Notes 17 largely plagiarized by %khc

1 Laplace Transforms

The Fourier transform allowed us to determine the frequency content of a signal, and the Fourier transform of an impulse response gave us the frequency response of a system. Likewise, the Laplace transform can be thought of as permitting us to determine the "complex exponential content" [the frequency content, appropriately weighted] of a signal, and the Laplace transform of the impulse response gives us the transfer function of a system.

We begin with the Fourier transform (FT), defined as:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Note the notation change: $X(\omega)$ will now be written as $X(j\omega)$ to emphasize the relationship between the LT and the FT.

But let's not restrict ourselves to the $j\omega$ axis. Instead, let's consider the entire complex plane. We then have the bilateral Laplace transform (BLT):

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

where $s = \sigma + j\omega$.

There is an alternate form of the Laplace transform, known as the unilateral Laplace transform (ULT):

$$X(s) = \int_{0-}^{\infty} x(t)e^{-st}dt$$

Note that it is quite similar to the BLT, except for the lower limit of integration.

Unlike the FT, both the ULT and the BLT have regions of convergence (ROCs) over which the transform exists. A ROC is the range of values over which the integral converges.

The ULT is only useful for causal signals, a causal signal being defined as one that is nonzero for only nonnegative values of t;[1] to see this, just look at the limits of integration. The BLT permits us to consider any signal. For real-world systems, the ULT is sufficient for our needs, since all real-world systems are causal and input signals can be thought of as beginning from some arbitrary initial time, which might as well be $t = 0$.

2 ULT Properties

This is going to look very, very familiar (at least to those who are reading carefully). Compare to the FT properties.

Linearity: $\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(s) + \beta Y(s)$, ROC: at least the intersection of the ROC of $x(t)$ and the ROC of $y(t)$
The Laplace transform is a linear operator (as is the Fourier transform), following from properties of integration.

Convolution: $x(t) * y(t) \leftrightarrow X(s)Y(s)$, ROC: at least the intersection of the ROC of $x(t)$ and the ROC of $y(t)$
Convolution in time domain is equivalent to multiplication in the frequency domain. For causal signals, the convolution $x(t) * y(t)$ can be written as $\int_{0-}^{\infty} x(t-\tau)y(\tau)d\tau$. Taking the LT of this gives:

$$\begin{aligned}\mathcal{L}[x(t) * y(t)] &= \int_{0-}^{\infty}[\int_{0-}^{\infty} x(t-\tau)y(\tau)d\tau]e^{-st}dt \\ &= \int_{0-}^{\infty} y(\tau)[\int_{0-}^{\infty} x(t-\tau)e^{-st}dt]d\tau\end{aligned}$$

[1] Note that before we said that linearity, time-invariance, causality, memorylessness, and BIBO stability applied only to systems. Now we're going to stretch the definition of causality, and apply that to signals. But that's the only definition we're going to do that to.

$$
\begin{aligned}
&= \int_{0^-}^{\infty} y(\tau)[\int_{-\tau}^{\infty} x(t')e^{-s(t'+\tau)}dt']d\tau \\
&= \int_{0^-}^{\infty} y(\tau)e^{-s\tau}[\int_{0^-}^{\infty} x(t')e^{-st'}dt']d\tau \\
&= X(s)\int_{0^-}^{\infty} y(\tau)e^{-s\tau}d\tau \\
&= X(s)Y(s)
\end{aligned}
$$

Note the change of variables $t' = t - \tau$ and the fact that since τ is positive, the quantity in the parentheses in the fourth line reduces to $X(s)$.

Delay $x(t-T) \leftrightarrow X(s)e^{-sT}$, for $T > 0$, ROC: same as that of $x(t)$
The delay property can also be directly derived:

$$
\begin{aligned}
\mathcal{L}[x(t-T)] &= \int_{0^-}^{\infty} x(t-T)e^{-st}dt \\
&= \int_{-T}^{\infty} x(t')e^{-s(t'+T)}dt' \\
&= e^{-sT}\int_{-T}^{\infty} x(t')e^{-st'}dt' \\
&= e^{-sT}X(s)
\end{aligned}
$$

using the change of variables $t' = t - T$. Note that for negative T, the integral on the third line cannot be indentified as $X(s)$, since the lower limit will not include the portion of time between 0^- and T.

