Demo 4 Solutions

(1) Problem 3.40 of OWN:

(a) \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

where \( \omega_0 = \frac{2\pi}{T} \). Thus, we have

\[
x(t - t_0) + x(t + t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t+t_0)}
\]

\[
= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{jk\omega_0 t_0}
\]

\[
= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \left[ e^{-jk\omega_0 t_0} + e^{jk\omega_0 t_0} \right]
\]

\[
= \sum_{k=-\infty}^{\infty} 2a_k \cos (k\omega_0 t_0) e^{jk\omega_0 t}
\]

Therefore the new Fourier coefficients are

\[
\hat{a}_k = 2a_k \cos (k\omega_0 t_0)
\]

(b) \[
\mathcal{E}v\{x(t)\} = \frac{x(t) + x(-t)}{2}
\]

\[
= \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t} \right]
\]

\[
= \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t} \right]
\]

\[
= \sum_{k=-\infty}^{\infty} \left[ \frac{a_k + a_{-k}}{2} \right] e^{jk\omega_0 t}
\]
Hence the Fourier series coefficients of \( \mathcal{E} \{ x(t) \} \) are given by

\[
\hat{a}_k = \left[ \frac{a_k + a_{-k}}{2} \right]
\]

(c)

\[
\mathcal{R} \{ x(t) \} = \frac{x(t) + x^*(t)}{2}
\]

\[
= \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} a_k e^{j \omega_0 t} + \sum_{k=-\infty}^{\infty} a_k^* e^{-j \omega_0 t} \right]
\]

\[
= \frac{1}{2} \left[ \sum_{k=-\infty}^{\infty} a_k e^{j \omega_0 t} + \sum_{k=-\infty}^{\infty} a_k^* e^{j \omega_0 t} \right]
\]

\[
= \sum_{k=-\infty}^{\infty} \left[ \frac{a_k + a_k^*}{2} \right] e^{j \omega_0 t}
\]

Hence the Fourier series coefficients of \( \mathcal{R} \{ x(t) \} \) are given by

\[
\hat{a}_k = \left[ \frac{a_k + a_{-k}}{2} \right]
\]

(d)

\[
\frac{d^2 x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} \frac{d^2 (a_k e^{j \omega_0 t})}{dt^2}
\]

\[
= \sum_{k=-\infty}^{\infty} a_k (j k \omega_0)^2 e^{j \omega_0 t}
\]

\[
= \sum_{k=-\infty}^{\infty} -a_k k^2 \omega_0^2 e^{j \omega_0 t}
\]

Hence the Fourier series coefficients of \( \frac{d^2 x(t)}{dt^2} \) are given by

\[
\hat{a}_k = -a_k k^2 \omega_0^2
\]

2
(e) \[ x(3t - 1) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (3t - 1)} \]
\[ = \sum_{k=-\infty}^{\infty} a_k e^{j3k\omega_0 t} e^{-jk\omega_0} \]
\[ = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0} e^{jk(3\omega_0)t} \]

Since \( x(t) \) has fundamental period \( T \), \( x(3t - 1) \) has fundamental period \( \frac{T}{3} \). Therefore, from (1) the Fourier coefficients of \( x(3t - 1) \) are
\[ \hat{a}_k = a_k e^{-jk\omega_0} \]

(2) Problem 3.44 of OWN:

Since \( x(t) \) is a real signal, \( a_k = a_k^* \). But from the given hypothesis, \( a_k = 0 \) for \( k > 2 \). This implies that \( a_{-k} = a_k^* = 0 \) for \( k > 2 \). Also, it is given that \( a_0 = 0 \). Therefore the only non-zero Fourier coefficients are \( a_1 \), \( a_{-1} = a_1^* \), \( a_2 \) and \( a_{-2} = a_2^* \). It is also given that \( a_1 \) is a positive real number, therefore \( a_{-1} = a_1 \). Thus we have,
\[ x(t) = a_1 \left[ e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t} \right] + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t} \]
\[ = 2a_1 \cos \left( \frac{2\pi}{T}t \right) + a_2 e^{j\frac{4\pi}{3}t} + a_2^* e^{-j\frac{4\pi}{3}t} \]
\[ = 2a_1 \cos \left( \frac{\pi}{3}t \right) + a_2 e^{j\frac{4\pi}{3}t} + a_2^* e^{-j\frac{4\pi}{3}t} \]

Since \( e^{j\frac{4\pi}{3}t} \) and \( e^{-j\frac{4\pi}{3}t} \) are both periodic with period 3, we have
\[ x(t - 3) = -2a_1 \cos \left( \frac{\pi}{3}t \right) + a_2 e^{j\frac{4\pi}{3}t} + a_2^* e^{-j\frac{4\pi}{3}t} \]

But, by given hypothesis we have \( x(t) = -x(t - 3) \), which implies that
\[ 2[a_2 e^{j\frac{4\pi}{3}t} + a_2^* e^{-j\frac{4\pi}{3}t}] = 0 \]

Therefore we have,
\[ x(t) = 2a_1 \cos \left( \frac{\pi}{3}t \right) \]
Finally, it is given that

\[
\frac{1}{6} \int_{-3}^{3} |x(t)|^2 dt = \frac{1}{2}
\]

\[
\Rightarrow \frac{4}{6} \int_{-3}^{3} a_1^2 \cos^2 \left( \frac{\pi}{3} t \right) dt = \frac{1}{2}
\]

\[
\Rightarrow a_1 = \frac{1}{2}
\]

Therefore, \( x(t) = \cos \left( \frac{\pi}{3} t \right) \) and the constants \( A = 1, \ B = \frac{\pi}{3} \) and \( C = 0 \).

