EECS 120 **Fall 2004**

Demo 4 Solutions

(1) Problem 3.40 of OWN:

(a)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$. Thus, we have

$$\begin{aligned}
x(t-t_0) + x(t+t_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-t_0)} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t+t_0)} \\
&= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{jk\omega_0 t_0} \\
&= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \left[e^{-jk\omega_0 t_0} + e^{jk\omega_0 t_0} \right] \\
&= \sum_{k=-\infty}^{\infty} 2a_k \cos(k\omega_0 t_0) e^{jk\omega_0 t}
\end{aligned}$$

Therefore the new Fourier coefficients are

$$\hat{a}_k = 2a_k \cos(k\omega_0 t_0)$$

(b)

$$\begin{aligned}
\mathfrak{Ev}\{x(t)\} &= \frac{x(t) + x(-t)}{2} \\
&= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t} \right] \\
&= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t} \right] \\
&= \sum_{k=-\infty}^{\infty} \left[\frac{a_k + a_{-k}}{2} \right] e^{jk\omega_0 t}
\end{aligned}$$

Hence the Fourier series coefficients of $\mathfrak{Ev}\{x(t)\}$ are given by

$$\hat{a}_k = \left[\frac{a_k + a_{-k}}{2}\right]$$

(c)

$$\begin{aligned}
\mathfrak{Re}\{x(t)\} &= \frac{x(t) + x^*(t)}{2} \\
&= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\right] \\
&= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}\right] \\
&= \sum_{k=-\infty}^{\infty} \left[\frac{a_k + a_{-k}^*}{2}\right] e^{jk\omega_0 t}
\end{aligned}$$

Hence the Fourier series coefficients of $\mathfrak{Re}\{x(t)\}$ are given by

$$\hat{a}_k = \left[\frac{a_k + a_{-k}^*}{2}\right]$$

(d)

$$\begin{aligned}
\frac{d^2x(t)}{dt^2} &= \sum_{k=-\infty}^{\infty} \frac{d^2(a_k e^{jk\omega_0 t})}{dt^2} \\
&= \sum_{k=-\infty}^{\infty} a_k (jk\omega_0)^2 e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} -a_k k^2 \omega_0^2 e^{jk\omega_0 t}
\end{aligned}$$

Hence the Fourier series coefficients of $\frac{d^2x(t)}{dt^2}$ are given by

$$\hat{a}_k = -a_k k^2 \omega_0^2$$

(e)

$$\begin{aligned} x(3t-1) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(3t-1)} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j3k\omega_0 t} e^{-jk\omega_0} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0} e^{jk(3\omega_0)t} \qquad (1) \end{aligned}$$

Since $x(t)$ has fundamental period T, $x(3t-1)$ has fundamental period $\frac{T}{3}$. Therefore, from (1) the Fourier coefficients of $x(3t-1)$ are

$$\hat{a}_k = a_k e^{-jk\omega_0}$$

(2) Problem 3.44 of OWN:

Since $x(t)$ is a real signal, $a_k = a^*_{-k}$. But from the given hypothesis, $a_k = 0$ for $k > 2$. This implies that $a_{-k} = a^*_k = 0$ for $k > 2$. Also, it is given that $a_0 = 0$. Therefore the only non-zero Fourier coefficients are a_1 , $a_{-1} = a^*_1$, a_2 and $a_{-2} = a^*_2$. It is also given that a_1 is a positive real number, therefore $a_{-1} = a_1$. Thus we have,

$$\begin{aligned} x(t) &= a_1 \left[e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t} \right] + a_2 e^{j\frac{4\pi}{T}t} + a^*_2 e^{-j\frac{4\pi}{T}t} \\ &= 2a_1 \cos\left(\frac{2\pi}{T}t\right) + a_2 e^{j\frac{4\pi}{T}t} + a^*_2 e^{-j\frac{4\pi}{T}t} \\ &= 2a_1 \cos\left(\frac{\pi}{3}t\right) + a_2 e^{j\frac{4\pi}{T}t} + a^*_2 e^{-j\frac{4\pi}{T}t} \end{aligned}$$

Since $e^{j\frac{4\pi}{T}t}$ and $e^{-j\frac{4\pi}{T}t}$ are both periodic with period 3, we have

$$x(t-3) = -2a_1 \cos\left(\frac{\pi}{3}t\right) + a_2 e^{j\frac{4\pi}{T}t} + a^*_2 e^{-j\frac{4\pi}{T}t}$$

