EECS 120 Fall 2004

Demo 13 Solutions

(1) Problem 10.45 of OWN:

(a) \( H(z) = \frac{(z^{-1})^2}{1 - \frac{z^{-1}}{2}} \). We can re-write this as \( H(z) = \frac{(z-1)^2}{z(z-\frac{1}{2})} \). Here, the degree of the numerator is same as the degree of the denominator, and hence the partial fraction expansion of this will look like

\[
H(z) = A_1 + \frac{A_2}{z} + \frac{A_3}{z - \frac{1}{2}}
\]

for some constants \( A_i, \ i = 1, 2, 3 \). The unit sample response is the inverse Z-transform of \( H(z) \). From the form of the partial fraction expansion, we can see that the inverse Z-transform (with respect to the ROC: \( |z| > \frac{1}{2} \)) of \( H(z) \) will be zero for \( n < 0 \), since the last two terms in the partial fraction expansion will be exponentials multiplied by \( u[n-1] \) and the constant \( A_1 \) will give \( A_1 \delta[n] \). Therefore the unit sample response will be 0 for \( n < 0 \).

(b) \( H(z) = \frac{(z-1)^2}{z^{-\frac{1}{2}}} \). Since the degree of the numerator is greater than the degree of the denominator by 1, the partial fraction expansion of \( H(z) \) will look like

\[
H(z) = A_1 z + A_2 + \frac{A_3}{z - \frac{1}{2}}
\]

for some constants \( A_i, \ i = 1, 2, 3 \) such that \( A_1 \) is non-zero. In this case, for any possible ROC, the term \( A_1 z \) will have an inverse Z-transform of \( \delta(n+1) \) which is non-zero for \( n = -1 \). Thus, this transfer function doesn’t satisfy the conditions of the problem.

(c) \( H(z) = \frac{(z-\frac{1}{2})^6}{(z-\frac{1}{2})^6} \). In this case, the transfer function is a proper rational function, i.e., the degree of the numerator is less than the degree of the denominator. Therefore, if we choose the ROC to be the region outside the outermost pole, then the inverse Z-transform will be a sum of exponentials multiplied by \( u[n] \) and hence the unit sample response will be causal.

(d) \( H(z) = \frac{(z-\frac{1}{2})^6}{(z-\frac{1}{2})^6} \). Again in this case, as in part (b), the degree of the numerator is larger than the degree of the denominator. Hence the partial fraction expansion will have a term of the following form: \( Az \), for some non-zero constant \( A \). Therefore, the inverse Fourier transform for any possible
ROC will look like $\delta[n + 1]$, which is non-causal. Hence, the unit sample response will be non-zero for at least one $n < 0$.

(2) Problem 10.50 of OWN:

(a) 

$$|H(e^{j\omega})|^2 = K \frac{|e^{j\omega} - 1/a|^2}{|e^{j\omega} - a|^2}$$

$$= K \frac{(e^{j\omega} - 1/a)(e^{-j\omega} - 1/a)}{(e^{j\omega} - a)(e^{-j\omega} - a)}$$

$$= K \frac{1 - 1/a(e^{j\omega} + e^{-j\omega}) + 1/a^2}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2}$$

$$= K \frac{1/a^2[a^2 - a(e^{j\omega} + e^{-j\omega}) + 1]}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2}$$

$$= K/a^2$$

Here, $K$ is some positive constant. Thus, this shows that $|H(e^{j\omega})|$ is constant.

(b) From figure P10.50b, using the law of cosines we get

$$\|v_1\|^2 = 1 + a^2 - 2a \cos(\omega)$$

where $\|v_1\|^2$ is the length of the vector $v_1$.

(c) As in part (b) we can write,

$$\|v_2\|^2 = 1 + 1/a^2 - 21/a \cos(\omega)$$

$$= \frac{[a^2 + 1 - 2a \cos(\omega)]}{a^2}$$

$$= \frac{\|v_1\|^2}{a^2}$$

From the above equation we can see that $\|v_2\|^2$ is proportional to $\|v_1\|^2$ independently of $\omega$. Hence $|H(e^{j\omega})|$ must be a constant.

(3) Problem 10.59 of OWN:

(a) Let the signal at the middle node on the top of figure P10.59 be denoted by $r[n]$. By going around the two loops in the block diagram we can
write the following equations

\[ x[n] - \frac{k}{3} r[n - 1] = r[n] \]
\[ r[n] - \frac{k}{4} r[n - 1] = y[n] \]

Writing the above equations in \( Z \) domain, we get

\[ X(z) - \frac{k}{3} z^{-1} R(z) = R(z) \]
\[ R(z) - \frac{k}{4} z^{-1} R(z) = Y(z) \]

Eliminating the auxiliary variable \( R(z) \) from the above equations we get

\[ \frac{X(z)}{1 + \frac{k}{3} z^{-1}} = \frac{Y(z)}{1 - \frac{k}{4} z^{-1}} \]
\[ \Rightarrow H(z) = \frac{1 - \frac{k}{4} z^{-1}}{1 + \frac{k}{3} z^{-1}} \]

From the above equation we can infer that the system has a pole at \(-\frac{k}{3}\) and a zero at \(\frac{k}{4}\). Since, the system is given to be causal, the region of convergence must be \(|z| > |\frac{k}{3}|\). Fig. 1 sketches the pole-zero plot of \(H(z)\).

