EECS120: Signals and Systems Practice Final
The real final will obviously not be
a copy of this, but it will also have more overlap with the previous two midterms.

Some useful information for you to use:

Trig	$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$	$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$
CTFT	$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$	$x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} X(\omega)e^{+j\omega t}d\omega$
CTFS	$X_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi k}{T}t}dt$	$x(t) = \displaystyle\sum_{k=-\infty}^{+\infty} X_k e^{+j\frac{2\pi k}{T}t}$
DTFT	$X(\omega) = \sum_{t=-\infty}^{+\infty} x[t]e^{-j\omega t}$	$x[t] = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} X(\omega)e^{+j\omega t}d\omega$
DFT	$X_k = \sum_{k=0}^{T-1} x[t]e^{-j\frac{2\pi k}{T}t}$	$x[t] = \dfrac{1}{T}\displaystyle\sum_{k=0}^{T-1} X_k e^{+j\frac{2\pi k}{T}t}$

$$\text{Z-Transform } X(z) = \sum_{t=-\infty}^{+\infty} x[t]z^{-t}$$

$$\text{Laplace-Transform } X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

If $X(\omega) = 1$ for $\omega \in [-\frac{\pi}{T}, +\frac{\pi}{T}]$ and 0 otherwise, then $x(t) = \frac{1}{T}\operatorname{sinc}(\frac{t}{T})$
If $x(t) = 1$ for $t \in [-\frac{T}{2}, +\frac{T}{2}]$ and 0 otherwise, then $X(\omega) = T\operatorname{sinc}(\omega\frac{T}{2\pi})$.
If $y(t) = e^{j\omega_0 t}x(t)$, then $Y(\omega) = X(\omega - \omega_0)$.
If $x(t) = \sum_{k=-\infty}^{\infty}\delta(t - kT)$, then $X(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\frac{2\pi}{T})$.
If $y(t) = e^{-at}x(t)$, then $Y(s) = X(s+a)$ with the RoC of Y given by translating the RoC of X by a in the complex plane as well.

If $X(s) = \frac{1}{s}$, then either $x(t) = u(t)$ with RoC $\{s|\text{Re}(s) > 0\}$ or $x(t) = -u(-t)$ with RoC $\{s|\text{Re}(s) < 0\}$.

For $n > 1$, if $X(s) = \frac{1}{s^n}$, then either $x(t) = \frac{t^{n-1}}{(n-1)!}u(t)$ with RoC $\{s|\text{Re}(s) > 0\}$ or $x(t) = \frac{-t^{n-1}}{(n-1)!}u(-t)$ with RoC $\{s|\text{Re}(s) < 0\}$.

If $y[t] = a^{-t}x[t]$, then $Y(z) = X(az)$ with the RoC of Y given by scaling the RoC of X by a in the complex plane as well.

If $X(z) = \frac{1}{z-1}$, then either $x[t] = u[t-1]$ with RoC $\{z||z| > 1\}$ or $x[t] = -u[-t]$ with RoC $\{z||z| < 1\}$.

For $n > 1$, if $X(z) = \frac{1}{(z-a)^n}$, then either $x[t] = \frac{a^{t-n}u[t-n]}{(n-1)!}\prod_{k=1}^{n-1}(t-k)$ with RoC $\{z||z| > |a|\}$ or $x[t] = \frac{(-1)^n a^{t-n}u[-t]}{(n-1)!}\prod_{k=1}^{n-1}(k-t)$ with RoC $\{z||z| < |a|\}$.

Problem 3.1 *Modeling and Feedback Control*

Justify your answers for full credit: use any combination of time-domain and transform-domain reasoning to show that your answer is correct.

a. 10pts *You are given a physical system that is modeled by the following differential equation:*

$$\dot{y} = x$$

where x represents the input signal and y is the output signal.

What is the transfer function of the system?

b. 20pts *Given the system in part (a), please draw a pole-zero diagram illustrating the locations of any poles and/or zeros in the s-plane. What are the possible RoC and impulse responses?*

c. 10pts *Assume now that the physical system is causal. Your goal is to have the output of the system approximately follow a slowly varying causally given signal $v(t)$ by designing a controller that you can place in feedback around the system. You are only allowed to use a single scalar gain K, an adder, and a subtracter.*

Draw a block diagram of the total system including the original physical system as a sub-block. The input is v and the output is y. What are the values for K for which the total system is BIBO stable?

d. 10pts *For the cases where the combined system is BIBO stable, please give the CTFT for combined system and roughly sketch the magnitude response. Is the combined system causal?*

Problem 3.2 *True/False. Do at least 2 of the following for full credit. Any more is extra credit.*

If the bold statement is true, give a proof for it. If the statement is false, show a counterexample or proof that it is false.

a. 20pts *Let $x(t)$ be a periodic continuous time signal that has period $1ms$ and is also bandlimited so that its CTFT $X(\omega) = 0$ for $|\omega| \geq 4000\pi$.* **It is possible to reconstruct all of $x(t)$ from a *finite* number of samples.**

b. 20pts *Let $h[t]$ be a discrete time FIR signal that is nonzero only for $t \in [K_1, K_2]$ where $K_1 < K_2$ are both integers.* **$H(z)$ exists and the RoC includes the unit circle.**

c. 20pts **If $X(s)$ is a rational function of s, then there exists at least one possible RoC so that $X(s)$ is the Laplace transform of a signal $x(t)$ that has a CTFT.**

Problem 3.3 *Zero-order hold*

Let $x(t)$ be a bandlimited real signal which has $X(\omega) = 0$ for $|\omega| \geq 4000\pi$.

a. *20pts* *Assume that your only access to $x(t)$ is through a discrete time signal $y[k] = x(kT_s)$ where T_s is given.*

Express the DTFT $Y(\omega)$ in terms of the CTFT $X(\omega)$ and illustrate the following cases by plots:

1. $T_s > \frac{1}{4000}$
2. $T_s = \frac{1}{4000}$
3. $T_s < \frac{1}{4000}$

b. *20pts* *Suppose $T_s = \frac{1}{8000}$. You now want to reconstruct $x(t)$ from $y[k]$, but all you have access to is a "zero-order-hold" DAC. (i.e. The output $\hat{x}(t) = y[k]$ for $t \in [(k - \frac{1}{2})T_h, (k + \frac{1}{2})T_h))$ The "hold time" T_h in this part is set to be the same as T_s.*

Express the CTFT $\hat{X}(\omega)$ in terms of the CTFT $X(\omega)$ and illustrate with a plot.

c. *20pts* *Suppose that the discrete time signal $y[k]$ was given as in part (b), but you were allowed additional blocks to help you get a better reconstruction of $x(t)$.*

You are now allowed to adjust the hold time T_h as long as $T_h \geq \frac{1}{64000}$ and do any discrete time processing that you would like (e.g. $z[k] = \frac{y[\lfloor \frac{k}{2} \rfloor] + y[\lceil \frac{k}{2} \rceil]}{2}$ would be linearly interpolating between points to double the rate.) before sending the processed discrete time signal $z[k]$ into the zero-order hold.

Finally, you are permitted to apply a simple averaging continuous-time LTI filter to the output of the zero-order hold. This must have an impulse response of the form $h(t) = \frac{1}{T_a}$ for $t \in [-\frac{1}{2T_a}, +\frac{1}{2T_a}]$ and zero outside of that interval. You can adjust T_a as you like.

Use this to give an algorithm for a better reconstruction $\hat{x}(t)$ than you were able to get in part (b) and use time-domain and/or frequency domain arguments (with illustrative plots if desired) to argue why it is better and by how much.