But, by given hypothesis we have $x(t) = -x(t-3)$, which implies that

$$2[a_2 e^{j\frac{4\pi}{T}t} + a^*_2 e^{-j\frac{4\pi}{T}t}] = 0$$

Therefore we have,

$$x(t) = 2a_1 \cos\left(\frac{\pi}{3}t\right)$$

Finally, it is given that

$$\begin{aligned}
\frac{1}{6}\int_{-3}^{3}|x(t)|^2 dt &= \frac{1}{2} \\
\Rightarrow \frac{4}{6}\int_{-3}^{3} a_1^2 \cos^2\left(\frac{\pi}{3}t\right)dt &= \frac{1}{2} \\
\Rightarrow a_1 = \frac{1}{2}
\end{aligned}$$

Therefore, $x(t) = \cos\left(\frac{\pi}{3}t\right)$ and the constants $A = 1$, $B = \frac{\pi}{3}$ and $C = 0$.

(3) Problem 3.52 of OWN:

(a) The discrete time Fourier series representation of $x[n]$ is

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \tag{2}$$

Taking the complex conjugate of the above equation we get

$$\begin{aligned}
x^*[n] &= \sum_{k=0}^{N-1} a_k^* e^{-jk\frac{2\pi}{N}n} \\
&= \sum_{k=0}^{N-1} a_{k-N}^* e^{-jk\frac{2\pi}{N}n} \\
&= \sum_{k=-(N-1)}^{0} a_{-k-N}^* e^{jk\frac{2\pi}{N}n} \\
&= \sum_{k=1}^{N} a_{-k}^* e^{j(k-N)\frac{2\pi}{N}n} \\
&= \sum_{k=1}^{N} a_{-k}^* e^{jk\frac{2\pi}{N}n} \\
&= \sum_{k=0}^{N-1} a_{-k}^* e^{jk\frac{2\pi}{N}n} \qquad (3)
\end{aligned}$$

The second step can be justified from the fact that a_k is periodic with period N. The third equation in the chain comes from a change of variable from k to $-k$, the fourth step from the change of variable from k to $k+N$. Finally we use that fact that $a_0 = a_N$ to arrive at the last equation.

Since, $x[n]$ is real, $x^*[n] = x[n]$. Thus, equating (2), (3) and comparing the coefficients of $e^{jk\frac{2\pi}{N}n}$ we get

$$\begin{aligned} a_k^* &= a_{-k} \\ \Rightarrow b_k - jc_k &= b_{-k} + jc_{-k} \\ \Rightarrow b_k = b_{-k} &\quad ; \quad c_k = -c_{-k} \end{aligned}$$

(b) Now, suppose that N is even. Then since the Fourier series coefficients are periodic with period N, $a_{-N/2} = a_{N-N/2} = a_{N/2}$. From part (a), we also know that $a_{-N/2} = a_{N/2}^*$. Using both the above facts we can see that $a_{N/2} = a_{N/2}^*$, which implies that $a_{N/2}$ is real.

(c) Using the Fourier series expansion we can write $x[n]$ as

$$x[n] = \sum_{-\lceil \frac{N}{2} \rceil}^{\lfloor \frac{N}{2} \rfloor} a_k e^{jk\frac{2\pi}{N}n} \tag{4}$$

From part (a) we know that $a_k^* = a_{-k}$. Using the above relation in (4) we get

$$\begin{aligned} x[n] &= a_0 + \sum_{k=1}^{(N-1)/2} (a_k e^{jk\frac{2\pi}{N}n} + a_{-k} e^{-jk\frac{2\pi}{N}n}) \\ &= a_0 + \sum_{k=1}^{(N-1)/2} (a_k e^{jk\frac{2\pi}{N}n} + a_k^* e^{-jk\frac{2\pi}{N}n}) \\ &= a_0 + \sum_{k=1}^{(N-1)/2} 2\mathfrak{Re}(a_k e^{jk\frac{2\pi}{N}n}) \\ &= a_0 + 2\sum_{k=1}^{(N-1)/2} b_k \cos\left(\frac{2\pi kn}{N}\right) - c_k \sin\left(\frac{2\pi kn}{N}\right) \end{aligned} \tag{5}$$

