vectors are orthogonal, their inner product is zero. We can extend that concept to functions, noting that the inner product of two complex functions $x(\cdot)$ and $y(\cdot)$[2] is $\int x(t)y(t)^* dt$. So if $e^{jm\omega_0 t}$ and $e^{jn\omega_0 t}$ [m and n integer] are orthogonal, then over period $T \triangleq \frac{2\pi}{\omega_0}$, if $m \neq n$:

$$\begin{aligned}
\int_T x(t)y(t)^* dt &= \int_T e^{jm\omega_0 t} e^{-jn\omega_0 t} dt \\
&= \int_T e^{j(m-n)\omega_0 t} dt \\
&= \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t} |_{-T/2}^{T/2} \\
&= \frac{2}{(m-n)\omega_0} \sin(m-n)\omega_0 \frac{T}{2} \\
&= \frac{2}{(m-n)\omega_0} \sin(m-n)\pi \\
&= 0
\end{aligned}$$

where we have used

- $\sin x = \frac{1}{2j}[e^{jx} - e^{-jx}]$ on the fourth line.
- $\sin k\pi = 0$ for integer k on the fifth line.

If $m = n$:

$$\int_T x(t)y(t)^* dt = \int_T e^{jm\omega_0 t} e^{-jn\omega_0 t} dt$$

[2]The notation $\cdot$ indicates that there should be a variable in the parentheses, but we don't really care what that variable is.

$$
\begin{aligned}
&= \int_T dt \\
&= T
\end{aligned}
$$

So we have shown that the family of complex exponentials $\{e^{jm\omega_0 t}\}_{m=-\infty}^{m=\infty}$ is orthogonal. So what?

A periodic signal $x(t)$ with period $T \triangleq \frac{2\pi}{\omega_0}$ can be represented as the sum of scaled complex exponentials. In symbols, $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ But how do we determine those scaling factors a_k? We use the fact that we just proved above— that complex exponentials are orthogonal. If we multiply the above equation by $e^{-jm\omega_0 t}$ and integrate over T:

$$
\begin{aligned}
\int_T x(t)e^{jm\omega_0 t}dt &= \int_T [\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}] e^{-jm\omega_0 t}dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_T e^{jk\omega_0 t} e^{-jm\omega_0 t}dt \\
&= a_m \int_T e^{jm\omega_0 t} e^{-jm\omega_0 t}dt \\
&= T a_m
\end{aligned}
$$

Note that mathematical lossage from the second line to the third. Major term death from orthogonality! This tells us that our formula for the scaling factor a_k is:

$$
a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt
$$

Of course, we assumed that we could find Fourier series for any periodic signal. That's not necessarily true. It turns out that any function $x(t)$ which meets the Dirichlet conditions can be represented by a Fourier series:

- $x(t)$ is single-valued over the period T.
- $x(t)$ has a finite number of minima and maxima.
- $x(t)$ has a finite number of discontinuities.

Proof is beyond the scope of this course (thus preventing some premature pattern baldness).

Exercises Reread this section. Understand why having orthogonal components is useful. Find the Fourier series coefficients a_k for a square wave with period T, nonzero with period $2T_1$. Find the Fourier series coefficients for a triangle wave with period T. Find the Fourier series coefficients for the comb function $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

4 Interpretations

Any signal can be decomposed into a sum of appropriately scaled basis functions. Recall linear algebra if you ever took it. A basis is a linearly independent set of vectors that spans a vector space; the number of vectors in that linearly independent set is the dimension of the vector space. We can think of a vector space of functions and imagine a set of basis functions for that space. We could try $\{1, t, t^2, t^3, \ldots\}$— this is one basis that you have seen already (think Taylor series). But for the Fourier series, we use the complex exponentials, each an integer multiple of some fundamental frequency.

Of course, we could always use another set of basis functions, but complex exponentials will suffice for now.

One good thing about making the basis functions orthogonal is that the presence or absence of a given basis function does not affect the contribution of the other basis functions, so we can add or delete the contribution from a given basis function without changing the coefficients associated with the other basis functions. That means all we have to do is calculate a given a_k once and only once.

Another thing to keep in mind— Fourier series are valid only for periodic signals. The periodicity in time forces the FS coefficients to be discrete in frequency. Periodicity in one domain forces discreteness in the other domain. This fact will show up again later on down the road when we talk about the discrete-time Fourier transform (DTFT) and the discrete Fourier transform (DFT).

5 Random Facts, Parseval, and Symmetry

The DC level of periodic signal $x(t)$ is given by its a_0. This is easily seen by just checking out the formula for a_0:

$$a_0 = \int_T x(t)dt$$

If you gave me a Fourier series of a signal $x(t)$, what could i tell you about that signal? Well, each component represents some portion of the signal at the frequency $\omega = k\omega_0$. In fact, we have negative frequencies. Brainwarp!

We call the components at $\omega = \omega_0$ and $\omega = -\omega_0$ the fundamental. The components at $\omega = k\omega_0$ and $\omega = -k\omega_0$ comprise the kth harmonic.

In fact, if we consider the average power in the periodic signal $x(t)$:

$$\begin{aligned}
\frac{1}{T}\int_T x^2(t)dt &= \frac{1}{T}\int_T x(t) \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\,dt \\
&= \sum_{k=-\infty}^{\infty} a_k \frac{1}{T}\int_T x(t)e^{jk\omega_0 t}dt \\
&= \sum_{k=-\infty}^{\infty} a_k a_k^* \\
&= \sum_{k=-\infty}^{\infty} |a_k|^2
\end{aligned}$$

where we have made liberal use of the formula for a_k. This tells us that the power in the signal is just the sum of the magnitudes squared of the components. This is also known as Parseval's theorem for periodic signals.

What if $x(t)$ is periodic and real (as in real life$^{\text{TM}}$)? What happens to the FS coefficients?

$$\begin{aligned}
a_k^* &= \frac{1}{T}\int_T x^*(t)e^{+jk\omega_0 t}dt \\
&= \frac{1}{T}\int_T x(t)e^{-j(-k)\omega_0 t}dt \\
&= a_{-k}
\end{aligned}$$

This says that the negative frequency coefficients are the complex conjugates of the positive frequency coefficients. So their magnitudes are equivalent, but their phases are the negative of each other.

Exercises Prove that if $x(t)$ is even, a_k is purely real. Prove that if $x(t)$ is odd, a_k is purely imaginary.

6 So What?

This section is reproduced from notes04. It is included to remind you of why we bother to study Fourier series.

If you gave me a system with impulse response $h(t)$ and input $x(t)$ and told me to find $y(t)$, i could always convolve and give you an answer. But since $e^{j\omega t}$ is an eigenfunction for the convolution operator, if i can

1. represent the input $x(t)$ as the sum of complex exponentials
2. determine $H(\omega)$ for the impulse response $h(t)$

then i can give you the output $y(t)$ as the sum of complex exponentials, scaled by H at the appropriate values of ω. That is, if

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

then

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} a_k H(k\omega_0) e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} a_k |H(k\omega_0)| e^{j(k\omega_0 t + \angle H(k\omega_0))} \end{aligned}$$

Notice that i didn't have to do any convolution. What a feature.

7 A Look Ahead

But where are we going to use Fourier series? Some examples in the next set of notes.

Then we can extend the Fourier series to the much-promised Fourier transform.