Differentiation $\dot{x}(t) \leftrightarrow sX(s) - x(0^-)$, ROC: at least that of $x(t)$
This property can be derived using integration by parts:

$$
\begin{aligned}
\mathcal{L}[\dot{x}(t)] &= \int_{0^-}^{\infty} \dot{x}(t)e^{-st}dt \\
&= x(t)e^{-st}|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}dt \\
&= \lim_{t\to\infty} x(t)e^{-st} - x(0^-) + sX(s) \\
&= sX(s) - x(0^-)
\end{aligned}
$$

assuming that $\lim_{t\to\infty} x(t)e^{-st}$ is zero. Fortunately, this will not reduce the ROC of the original transform $X(s)$. This property can be generalized to the nth derivative by considering the transform of $\frac{d}{dt}[\frac{d^{(n-1)}}{dt^{(n-1)}}]$.

Integration $\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau$, ROC: at least the intersection of that of $x(t)$ and $\mathcal{R}e(s) > 0$
Proof by integration by parts.

Multiplication by t $tx(t) \leftrightarrow -\frac{dX(s)}{ds}$, ROC: same as that of $x(t)$
If we differentiate $X(s)$, we get:

$$
\begin{aligned}
\frac{dX(s)}{ds} &= \frac{d}{ds}[\int_{0^-}^{\infty} x(t)e^{-st}dt] \\
&= \int_{0^-}^{\infty} x(t)\frac{d}{ds}[e^{-st}]dt \\
&= \int_{0^-}^{\infty} x(t)(-t)e^{-st}dt \\
&= -\mathcal{L}[tx(t)]
\end{aligned}
$$

Multiplication by $e^{-\alpha t}$ $e^{-\alpha t}x(t) \leftrightarrow X(s+\alpha)$, ROC: same as that of $x(t)$, shifted by $\mathcal{R}e\ (\alpha)$

$$\begin{aligned} \mathcal{L}[e^{-\alpha t}x(t)] &= \int_{0-}^{\infty} e^{-\alpha t}x(t)e^{-st}dt \\ &= \int_{0-}^{\infty} x(t)e^{-(s+\alpha)t}dt \\ &= X(s+\alpha) \end{aligned}$$

Initial value theorem $\lim_{s\to\infty} sX(s) = x(0^-)$
From the differentiation property, we have:

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0^-)$$

Taking the limit as $s \to \infty$, on the left-hand side we have:

$$\begin{aligned} \lim_{s\to\infty} \mathcal{L}[\dot{x}(t)] &= \lim_{s\to\infty} \int_{0-}^{\infty} \dot{x}(t)e^{-st}dt \\ &= \int_{0-}^{\infty} \lim_{s\to\infty} [\dot{x}(t)e^{-st}]dt \\ &= 0 \end{aligned}$$

if s falls within the ROC. This then implies that:

$$\lim_{s\to\infty} sX(s) = x(0^-)$$

as desired. This assumes that $x(t)$ does not have a second order discontinuity at $t = 0$ [ie the IVT will work if $x(t)$ has a step-like discontinuity at $t = 0$, such as $f(t)u(t)$, but not if $x(t)$ has a delta function at $t = 0$].

