Notes 12 largely plagiarized by %khc

1 Fourier Series Meets Fourier Transform

Let $x_p(t)$ be a periodic signal, and $x_{ap}(t)$ be equal to $x_p(t)$ over one period of $x_p(t)$, while being equal to zero elsewhere. From the definition of the Fourier series analysis integral, we have:

$$a_k = \frac{1}{T}\int_T x_p(t)e^{-jk\omega_0 t}dt$$

But since $x_{ap}(t)$ is equal to $x_p(t)$ over a single period, we can rewrite this as:

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x_{ap}(t)e^{-jk\omega_0 t}dt$$

But this looks quite a bit like the Fourier transform $X_{ap}(\omega)$ of $x_{ap}(t)$:

$$a_k = \frac{1}{T}X_{ap}(\omega)|_{\omega=k\omega_0}$$

We have found one relationship between the Fourier series coefficients and the Fourier transform of the underlying aperiodic signal.

Earlier we derived the Fourier transform of a complex exponential: $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$. Because any given Fourier series is the sum of scaled exponentials and the Fourier transform is a linear operator:

$$\begin{aligned}\mathcal{F}[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}] &= \sum_{k=-\infty}^{\infty} a_k \mathcal{F}[e^{jk\omega_0 t}] \\ &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)\end{aligned}$$

Of particular interest is the fact that an impulse train in time transforms into another impulse train in frequency:

$$\begin{aligned}\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t-nT)] &= \mathcal{F}[\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}] \\ &= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)\end{aligned}$$

We can then think of any periodic signal $x_p(t)$ as the convolution of the waveform for a single period $x_{ap}(t)$ convolved with an impulse train with the appropriate period:

$$x_p(t) = x_{ap}(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

In the frequency domain, we then have $X_{ap}(\omega)$ multiplied by another impulse train. But multiplying by an impulse train just samples $X_{ap}(\omega)$ to give $X_p(\omega)$:

$$\begin{aligned}X_p(\omega) &= X_{ap}(\omega)\frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \\ &= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} X_{ap}(k\omega_0)\delta(\omega - k\omega_0) \\ &= 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)\end{aligned}$$

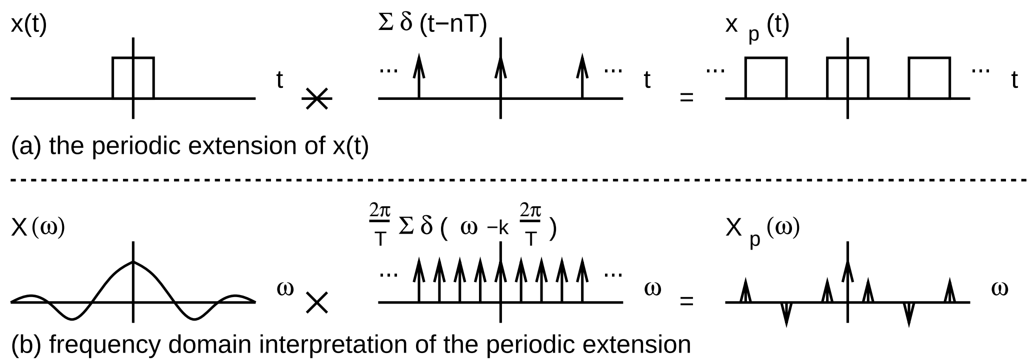

Figure 1: Another interpretation of the Fourier transform of a Fourier series.

where in the last line we used our formula that we derived above relating the FS coefficients a_k with the FT of $x_{ap}(t)$. This process is illustrated in Figure 1.

So making something periodic in time corresponds to making its transform discrete in frequency. The converse is also true: making something discrete in time corresponds to making its transform periodic in frequency. More on this when we talk about discrete systems.

2 Filtering

Before, we briefly addressed the problem of constructing a realizable filter from an ideal filter. We must do two things: truncate the filter's impulse response in time so that it is no longer of infinite extent (really long convolutions are an extreme drag) and delay the filter's impulse response so that it is causal. But what does that do to the filter's frequency response?

Let's start with an ideal low pass filter with impulse response $h_i(t)$ and frequency response $H_i(\omega)$. The truncation in time corresponds to multiplying $h_i(t)$ by a pulse in time to give $h_t(t)$. So we end up convolving our ideal pulse $H_i(\omega)$ by a sinc in frequency. This introduces some ripple into our resulting frequency response $H_t(\omega)$.

Now we need to delay $h_t(t)$ by an appropriate time τ to make our filter causal. This corresponds to multiplying our frequency response $H_t(\omega)$ by $e^{-j\omega\tau}$, introducing a linear phase factor into the entire mess. Sketching the magnitude response $|H_{td}(\omega)|$ of our filter is relatively straightforward. However, the phase response $\angle H_{td}(\omega)$ is slightly more difficult. Not only do we have a linear phase factor, but the negative going parts of the sinc contribute π radians for positive ω and $-\pi$ radians for negative ω (by convention, we choose $+\pi$ for positive ω and $-\pi$ for negative ω).

This entire process is illustrated in Figure 2.

3 Sample Problems

[from fall94, midterm II] The signal $x(t)$ is passed through a low pass filter with frequency response $H(\omega)$ [as in Figure 3]. The signal $x(t)$ contains a sinusoidal component at 100kHz. Sketch $y(t)$, the output in time of the low pass filter for the input $x(t)$.

Because 100kHz is greater than the cutoff frequency of our filter, the sinusoid goes away. However, since high frequencies are cut off, the jumps at $t = 0.1ms$ and $t = 0.2ms$ are rounded off. The filter also introduces some ripple into the output $y(t)$. Finally, we don't know if our filter is causal, so we can sketch the output either causal or acausal.