(b) For the system to be stable, the region of convergence must include the unit circle. Thus, the system is stable iff \(|\frac{k}{3}| < 1\), i.e., \(|k| < 3\).

(c) We know that the output of an LTI system corresponding to the input \(x[n] = z_0^n\) is given by \(y[n] = H(z_0)z_0^n\). Using this, we get that the output corresponding to the input \(x[n] = (\frac{2}{3})^n\) will be, \(y[n] = H(\frac{2}{3}) (\frac{2}{3})^n\). When \(k = 1\), \(H(\frac{2}{3}) = \frac{5}{12}\). Therefore, \(y[n] = \frac{5}{12} (\frac{2}{3})^n\).

(4a) Problem 10.63 of OWN:

(a) \(X(z) = \log(1 - 2z)\), \(|z| < \frac{1}{2}\). Using the power-series expansion given in the statement of the problem we get,

\[ X(z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i} \]
\[ = -\sum_{i=1}^{\infty} \frac{2^i (z^{-1})^{-i}}{i} \]
Figure 1: Pole-zero plot of $H(z)$ is sketched for $|k| < 3$. The dotted line represents the unit circle. The ROC as mentioned in the figure is outside the darker circle.
\[
\begin{align*}
&= - \sum_{i=-\infty}^{-1} \frac{2^{-i}(z^{-1})^i}{-i} \\
&= \sum_{i=-\infty}^{-1} \left( \frac{2^{-i}}{i} \right) z^{-i}
\end{align*}
\]

From the last equation, we can conclude that \(x[n]\)

\[
x[n] = \left( \frac{2^{-n}}{n} \right) u[-n - 1]
\]

(b) \(X(z) = \log(1 - \frac{1}{2}z^{-1}), \ |z| > \frac{1}{2}\). Again using the power-series expansion we get

\[
X(z) = - \sum_{i=1}^{\infty} \frac{(\frac{1}{2}z^{-1})^i}{i}
\]

\[
= \sum_{1}^{\infty} - \left( \frac{1}{2} \right)^i z^{-i}
\]

Again, by inspection we get

\[
x[n] = - \left( \frac{1}{2} \right)^n u[n - 1]
\]

(4b) Problem 10.64 of OWN:

(a) \(X(z) = \log(1 - 2z), \ |z| < \frac{1}{2}\). By differentiation we get

\[
\begin{align*}
\frac{dX(z)}{dz} &= - \frac{2}{1 - 2z}, \quad |z| < \frac{1}{2} \\
\Rightarrow -z \frac{dX(z)}{dz} &= \frac{2z}{1 - 2z}, \quad |z| < \frac{1}{2} \\
&= - \frac{1}{(1 - \frac{1}{2}z^{-1})}, \quad |z| < \frac{1}{2}
\end{align*}
\]

Taking the inverse Z-transform of the last equation, we get

\[
\begin{align*}
nx[n] &= \left( \frac{1}{2} \right)^n u[-n - 1] \\
\Rightarrow x[n] &= \frac{(\frac{1}{2})^n}{n} u[-n - 1]
\end{align*}
\]
which is the same as the answer in part (a) of problem (4a).

(b) \( X(z) = \log(1 - \frac{1}{2} z^{-1}), \ |z| > \frac{1}{2}. \) By differentiation we get

\[
\frac{dX(z)}{dz} = \frac{1}{2} z^{-2} \quad |z| > \frac{1}{2}
\]

\[
\Rightarrow -z \frac{dX(z)}{dz} = \frac{-1}{2} z^{-1} \quad |z| > \frac{1}{2}
\]

\[
\Rightarrow \mathcal{Z}\{nx[n]\} = \frac{-1}{2} z^{-1} \mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\}
\]

\[
\Rightarrow \mathcal{Z}\{nx[n]\} = \frac{-1}{2} \mathcal{Z}\left\{\left(\frac{1}{2}\right)^{n-1} u[n-1]\right\}
\]

\[
\Rightarrow nx[n] = \frac{-1}{2} \left(\frac{1}{2}\right)^{n-1} u[n-1]
\]

\[
\Rightarrow x[n] = -\frac{\left(\frac{1}{2}\right)^n}{n} u[n-1]
\]

which is same as the answer obtained in part (b) of problem (4a).