Final value theorem $\lim_{s\to 0} sX(s) = x(\infty)$ if $\lim_{t\to\infty} x(t)$ exists
Starting once again from the differentiation property, but taking the limit as $s \to 0$, on the left hand side we have:

$$\begin{aligned} \lim_{s\to 0} \mathcal{L}[\dot{x}(t)] &= \lim_{s\to 0} \int_{0-}^{\infty} \dot{x}(t)e^{-st}dt \\ &= \int_{0-}^{\infty} \dot{x}(t) \lim_{s\to 0}[e^{-st}]dt \\ &= \int_{0-}^{\infty} \dot{x}(t)dt \\ &= \lim_{t\to\infty} x(t) - x(0^-) \end{aligned}$$

The total equation is then:

$$\begin{aligned} \lim_{t\to\infty} x(t) - x(0^-) &= \lim_{s\to 0} sX(s) - x(0^-) \\ \lim_{s\to 0} sX(s) &= \lim_{t\to\infty} x(t) \end{aligned}$$

Unfortunately, if $\lim_{t\to\infty} x(t)$ does not exist, $\lim_{s\to 0} sX(s)$ will give you a bogus answer. This is because if you put poles in the right half plane, there will be exponentially growing terms instead.
Exercise Familiarize yourself with these properties. Derive the proof of the integration property. Make sure that the changes to the ROCs make sense.

3 Some Useful ULTs

$\delta(t) \leftrightarrow 1$**, ROC: all** s

$$\begin{aligned} X(s) &= \mathcal{L}[\delta(t)] \\ &= \int_{0^-}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} \delta(t) dt \\ &= 1 \end{aligned}$$

Note that this is why we need to define the lower limit as 0^-.

$u(t) \leftrightarrow \frac{1}{s}$**, ROC:** $\mathcal{R}e(s) > 0$

$$\begin{aligned} X(s) &= \mathcal{L}[u(t)] \\ &= \int_{0^-}^{\infty} u(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} e^{-st} dt \\ &= -\frac{1}{s} e^{-st} |_{0^-}^{\infty} \\ &= \frac{1}{s} \end{aligned}$$

This can also be performed by the integration property.

$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$**, ROC:** $\mathcal{R}e\ (s) > -\mathcal{R}e\ (a)$

$$\begin{aligned} X(s) &= \mathcal{L}[e^{-at}u(t)] \\ &= \int_{0^-}^{\infty} e^{-at} u(t) e^{-st} dt \\ &= \int_{0^-}^{\infty} e^{-(s+a)t} dt \\ &= -\frac{1}{s+a} e^{-(s+a)t} |_{0^-}^{\infty} \\ &= \frac{1}{s+a} \end{aligned}$$

$\sin \omega_0 t u(t) \leftrightarrow \frac{\omega_0}{s^2+\omega_0^2}$**, ROC:** $\mathcal{R}e(s) > 0$

$$\begin{aligned} X(s) &= \mathcal{L}[\sin \omega_0 t u(t)] \\ &= \frac{1}{2j}\mathcal{L}[e^{j\omega_0 t}u(t)] - \frac{1}{2j}\mathcal{L}[e^{-j\omega_0 t}u(t)] \\ &= \frac{1}{2j}\frac{1}{s - j\omega_0} - \frac{1}{2j}\frac{1}{s + j\omega_0} \\ &= \frac{\omega_0}{s^2 + \omega_0^2} \end{aligned}$$

Note that the ROC comes from the transforms of the complex exponentials.

$\cos\omega_0 t u(t) \leftrightarrow \frac{s}{s^2+\omega_0^2}$, **ROC:** $\mathcal{R}e(s) > 0$

$$\begin{aligned}
\frac{d}{dt}\sin\omega_0 t u(t) &= \omega_0\cos\omega_0 t u(t) + \sin\omega_0 t\delta(t) \\
&= \omega_0\cos\omega_0 t u(t) \\
\mathcal{L}[\frac{d}{dt}\sin\omega_0 t u(t)] &= \omega_0\mathcal{L}[\cos\omega_0 t u(t)] \\
\mathcal{L}[\cos\omega_0 t u(t)] &= \frac{1}{\omega_0}\mathcal{L}[\frac{d}{dt}\sin\omega_0 t u(t)] \\
&= \frac{1}{\omega_0}\frac{\omega_0 s}{s^2+\omega_0^2} \\
&= \frac{s}{s^2+\omega_0^2}
\end{aligned}$$