[from fall94, ps6] An LTI system has impulse response $h(t) = \frac{\sin(400\pi t)}{t}$. Find the time output of the system for an input of 60Hz even square wave with 50% duty cycle with zero average value and ± 1 peak to peak amplitude. Find the time output of the system for an input $x(t) = \frac{\sin 1000\pi t}{t}$.

We've already done the response of an LTI system to a periodic signal.

Exercise Verify that the system output $y(t) = 4\cos\frac{t}{60} - \frac{4}{3}\cos\frac{t}{180}$ in response to the square wave input.

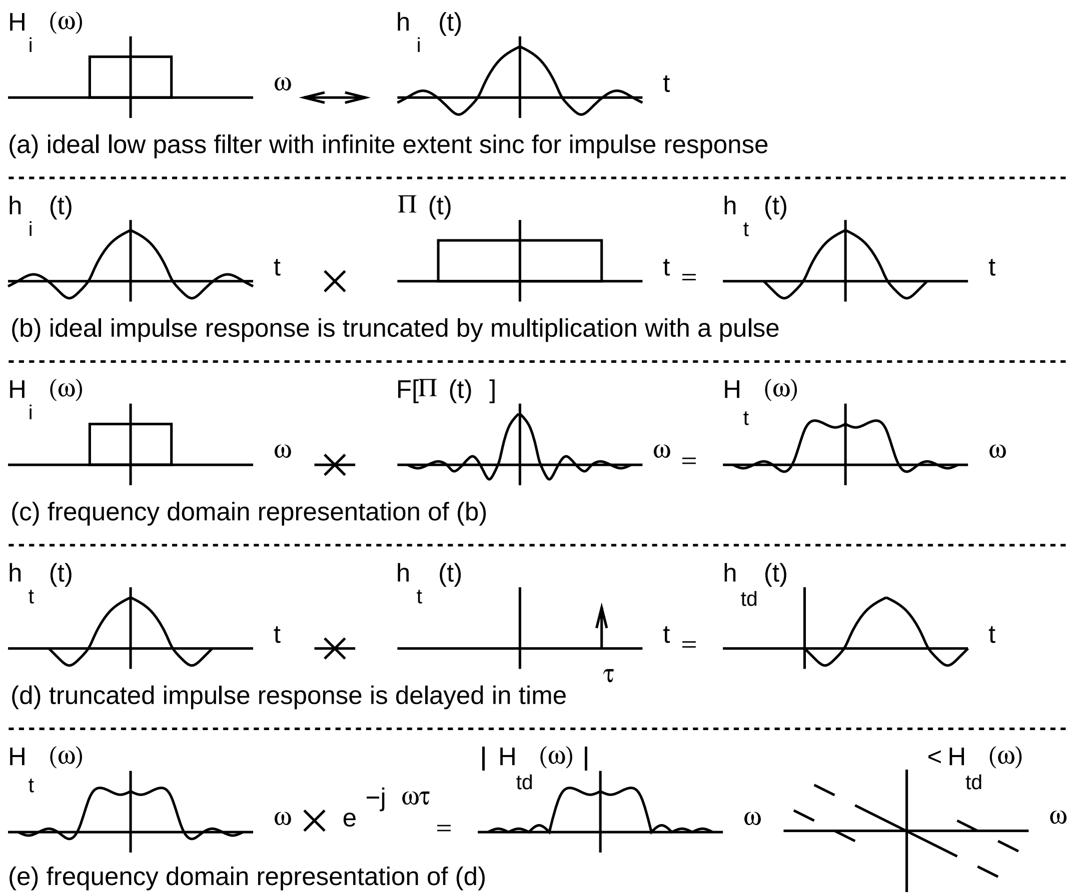

Figure 2: The creation of a realizable filter.

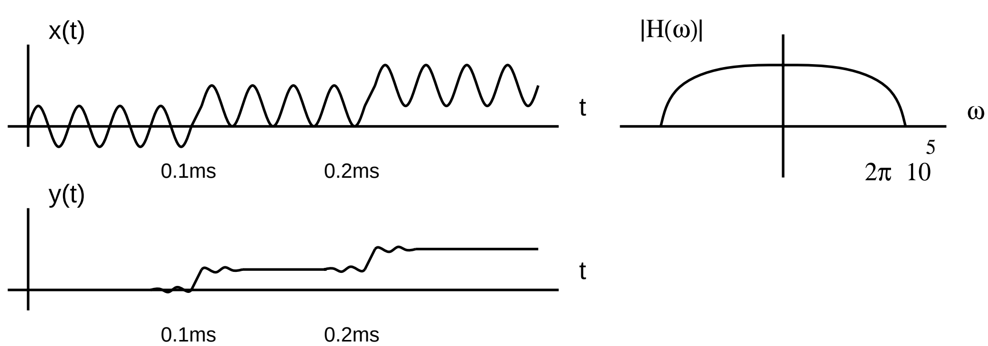

Figure 3: A sample midterm problem.

For the sinc input, we can take the Fourier transform of the input, get the frequency response of the system, multiply, and then inverse Fourier transform.

$$
\begin{aligned}
X(\omega) &= \pi\Pi(\frac{\omega}{800\pi}) \\
H(\omega) &= \pi\Pi(\frac{\omega}{2000\pi}) \\
Y(\omega) &= H(\omega)X(\omega) \\
&= \pi H(\omega) \\
y(t) &= \pi\frac{\sin(400\pi t)}{t}
\end{aligned}
$$