

Notes 23 largely plagiarized by %khc

1 ZT Problems

[93 final]: Given the following difference equation, find $H(z)$:

$$y[n - 2] + 3y[n - 1] + 2y[n] = x[n - 1]$$

Also, for $y[-1] = 1$, $y[-2] = 0$, and $x[n] = u[n]$, find the ZIR and ZSR.

Take the [unilateral] Z transform and solve for $\frac{Y(z)}{X(z)}$, which is $H(z)$:

$$\begin{aligned} z^{-2}Y(z) + z^{-1}y[-1] + y[-2] + 3z^{-1}Y(z) + 3y[-1] + 2Y(z) &= z^{-1}X(z) + x[-1] \\ z^{-2}Y(z) + 3z^{-1}Y(z) + 2Y(z) &= z^{-1}X(z) \\ \frac{Y(z)}{X(z)} &= \frac{z^{-1}}{z^{-2} + 3z^{-1} + 2} \\ H(z) &= \frac{z^{-1}}{z^{-2} + 3z^{-1} + 2} \end{aligned}$$

For the ZIR, kill the input and solve for $Y(z)$:

$$\begin{aligned} z^{-2}Y(z) + z^{-1}y[-1] + y[-2] + 3z^{-1}Y(z) + 3y[-1] + 2Y(z) &= 0 \\ Y(z)[z^{-2} + 3z^{-1} + 2] &= -z^{-1} - 3 \\ Y(z) &= \frac{-z^{-1} - 3}{z^{-2} + 3z^{-1} + 2} \\ &= \frac{-z^{-1} - 3}{(z^{-1} + 1)(z^{-1} + 2)} \\ &= \frac{-2}{z^{-1} + 1} + \frac{1}{z^{-1} + 2} \\ y[n] &= -2(-1)^n u[n] + \frac{1}{2}(-\frac{1}{2})^n u[n] \end{aligned}$$

For the ZSR, kill the initial conditions and solve for $Y(z)$:

$$\begin{aligned} z^{-2}Y(z) + 3z^{-1}Y(z) + 2Y(z) &= z^{-1}X(z) + x[-1] \\ Y(z) &= \frac{z^{-1}}{z^{-2} + 3z^{-1} + 2} X(z) \\ &= \frac{z^{-1}}{(1 - z^{-1})(z^{-2} + 3z^{-1} + 2)} \\ &= \frac{\frac{1}{6}}{1 - z^{-1}} + \frac{\frac{2}{3}}{z^{-1} + 2} + \frac{-\frac{1}{2}}{z^{-1} + 1} \\ y[n] &= \frac{1}{6}u[n] + \frac{1}{3}(-\frac{1}{2})^n u[n] - \frac{1}{2}(-1)^n u[n] \end{aligned}$$

[93 final]: The impulse response $h[n]$ of a DT LTI system is given as in Figure 1. Find $g[n]$, the impulse response of the inverse system; that is, find a $g[n]$ such that $g[n] * h[n] = \delta[n]$. Note that $g[n]$ should be in closed form.

We can either find something that convolves out with $h[n]$ to be the unit sample, or we can use Z transforms, with the idea that $G(z)H(z) = 1$ [by just transforming the equation above for the inverse system]. Let's use the Z approach. First thing to note is that the impulse response can be written as $h[n] = \delta[n] + (\frac{1}{2})^{n-1}u[n - 1]$. So

$$H(z) = 1 + \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

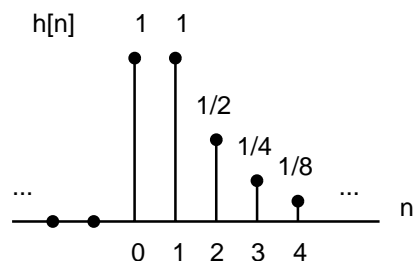


Figure 1: Impulse response.