$te^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^2}$, **ROC:** $\mathcal{R}e(s) > -a$ The multiplication by t property applied to $\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}$ gives:

$$\begin{aligned}
\mathcal{L}[e^{-at}u(t)] &= -\frac{d}{ds}\frac{1}{s+a} \\
&= \frac{1}{(s+a)^2}
\end{aligned}$$

$e^{-at}\sin\omega_0 t \leftrightarrow \frac{\omega_0}{(s+a)^2+\omega_0^2}$, **ROC:** $\mathcal{R}e(s) > -a$ The multiplication by e^{-at} property can be applied to the Laplace transform of $\sin\omega_0 t u(t)$ to give the above result.

$e^{-at}\cos\omega_0 t \leftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2}$, **ROC:** $\mathcal{R}e(s) > -a$ Another application of the multiplication by e^{-at} property.

4 Utility Value

Now, all this transform stuff is nice and mathematical, but what good is it?

Well, the convolution property allows us to study systems. From the third week of this class, we saw that the output of a system $y(t)$ is the convolution of the impulse response $h(t)$ with the input $x(t)$:

$$y(t) = h(t) * x(t)$$

If we take the Laplace transform:

$$Y(s) = H(s)X(s)$$

or

$$H(s) = \frac{Y(s)}{X(s)}$$

Nothing new; we already saw this with Fourier transforms. However, we can now put in signals for which the Fourier transform does not exist, such as $e^t u(t)$ and $r(t)$.

So starting from an LDE-CC [linear differential equation with constant coefficients], we can derive the transfer function $H(s)$ by simply taking the LT of both sides, with $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ and $y(t) \overset{\mathcal{L}}{\leftrightarrow} Y(s)$:

$$\begin{aligned}
\sum_{k=0}^{N} a_k y^{(k)} &= \sum_{l=0}^{M} b_l x^{(l)} \\
\sum_{k=0}^{N} a_k s^{(k)} Y(s) &= \sum_{l=0}^{M} b_l s^{(l)} X(s) + A(s)
\end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{1}{\sum_{k=0}^{N} a_k s^{(k)}}[\sum_{l=0}^{M} b_l s^{(l)} X(s) + A(s)] \\ \text{where } A(s) &= a_N[s^{N-1}y(0^-) + s^{N-2}y^{(1)}(0^-) + \ldots + y^{(N-1)}(0^-)] \\ &+ a_{N-1}[s^{N-2}y(0^-) + s^{N-3}y^(1)(0^-) + \ldots + y^{(N-2)}(0^-)] \\ &+ \ldots + a_1 y(0^-) \end{aligned}$$

If we take the inverse transform of this, the term to the left on the right hand side will give you the ZSR. The term to the right on the right hand side will give you the ZIR. [Some may think this easier than solving the LDE, but you end up doing a lot of algebra; your mileage may vary.] If we zero all the initial conditions:

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} \\ &= \frac{\sum_{l=0}^{M} b_l s^l}{\sum_{k=0}^{N} a_k s^k} \end{aligned}$$

So what is the relationship between the Fourier transform and the Laplace transform? For a causal signal/system [this means that the ROC will be to the right of the rightmost pole; more on this later], if the ROC contains the $j\omega$ axis, then then evaluating the ULT on the $j\omega$ axis will give you the FT:

$$X(j\omega) = X(s)|_{s=j\omega}$$

If the $j\omega$ axis is not in the ROC, then evaluating the LT on the $j\omega$ axis and taking the inverse FT gives you a time function completely different from the one of which you took the Laplace transform.