if N is odd, and

$$\begin{aligned} x[n] &= a_0 + a_{N/2} e^{j\pi n} + \sum_{k=1}^{(N-2)/2} (a_k e^{jk\frac{2\pi}{N}n} + a_{-k} e^{-jk\frac{2\pi}{N}n}) \\ &= a_0 + a_{N/2}(-1)^n + \sum_{k=1}^{(N-2)/2} (a_k e^{jk\frac{2\pi}{N}n} + a_k^* e^{-jk\frac{2\pi}{N}n}) \end{aligned}$$

$$
\begin{aligned}
&= a_0 + a_{N/2}(-1)^n + \sum_{k=1}^{(N-2)/2} 2\mathfrak{Re}(a_k e^{jk\frac{2\pi}{N}n}) \\
&= a_0 + a_{N/2}(-1)^n + 2\sum_{k=1}^{(N-2)/2} b_k \cos\left(\frac{2\pi kn}{N}\right) - c_k \sin\left(\frac{2\pi kn}{N}\right) \quad (6)
\end{aligned}
$$

if N is even.

(d) Substituting $a_k = A_k e^{j\theta_k}$, in (5) and (6) we get,

$$
\begin{aligned}
x[n] &= a_0 + 2\sum_{k=1}^{(N-1)/2} A_k \cos(\theta_k)\cos\left(\frac{2\pi kn}{N}\right) + A_k \sin(\theta_k)\sin\left(\frac{2\pi kn}{N}\right) \\
&= a_0 + 2\sum_{k=1}^{(N-1)/2} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)
\end{aligned}
$$

if N is odd and,

$$
\begin{aligned}
x[n] &= a_0 + a_{N/2}(-1)^n + 2\sum_{k=1}^{(N-2)/2} A_k \cos(\theta_k)\cos\left(\frac{2\pi kn}{N}\right) + A_k \sin(\theta_k)\sin\left(\frac{2\pi kn}{N}\right) \\
&= a_0 + a_{N/2}(-1)^n + 2\sum_{k=1}^{(N-2)/2} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right)
\end{aligned}
$$

if N is even.

(e) Here, we can write $y[n]$ as

$$
\begin{aligned}
y[n] &= a_0 - d_0 + 2\sum_{k=1}^{3}\left\{d_k \cos\left(\frac{2\pi kn}{7}\right) + (f_k - c_k)\sin\left(\frac{2\pi kn}{7}\right)\right\} \\
&= a_0 - 2d_0 + \left\{d_0 + 2\sum_{k=1}^{3} d_k \cos\left(\frac{2\pi kn}{7}\right)\right\} + 2\sum_{k=1}^{3} f_k \sin\left(\frac{2\pi kn}{7}\right) - 2\sum_{k=1}^{3} c_k \sin\left(\frac{2\pi kn}{7}\right) \\
&= a_0 - 2d_0 + \mathbf{Ev}\{z[n]\} - \mathbf{Od}\{z[n]\} + \mathbf{Od}\{x[n]\} \\
&= a_0 - 2d_0 + z[-n] + \mathbf{Od}\{x[n]\}
\end{aligned}
$$

From the plot of the functions we can easily calculate a_0 and d_0, since they are the average of the function over one period. Therefore, $a_0 = 1$, and $d_0 = 1$. Also, we can calculate the even and odd parts of both $x[n]$ and $z[n]$ from its plot, and hence we have the following plot of $y[n]$.

