Notes 18 largely plagiarized by %khc

1 Pole-Zero Diagram to Magnitude/Phase Plot

For particular arrangements of the poles and zeros, we can arrive at various magnitude and phase responses. These magnitude and phase responses can be divided into five major classes:

- all pass
- band pass
- high pass
- low pass
- band stop or notch

all pass The magnitude response is unity for all frequencies. This doesn't appear to be useful, but we actually design these filters for a desired phase response instead. Note that the poles and zeros must appear in complex conjugate pairs if this is supposed to be a realizable system [this means that the coefficients in the transfer function will be real]. Each pole also has a corresponding zero appearing as a mirror image across the $j\omega$ axis. This means that the contribution to the magnitude of the vector from the pole to the $j\omega$ test point is balanced by the contribution of the vector from the zero to the $j\omega$ test point.

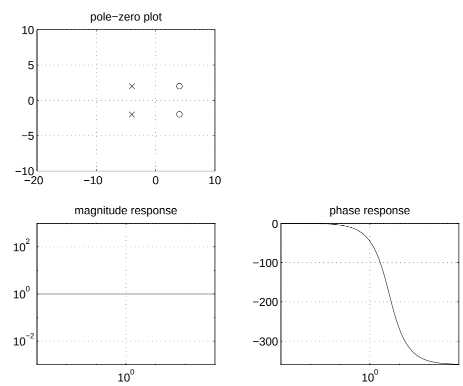

Figure 1: All pass system.

band pass We'd like a magnitude response that is zero for all frequencies except the ones that we are interested in passing. So we can put a bunch of poles around the frequencies that we want to pass, pulling up the magnitude response in that range, and if we really wanted the magnitude response to drop sharply around that, we would throw zeros above and below that range.

low pass To have a magnitude response that is large for low frequencies, we'll need to have some number of poles at low frequencies. If we wish the magnitude response to drop steeply around the cutoff frequency, we could throw zeros above the cutoff frequency.

high pass To have a magnitude response that is constant for high frequencies, we'll need to have the same number of poles and zeros. To pull down the magnitude response for low frequencies, a smattering of zeros scattered around low frequencies is useful.

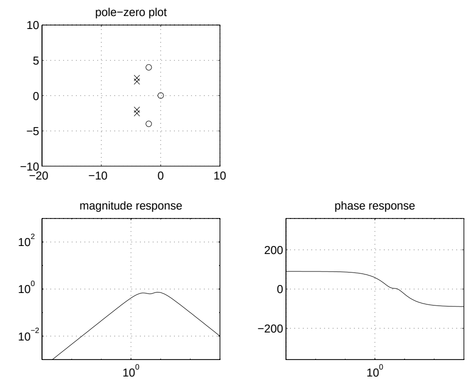

Figure 2: Band pass system.

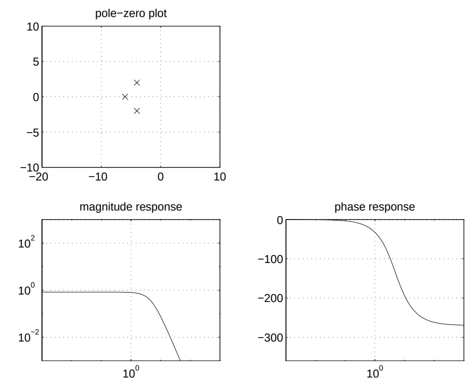

Figure 3: Low pass system.

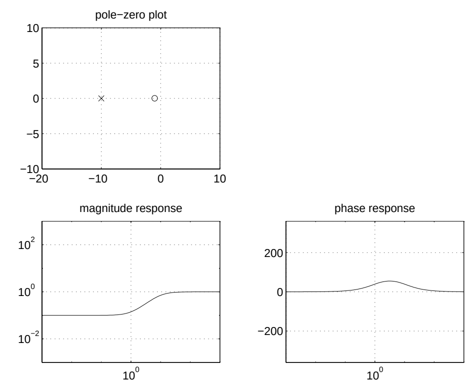

Figure 4: High pass system.

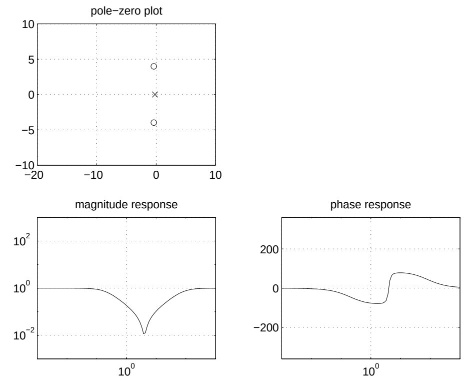

Figure 5: Band stop system.

band stop We want a magnitude response that is constant for all frequencies except the one that we want to kill. For high frequencies, the magnitude response should be constant, so the number of poles and zeros should be the same. The poles should be placed on the real axis and the zeros should be placed in complex conjugate pairs. One such band stop filter has transfer function:

$$H(s) = \frac{s^2 + 2\zeta_z \omega_n s + \omega_n^2}{s^2 + 2\zeta_p \omega_n s + \omega_n^2}$$

2 Minimum Phase, Inverse Systems

At some point in time, you might run across someone who walks up to you and mentions the term "minimum phase". If a system is referred to as being minimum phase, that means that the poles and zeros of the transfer function are all in the left half plane. So zeros contribute 0 to 90 degrees to phase and poles contribute 0 to -90 degrees. If a zero were in the right half plane, it would contribute 180 to 90 degrees of phase, which is certainly more than a zero in the left half plane. Similarly, if a pole were in the right half plane, it would contribute -180 to -90 degrees of phase.

The other thing about minimum phase systems is that the corresponding inverse system [such that $H(s)H^{-1}(s) = 1$] will also be of minimum phase. This can easily be seen by inverting the transfer function; if the poles and zeros of the original system were in the left half plane, the poles and zeros of the inverse system would also be in the left half plane. So causal stable minimum phase systems will have causal stable minimum phase inverse systems.

3 BIBO Stability, Revisited

In the frequency domain, we have two criteria for stability:

- The transfer function must be proper. Otherwise the impulse response would contain the doublet or other higher order derviatives of $\delta(t)$, which would lead to an unbounded step response. Impulses in an impulse response are acceptable [plausibility argument: if $h(t) = \delta(t)$ wasn't the impulse response of a stable system, then everything would be blowing up around us, since $h(t) = \delta(t)$ is the impulse response of an ideal wire].
- The poles must be in the open left half plane. Otherwise when we consider, say, a step response, and do a partial fraction expansion, the terms over the right half plane poles would inverse transform into nasty unbounded terms [plausibility argument: consider $H(s) = \frac{1}{s-1}$; step response has a $\frac{1}{s-1}$ term in it, which transforms into an $e^t u(t)$ term that is unbounded].

In the time domain, if the impulse response must be absolutely integrable. This can be seen by considering the definition of BIBO stabilty; $x(t)$ and $y(t)$ are both finite for all values of t:

$$|y(t)| = |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau|$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)| d\tau \\
&\leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau \\
&\leq K \int_{-\infty}^{\infty} |h(\tau)| d\tau
\end{aligned}
$$

assuming $|x(t)| \leq K$ for all t. If $|y(t)| \leq \infty$, then $\int_{-\infty}^{\infty} |h(\tau)| d\tau \leq \infty$, which is the absolute integrability condition.
Exercise Consider going back to those systems in Notes 02 and checking whether or not they are BIBO stable.