5 Magnitude and Phase Plots

Let's consider an $H(s)$ of the form (generalize in the privacy of your own dorm room):

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}|_{s=j\omega}$$

where z_1, z_2, p_1, and p_2 are real and s is restricted to the $j\omega$ axis. If we just look at $j\omega + z_1$, in polar this is $\sqrt{z_1^2 + \omega^2} e^{j \arctan \frac{\omega}{z_1}}$ where the square root junk is the magnitude and the arctan garbage is the phase. If we do this for each set of expressions in parentheses in $H(j\omega)$, we get a whole bunch of polar things on the top and another whole bunch of polar things on the bottom. If we then look for $|H(j\omega)|$, this is just the product of all the magnitudes on the top divided by the product of all the magnitudes on the bottom. If we look for $\angle H(j\omega)$, this is the sum of all the angles on the top, minus the sum of all the angles on the bottom (since the angles are in the exponents, they add and subtract, instead of getting multiplied and divided).

We can then plot the magnitude on a linear or log-log scale, and the phase on a linear or log-linear scale. If we stick to a log-log scale in magnitude and a log-linear scale in phase, we have Bode plots, as in ee40.

Bode plots are useful things to know. However, since they are pretty painful to draw out, i'll just direct you to the review modules and your old ee40 book.

The one caveat is that the techniques that you were taught for Bode plots do not work correctly if the poles are complex-valued. More on this in a bit.

6 Poles and Zeros

For the large majority of systems that we will study, the transfer function can be written in terms of a numerator polynomial $N(s)$ and a denominator polynomial $D(s)$. The roots of $N(s)$ are called zeros and the roots of $D(s)$ are called poles.

A plot of these poles and zeros, called a pole-zero diagram, uniquely describes a transfer function, up to a constant. The zeros are marked on the complex s plane as circles; poles are marked on the same plane as crosses.

The denominator polynomial, sometimes called the characteristic polynomial, determines a large portion of the behavior of the system. One way to see this is to take the inverse transform the output transform $Y(s)$ for a given input transform $X(s)$. Since $Y(s) = H(s)X(s)$, factoring $D(s)$ and taking the partial fraction expansion is going to give you a bunch of terms; the time function that corresponds to some of these terms is determined by the roots of the denominator of $X(s)$, but other terms will be determined by the roots of the denominator of $H(s)$. More on this in a bit.

7 Vectorial Interpretation

On a pole-zero diagram, draw a vector from the origin along the $j\omega$ axis. Call this vector (creatively) $j\omega_0$.

- (a) Draw vectors from the zeros to the head the $j\omega_0$ vector. The product of the magnitudes of these vectors is the magnitude of the numerator of the transfer function at $\omega = \omega_0$. The sum of the angles of these vectors is the phase of the numerator at $\omega = \omega_0$.
- (b) Draw vectors from the poles to the head the $j\omega_0$ vector. The product of the magnitudes of these vectors is the magnitude of the denominator of the transfer function at $\omega = \omega_0$. The sum of the angles of these vectors is the phase of the denominator at $\omega = \omega_0$.
- (c) Divide the magnitude of the numerator by the magnitude of the denominator. This is the magnitude of the transfer function at $\omega = \omega_0$. Subtract the phase of the denominator from the phase of the numerator. This is the phase of the transfer function at $\omega = \omega_0$.

Because i'm sort of pressed for time right now, please take a look at 590-594 in your textbook.

For an analog method of finding the magnitude response from the pole-zero diagram, get a rubber sheet (the same one you used in physics for imagining how gravity affects space and time), some tent poles, and some thumbtacks. Find a plot of ground, and draw the real and imaginary axes on it (this is the complex plane). Stretch the sheet over the ground. Push up the sheet at the pole locations using the tent poles. Take the thumbtacks and shove them through the sheet and into the ground at the zero locations. Admire your handiwork. Do not use this for a trampoline. You will damage the pole locations.

8 Stability and Causality

The poles of the transfer function in combination with the region of convergence determine the stability and casuality of the system.

Causal systems will have a region of convergence heading off to the right. Anticausal systems will have a region of convergence heading off the to the left. [justification and other fun facts about ROCs in section 9.2 of OWY.]

By definition, no poles can appear in the ROC [otherwise it wouldn't be called a ROC]. Stable systems will have ROCs containing the $j\omega$ axis. Unstable systems will have ROCs that do not contain the axis.