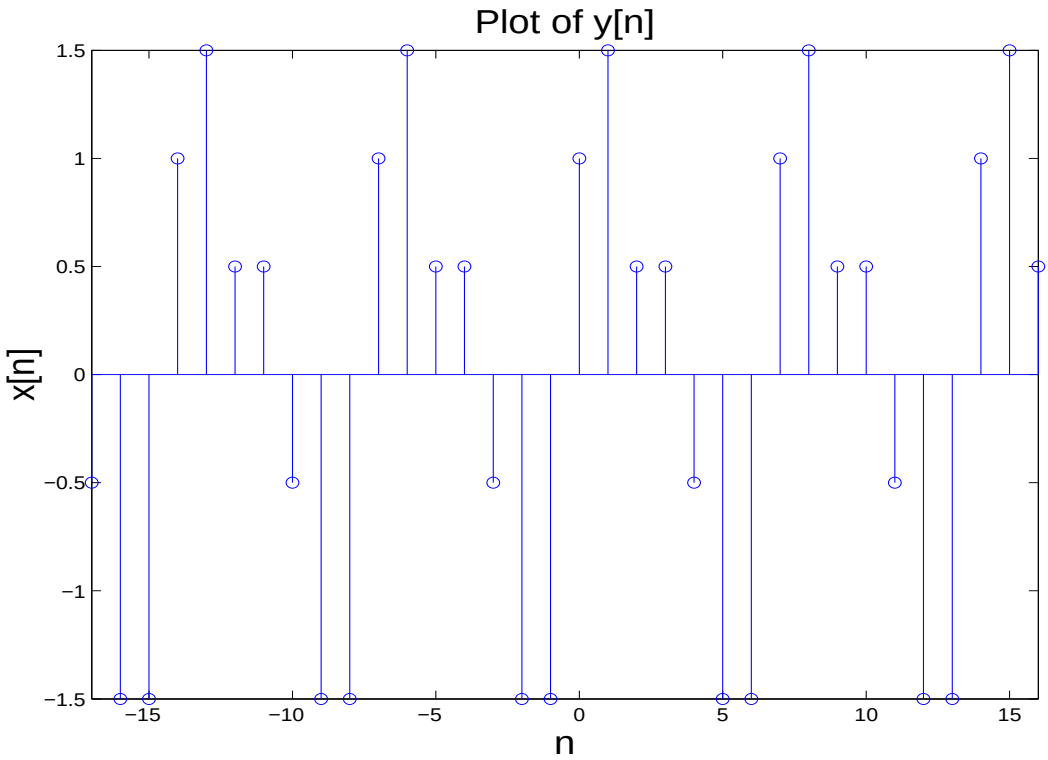

Figure 1: The stem plot of $y[n]$.

(4) Problem 3.65 of OWN:

(a-i)

$$\begin{aligned}\int_0^4 u(t)v^*(t)dt &= \int_0^1 1\cdot 1dt + \int_1^2 (-1)\cdot 1dt + \int_2^3 (-1)\cdot(-1)dt + \int_3^4 (1)\cdot(-1)dt \\ &= 1-1+1-1 \\ &= 0\end{aligned}$$

Hence the above signals are orthogonal.

(a-ii)

$$\begin{aligned}\int_0^4 u(t)v^*(t)dt &= \int_0^1 (-3e^{-t})(-3e^{-t})dt + \int_1^2 (3e^{-(t-1)})(-3e^{-(t-1)})dt \\ &+ \int_2^3 (3e^{-(t-2)})(3e^{-(t-2)})dt + \int_3^4 (3e^{-(t-3)})(-3e^{-(t-3)})dt \\ &= \int_0^1 (-3e^{-t})(-3e^{-t})dt + \int_0^1 (3e^{-t})(-3e^{-t})dt\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 (3e^{-t})(3e^{-t})dt + \int_0^1 (3e^{-t})(-3e^{-t})dt \\
&= 0 \qquad (7)
\end{aligned}
$$

Thus in this case also the two signals are orthogonal.

(a-iii)

$$
\begin{aligned}
\int_0^4 u(t)v^*(t)dt &= \int_0^4 \sin\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi t}{2}+\frac{\pi}{4}\right)dt \\
&= \frac{1}{2}\int_0^4 \left[\cos\left(\frac{\pi}{4}\right) - \cos\left(\pi t + \frac{\pi}{4}\right)\right] dt \\
&= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\int_0^4 dt - 0 \\
&= \sqrt{2} > 0
\end{aligned}
$$

Therefore, the two signals are not orthogonal.

(a-iv)

$$
\begin{aligned}
\int_0^4 u(t)v^*(t)dt &= \int_1^2 \pi\cdot\pi dt \\
&= \pi^2 > 0
\end{aligned}
$$

Therefore in this case also the signals are not orthogonal.