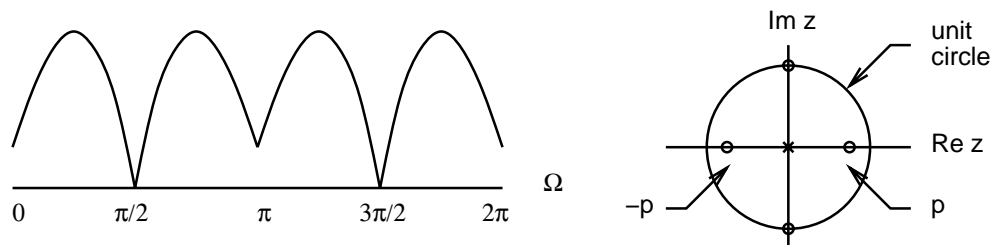


Figure 2: Magnitude spectrum and pole-zero plot.

$$\begin{aligned}
 &= \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}} \\
 G(z) &= \frac{1}{H(z)} \\
 &= \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}} \\
 &= 1 - \frac{z^{-1}}{1 + \frac{1}{2}z^{-1}} \\
 g[n] &= \delta[n] - \left(-\frac{1}{2}\right)^{n-1}u[n-1]
 \end{aligned}$$

[93 final]: Sketch a pole-zero diagram in the z -plane for a stable, causal FIR filter with the magnitude spectrum as in Figure 2. You may make reasonable engineering assumptions for your diagram. [Hint: the filter has four poles.] Don't worry about the exact height at 0 or π . You may leave locations in terms of parameter p . Then determine the unit pulse response $h[n]$ of the filter.

Terminology: what's an FIR filter? FIR stands for "finite impulse response"; the impulse response is of finite duration. If the system is causal and has an impulse response beginning at index 0 and limited to $N + 1$ samples, its Z transform can be written in the form:

$$\begin{aligned}
 H(z) &= \sum_{k=0}^N h[k]z^{-k} \\
 &= \frac{1}{z^N} \sum_{k=0}^N h[k]z^{N-k}
 \end{aligned}$$

Where are the poles of this transfer function? All at the origin. It is the zero locations that determine the frequency response of this system.

Using the vectorial interpretation of the transfer function [as on page 646 of your textbook], we can deduce that there are two zeros at $z = \pm j$, since the magnitude response goes to zero at those points. Around $z = 0$ and $z = \pi$,

the response goes down, but does not reach zero. So there must be some zeros on the real axis. To make the filter of minimum phase [so that a stable inverse exists], I choose the zeros at locations $z = \pm p$. We know that the pole-zero diagram determines the transfer function up to a constant gain; if we assume that the gain is unity, the transfer function is then:

$$\begin{aligned} H(z) &= \frac{1}{z^4}(z+j)(z-j)(z+p)(z-p) \\ &= \frac{1}{z^4}(z^2+1)(z^2-p^2) \\ &= \frac{1}{z^4}[z^4 + (1-p^2)z^2 - p^2] \end{aligned}$$

The corresponding impulse response is then:

$$h[n] = \delta[n] + (1-p^2)\delta[n-2] - p^2\delta[n-4]$$

What kind of filter is this? It's sort of a bandpass filter, attenuating both high and low frequencies [with normalized frequencies, remember that stuff around $\omega = 0$ can be thought of as low frequencies, and stuff around $\omega = \pi$ can be thought of as high frequencies]. You can actually implement this filter on a computer: take the transfer function, set $H(z) = \frac{Y(z)}{X(z)}$, cross-multiply, and inverse transform to get an LDE [linear difference equation]:

$$\begin{aligned} H(z) &= \frac{1}{z^4}[z^4 + (1-p^2)z^2 - p^2] \\ &= 1 + (1-p^2)z^{-2} - p^2z^{-4} \\ \frac{Y(z)}{X(z)} &= 1 + (1-p^2)z^{-2} - p^2z^{-4} \\ Y(z) &= X(z) + (1-p^2)z^{-2}X(z) - p^2z^{-4}X(z) \\ y[n] &= x[n] + (1-p^2)x[n-2] - p^2x[n-4] \end{aligned}$$

Note that the LDE only has one $y[n]$ term. This means that there is no feedback in an implementation of this system using just adders and delay blocks. FIR filters do not utilize feedback [infinite impulse response (IIR) filters do].