One of the measures of stability is bounded input, bounded output stability (BIBO stability). This means that if i make a signal that is finite for all time and put it into a system, the output is finite for all time.

We can then determine BIBO stability from the transfer function. All the poles must be in the open left half plane [$\mathcal{R}e\ (s) < 0$], and the numerator must have degree less then or equal to that of the denominator.

Exercise Verify the statement in the last paragraph. Consider a system with a pole in the right half plane [$\mathcal{R}e\ (s) \geq 0$] and put in a unit step into the system. Look at the system's time response. Also consider a system with the degree of the numerator greater than the degree of the denominator. Notice that there will be derivatives of the delta function in the impulse response, giving a time response that will be unbounded for a step input.

9 First Order Systems

If we consider the order of a system as the order of the denominator polynomial, a first order system would have a transfer function of the form:

$$H(s) = \frac{1}{\tau s + 1}$$

This system will have a gain of unity at low frequencies, the gain rolling off at 20dB/decade beginning at $s = \frac{1}{\tau}$.

For stable, causal, and real first order systems, this corresponds to an impulse response of:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

Integrating this impulse response from $-\infty$ to t gives us a step response of

$$y_{\text{step}}(t) = [1 - e^{-t/\tau}]u(t)$$

Note that the step response exponentially decays to 1, leading some to call the first order system a first order lag.

Note that if the pole is in the right half plane [such that $\tau < 0$], then the system will be unstable, since the output will contain an exponentially increasing term $e^{-t/\tau}$ for any input.

10 Second Order Systems

Second order systems are more fun, since there is more than one way to arrange the poles. Let's assume that the system has a transfer function of the form:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ζ is called the damping ratio and ω_n is the natural frequency.

Stable, causal, and real second order systems of this form can be classified into three different types, depending on pole location.

- poles on real axis (overdamped, $\zeta > 1$)
- poles at same location on real axis (critically damped, $\zeta = 1$)
- poles in complex conjugate pairs (real system implied, underdamped, $\zeta < 1$)

The best way to see how ζ affects the poles is to use the quadratic formula on the characteristic polynomial:

$$\begin{aligned} s &= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned}$$

For $\zeta > 1$, the poles are on the real axis. For $\zeta = 1$, both of the poles are in the same place. For $0 < \zeta < 1$, the poles show up as a complex conjugate pair.

Some interesting facts: as ζ is varied from 0 to 1, the poles move in a semicircle with radius ω_n centered on the origin. As ζ is varied from 1 to ∞, the poles stick to the real axis. One goes flying out to $-\infty$, and the other approaches the origin.

Exercise Verify these facts. [To notice the semicircle, take the magnitude of the poles for $0 \leq \zeta \leq 1$ and watch the dependence on ζ disappear.]

Another more fun thing to do is study the step responses of these systems. Let's do this by example instead, since it will make life easier. Let $w_n = 4$ and examine step responses for $\zeta = \frac{1}{2}$, $\zeta = 1$, and $\zeta = \frac{5}{4}$.

For $\zeta = \frac{5}{4}$, the characteristic polynomial becomes $s^2 + 10s + 16$, which can be factored into $(s+2)(s+8)$. The transfer function is:

$$H(s) = \frac{16}{(s+2)(s+8)}$$

If we're looking for the step response, $x(t) = u(t)$, which means $X(s) = \frac{1}{s}$. $y(t)$ is then the inverse transform of $Y(s)$ [although you need to perform the proper partial fraction expansion]:

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{16}{(s+2)(s+8)s} \\ &= \frac{-4/3}{s+2} + \frac{1/3}{s+8} + \frac{1}{s} \\ y(t) &= [-\frac{4}{3}e^{-2t} + \frac{1}{3}e^{-8t} + 1]u(t) \end{aligned}$$