(b) Let $u(t) = \sin m\omega_0 t$, and $v(t) = \sin n\omega_o t$. We have,

$$
\begin{aligned}
\int_0^T u(t)v^*(t)dt &= \int_0^T \sin(m\omega_0 t)\sin(n\omega_0 t)dt \\
&= \frac{1}{2}\left[\int_0^T \cos(m-n)\omega_0 t dt - \int_0^T \cos(m+n)\omega_0 t\right] dt
\end{aligned}
$$

If $m = n$, then the first integral is $\frac{T}{2}$ and the second integral is 0. Hence in this case, the two signals are not orthogonal. When $m \neq n$, then both $\cos(m-n)\omega_0 t$ and $\cos(m+n)\omega_0 t$ are periodic with period T and hence both the integrals are 0. Therefore, in this case the two signals are orthogonal. Further

$$
\begin{aligned}
\int_0^T \sin^2 m\omega_0 t dt &= \frac{1}{2}\int_0^T [1-\cos 2\omega_0 t]dt \\
&= T/2
\end{aligned}
$$

Thus, the two signals are not orthonormal unless $T = 2$.

(c) Consider the two signals,

$$\begin{aligned}
\phi_n(t) &= \frac{1}{T}[\cos n\omega_0 t + \sin n\omega_0 t] \\
\phi_m(t) &= \frac{1}{T}[\cos m\omega_0 t + \sin m\omega_0 t]
\end{aligned}$$

As, in the previous part if $m = n$ then the $\phi_m(t) = \phi_n(t)$ and hence the signals are not orthogonal. Now, suppose $m \neq n$. In this case,

$$\begin{aligned}
\int_0^T \phi_m(t)\phi_n^*(t)dt &= \int_0^T \frac{1}{T^2}[\cos m\omega_0 t + \sin m\omega_0 t][\cos n\omega_0 t + \sin n\omega_0 t]dt \\
&= \frac{1}{T^2}\left[\int_0^T \cos m\omega_0 t \cos n\omega_0 t dt + \int_0^T \cos m\omega_0 t \sin n\omega_0 t dt\right] \\
&+ \frac{1}{T^2}\left[\int_0^T \sin m\omega_0 t \cos n\omega_0 t dt + \int_0^T \sin m\omega_0 t \sin n\omega_0 t dt\right] \\
&= 0
\end{aligned}$$

since each of the integrals are 0. Also,

$$\begin{aligned}
\int_0^T \phi_n^2(t)dt &= \frac{1}{T^2}\int_0^T [\cos n\omega_0 t + \sin n\omega_0 t]^2 dt \\
&= \frac{1}{T^2}\int_0^T [1 + \sin 2n\omega_o t]dt \\
&= \frac{1}{T}
\end{aligned}$$

Therefore the signals $\phi_n(t)$, $\phi_m(t)$ are orthonormal iff $T = 1$.

(d) Let $\phi_m(t) = e^{jm\omega_0 t}$ and $\phi_n(t) = e^{jn\omega_0 t}$. We now show that these functions are orthogonal on any interval of length $T = \frac{2\pi}{\omega_0}$ if $m \neq n$. We have,

$$\begin{aligned}
\int_\alpha^{\alpha+T} \phi_m(t)\phi_n^*(t) &= \int_\alpha^{\alpha+T} e^{jm\omega_0 t}e^{-jn\omega_0 t}dt \\
&= \int_\alpha^{\alpha+T} e^{j(m-n)\omega_0 t}dt \\
&= \frac{e^{j(m-n)\omega_0 t}|_\alpha^{\alpha+T}}{(m-n)\omega_0}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left[e^{j(m-n)\omega_0(\alpha+T)} - e^{j(m-n)\omega_0\alpha}\right]}{(m-n)\omega_0} \\
&= \frac{e^{j(m-n)\omega_0\alpha}\left[e^{j(m-n)\omega_0 T} - 1\right]}{(m-n)\omega_0} \\
&= \frac{e^{j(m-n)\omega_0\alpha}\left[e^{j(m-n)2\pi} - 1\right]}{(m-n)\omega_0} \\
&= 0 \qquad (8)
\end{aligned}
$$

Now consider the case when $m = n$. In this case also we evaluate the integral

$$
\begin{aligned}
\int_{\alpha}^{\alpha+T} \phi_m(t)\phi_n^*(t) &= \int_{\alpha}^{\alpha+T} e^{jm\omega_0 t} e^{-jm\omega_0 t} dt \\
&= \int_{\alpha}^{\alpha+T} 1 dt \\
&= T
\end{aligned}
$$

Thus the function $\phi_k(t)$ are orthogonal but not orthonormal, except for the case when $T = 1$.