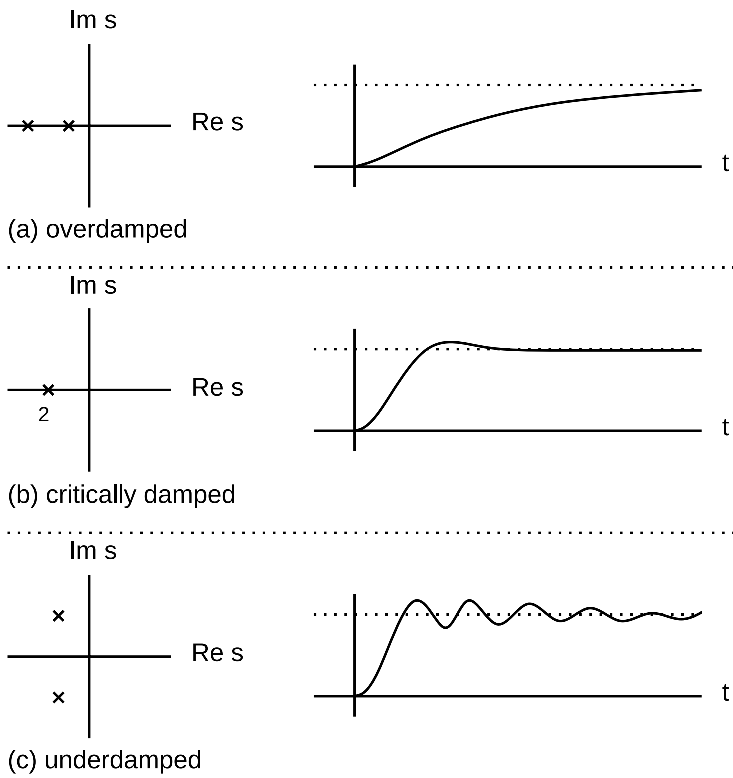

Figure 1: The classes of second order systems.

Note that the step response, sketched in Figure 1(a), has two exponentially decaying terms, in addition to the step. The step results from the input, but the exponentially decaying terms come about because of the particular poles. So the pole locations are very important. In fact, for large ζ, the pole closest to the $j\omega$ axis will give an exponentially decaying term that will dominate the time response.

Exercise Why?

For $\zeta = 1$, the characteristic polynomial becomes $s^2 + 8s + 16$, which can be factored into $(s + 4)^2$. The transfer function is:

$$H(s) = \frac{16}{(s+4)^2}$$

If we're looking for the step response, $x(t) = u(t)$, which means $X(s) = \frac{1}{s}$. $y(t)$ is then the inverse transform of $Y(s)$ [use the normal partial fractions trick to get the coefficients for the $\frac{1}{s}$ and $\frac{1}{(s+4)^2}$ terms, and the derivative trick for the $\frac{1}{s+4}$ term]:

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{16}{s(s+4)^2} \\ &= \frac{1}{s} + \frac{-1}{s+4} + \frac{-4}{(s+4)^2} \\ y(t) &= [1 - e^{-4t} - 4te^{-4t}]u(t) \end{aligned}$$

Note that the step response a term of the form te^{-t}. This is what makes having a critically damped system so great, since the time response is that sketched in Figure 1(b). It barely overshoots and is otherwise quite beautiful. Achieving such a time response is the holy grail for control people.

For $\zeta = \frac{1}{2}$, the characteristic polynomial becomes $s^2 + 4s + 16$, which cannot be elegantly factored into real terms.

The transfer function is:

$$H(s) = \frac{16}{s^2 + 4s + 16}$$

If we're looking for the step response, $x(t) = u(t)$, which means $X(s) = \frac{1}{s}$. $y(t)$ is then the inverse transform of $Y(s)$ [partial fraction expand with $\frac{1}{s}$ and $\frac{As+B}{s^2+4s+16}$, complete the square to get the denominator to be $(s+2)^2 + 12$, and use table lookup]:

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{16}{(s^2+4s+16)s} \\ &= \frac{1}{s} + \frac{-s-4}{(s+2)^2+12} \\ &= \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{(s+2)^2+12} \\ y(t) &= [1 - e^{-2t}\cos 2\sqrt{3}t - \frac{1}{\sqrt{3}}e^{-2t}\sin 2\sqrt{3}t]u(t) \end{aligned}$$

Note that the step response has is the sum of two exponentially decaying sinusoids. As $\zeta \to 0$, the time response, as sketched in Figure 1(c), approaches a purely sinusoidal response, since the poles end up on the $j\omega$ axis. In fact, as the poles get really close to the $j\omega$ axis, the magnitude of the frequency response becomes extremely large, and the phase response changes tremendously.

In fact, if we have a lot of poles and zeros, a decent first order approximation is to just to punt all the poles and zeros sufficiently far away enough from the $j\omega$ axis, and deal with the leftovers. Of course, we'll need to make sure that the stuff that we punted actually doesn't cause us a large headache later on.

11 Putting Everything Together

We have seen five major ways to represent a system:

- by transfer function
- by linear differential equation (LDE)
- by magnitude/phase response
- by pole-zero diagrams
- and by time response

You should be able to convert from one representation to another:

(a) transfer function to time response:
 use partial fraction expansion and then Laplace transform table lookup.

(a') time response to transfer function:
 Laplace transform the time response.

(b) transfer function to magnitude and phase response:
 let $s = j\omega$. Use Bode plotting tricks if poles and zeros on real axis.
 Otherwise: use (c), then (d).

(b') magnitude and phase response to transfer function:
 Determine number of poles and zeros by examining the behavior as $\omega \to 0$ and $\omega \to \infty$. Determine the ordering of the poles and zeros by examining all the stuff in between. If the the magnitude response goes up, there is a pole around there somewhere. If the magnitude response goes down, there may either be a zero around there, or you may be sufficiently far away from a pole not to see its effect. In general, poles and zeros closer to the $j\omega$ axis will have greater effect on the response than poles and zeros farther away. Choose and fine-tune locations of poles and zeros appropriately.

(c) transfer function to pole-zero diagram:
 Find roots of numerator of transfer function. These are zeros. Find roots of denominator of transfer function. These are poles. Plot.

(c') pole-zero diagram to transfer function:
Construct numerator from zeros. Construct denominator from poles. You may be off by some multiplicative factor (gain) though.
(d) pole-zero diagram to magnitude/phase response:
Let $s = j\omega$. Use the vectorial interpretation.
(d') magnitude/phase response to pole-zero diagram:
Use (b'), then (c').
(e) pole-zero diagram to time response:
Poles closest to $j\omega$ axis will dominate response if other poles are sufficiently far away. Because systems are restricted to be realizable ones, they will have real coefficients, so zeros and poles will exhibit complex conjugate symmetry. If the system is to be causal and stable, its poles should be in the open left half plane. If you have a single dominating pole on the real axis, you will have a first order time response. If you have a conjugate pair of dominating poles, you will have a second order time response.
For best results, however, find transfer function and then partial fraction expand.
(e') time response to pole-zero diagram:
Be familiar with comments in (e). If you have the equations for the time response, use (a'), then (c).
(f) LDE to transfer function:
Let $x(t) \leftrightarrow X(s)$ and $y(t) \leftrightarrow Y(s)$. Rewrite derivatives using differentiation property. Solve for $H(s) = \frac{Y(s)}{X(s)}$.
(f') transfer function to LDE:
Since $H(s) = \frac{Y(s)}{X(s)}$, cross multiply to obtain an equation in s, $X(s)$, and $Y(s)$. Then inverse Laplace transform.

12 A Look Ahead and Behind

Feedback and more on stability! Great stuff. Everybody should take ee128.

You might want to go back and review the relationship between the three transforms we have learned so far: FS [yes, you can think of it as a transform], FT, and LT.

- FT $\rightarrow$ FS: in time, make $x(t)$ periodic by convolving with an impulse train; in frequency, make $X(j\omega)$ discrete by multiplying with the transform of that impulse train, which is another impulse train.
- LT $\rightarrow$ FT: evaluate $X(s)$ on the $j\omega$ axis.