(e)

$$
\begin{aligned}
\int_{-T}^{T} x_0(t)x_e^*(t)dt &= \int_{-T}^{T} \left[\frac{x(t)+x(-t)}{2}\right]\left[\frac{x(t)-x(-t)}{2}\right]^* dt \\
&= \frac{1}{4}\left[\int_{-T}^{T} x(t)x^*(t) - x(-t)x^*(-t)dt + \int_{-T}^{T} x(-t)x^*(t) - x(t)x^*(-t)dt\right] \\
&= 0
\end{aligned}
$$

The last step follows from the fact that $\int_{-T}^{T} x(t)x^*(t)dt = \int_{-T}^{T} x(-t)x^*(-t)dt$ and $\int_{-T}^{T} x(-t)x^*(t)dt = \int_{-T}^{T} x(t)x^*(-t)dt$ which can be obtained by changing the variable of integration from t to $-t$. Hence the odd and even parts of $x(t)$ are orthogonal over the interval $(-T, T)$.

(f) Let $\phi_k(t)$ be an orthogonal set of signals over the interval (a, b). Now, consider the new set of signals $\frac{1}{\sqrt{A_k}}\phi_k(t)$, where $A_k = \int_a^b |\phi_k(t)|^2 dt$. For $m \neq n$,

$$
\int_a^b \frac{1}{\sqrt{A_m}}\phi_m(t)\frac{1}{\sqrt{A_n}^*}\phi_n^*(t)dt = \frac{1}{\sqrt{A_m}}\frac{1}{\sqrt{A_n}^*}\int_a^b \phi_m(t)\phi_n^*(t) \quad = \quad 0
$$

since $\phi_m(t)$ and $\phi_n(t)$ are orthogonal to start with. Now, for $m = n$, we have

$$
\int_a^b \frac{1}{\sqrt{A_m}}\phi_m(t)(\frac{1}{\sqrt{A_m}})^*\phi_m^*(t)dt \quad = \quad \frac{1}{\sqrt{A_m}}(\frac{1}{\sqrt{A_m}})^*\int_a^b \phi_m(t)\phi_m^*(t)
$$

$$
\begin{aligned}
&= \left|\frac{1}{\sqrt{A_m}}\right|^2 \int_a^b |\phi_m(t)|^2 \, dt \\
&= \frac{1}{A_m}\dot{A}_m \\
&= 1
\end{aligned}
$$

Thus, the new set of signals are orthonormal.

(g)

$$
\begin{aligned}
\int_a^b |x(t)|^2 dt &= \int_a^b x(t)x^*(t) \\
&= \int_a^b \left[\sum_i a_i\phi_i(t)\right]\left[\sum_j a_j\phi_j(t)\right]^* dt \\
&= \int_a^b \left[\sum_i a_i\phi_i(t)\right]\left[\sum_j a_j^*\phi_j^*(t)\right] dt \\
&= \int_a^b \left[\sum_{i\neq j} a_i a_j^*\phi_i(t)\phi_j^* + \sum_{i=j} a_i a_j^*\phi_i(t)\phi_j^*\right] dt \\
&= \sum_{i\neq j} a_i a_j^* \int_a^b \phi_i(t)\phi_j^* dt + \sum_i a_i a_i^* \int_a^b \phi_i(t)\phi_i^* dt \\
&= 0 + \sum_i |a_i|^2 \int_a^b |\phi_i^*(t)|^2 dt \\
&= \sum_i |a_i|^2
\end{aligned}
$$

Hence proved.

(h) $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are nonzero only in the time interval $0`t \leq T$ and are orthonormal over this interval. The output at time T when $\phi_j(t)$ is applied to a LTI system with system response $h_i(t)$ is given by

$$
\begin{aligned}
y(T) &= \int_{-\infty}^{\infty} \phi_j(\tau)h_i(T-\tau)d\tau \\
&= \int_0^T \phi_j(\tau)\phi_i(T-(T-\tau))d\tau \\
&= \int_0^T \phi_j(\tau)\phi_i(\tau)d\tau \\
&= \delta(i,j)
\end{aligned}
$$

where $\delta(i,j) = 1$ iff $i = j$. The last step follows from the fact that $\phi_i(t)$ and $\phi_j(t)$ are orthogonal in the time interval $[0, T]$.