(3) Problem 3.52 of OWN:

(a) The discrete time Fourier series representation of \( x[n] \) is

\[
x[n] = \sum_{k=0}^{N-1} a_k e^{j \frac{2 \pi}{N} n}
\]

(2)

Taking the complex conjugate of the above equation we get

\[
x^*[n] = \sum_{k=0}^{N-1} a_k^* e^{-j \frac{2 \pi}{N} n}
\]

\[
= \sum_{k=0}^{N-1} a_{k-N}^* e^{-j \frac{2 \pi}{N} n}
\]

\[
= \sum_{k=-(N-1)}^{0} a_{k-N}^* e^{j \frac{2 \pi}{N} n}
\]

\[
= \sum_{k=1}^{N} a_{-k}^* e^{j(k-N) \frac{2 \pi}{N} n}
\]

\[
= \sum_{k=1}^{N} a_{-k}^* e^{j \frac{2 \pi}{N} n}
\]

\[
= \sum_{k=0}^{N-1} a_{-k}^* e^{j \frac{2 \pi}{N} n}
\]

(3)

The second step can be justified from the fact that \( a_k \) is periodic with period \( N \). The third equation in the chain comes from a change of variable from \( k \) to \( -k \), the fourth step from the change of variable from \( k \) to \( k + N \). Finally we use that fact that \( a_0 = a_N \) to arrive at the last equation.
Since, $x[n]$ is real, $x^*[n] = x[n]$. Thus, equating (2), (3) and comparing the coefficients of $e^{jk \frac{2\pi}{N} n}$ we get

\[ a_k^* = a_{-k} \]
\[ \Rightarrow b_k - j c_k = b_{-k} + j c_{-k} \]
\[ \Rightarrow b_k = b_{-k} ; c_k = -c_{-k} \]

(b) Now, suppose that $N$ is even. Then since the Fourier series coefficients are periodic with period $N$, $a_{-N/2} = a_{N-N/2} = a_{N/2}$. From part (a), we also know that $a_{-N/2} = a_{N/2}^*$. Using both the above facts we can see that $a_{N/2} = a_{N/2}^*$, which implies that $a_{N/2}$ is real.

(c) Using the Fourier series expansion we can write $x[n]$ as

\[ x[n] = \sum_{k=-\lceil \frac{N}{2} \rceil}^{\lceil \frac{N}{2} \rceil} a_k e^{jk \frac{2\pi}{N} n} \]  

(4)

From part (a) we know that $a_k^* = a_{-k}$. Using the above relation in (4) we get

\[ x[n] = a_0 + \sum_{k=1}^{(N-1)/2} (a_k e^{jk \frac{2\pi}{N} n} + a_{-k} e^{-jk \frac{2\pi}{N} n}) \]
\[ = a_0 + \sum_{k=1}^{(N-1)/2} (a_k e^{jk \frac{2\pi}{N} n} + a_k^* e^{-jk \frac{2\pi}{N} n}) \]
\[ = a_0 + \sum_{k=1}^{(N-1)/2} 2 \Re(a_k e^{jk \frac{2\pi}{N} n}) \]
\[ = a_0 + 2 \sum_{k=1}^{(N-1)/2} b_k \cos \left( \frac{2\pi k n}{N} \right) - c_k \sin \left( \frac{2\pi k n}{N} \right) \]  

(5)

if $N$ is odd, and

\[ x[n] = a_0 + a_{N/2} e^{j\pi n} + \sum_{k=1}^{(N-2)/2} (a_k e^{jk \frac{2\pi}{N} n} + a_{-k} e^{-jk \frac{2\pi}{N} n}) \]
\[ = a_0 + a_{N/2}(-1)^n + \sum_{k=1}^{(N-2)/2} (a_k e^{jk \frac{2\pi}{N} n} + a_k^* e^{-jk \frac{2\pi}{N} n}) \]
\[ x[n] = a_0 + a_{N/2}(-1)^n + \sum_{k=1}^{(N-2)/2} 2\Re(a_k e^{j\frac{2\pi}{N}n}) \]
\[ = a_0 + a_{N/2}(-1)^n + 2 \sum_{k=1}^{(N-2)/2} b_k \cos \left( \frac{2\pi kn}{N} \right) - c_k \sin \left( \frac{2\pi kn}{N} \right) \] (6)

if \( N \) is even.

(d) Substituting \( a_k = A_k e^{j\theta_k} \), in (5) and (6) we get,
\[ x[n] = a_0 + 2 \sum_{k=1}^{(N-1)/2} A_k \cos (\theta_k) \cos \left( \frac{2\pi kn}{N} \right) + A_k \sin (\theta_k) \sin \left( \frac{2\pi kn}{N} \right) \]
\[ = a_0 + 2 \sum_{k=1}^{(N-1)/2} A_k \cos \left( \frac{2\pi kn}{N} + \theta_k \right) \]

if \( N \) is odd and,
\[ x[n] = a_0 + a_{N/2}(-1)^n + 2 \sum_{k=1}^{(N-2)/2} A_k \cos (\theta_k) \cos \left( \frac{2\pi kn}{N} \right) + A_k \sin (\theta_k) \sin \left( \frac{2\pi kn}{N} \right) \]
\[ = a_0 + a_{N/2}(-1)^n + 2 \sum_{k=1}^{(N-2)/2} A_k \cos \left( \frac{2\pi kn}{N} + \theta_k \right) \]

if \( N \) is even.

(e) Here, we can write \( y[n] \) as
\[ y[n] = a_0 - d_0 + 2 \sum_{k=1}^3 \left\{ d_k \cos \left( \frac{2\pi kn}{7} \right) + (f_k - c_k) \sin \left( \frac{2\pi kn}{7} \right) \right\} \]
\[ = a_0 - 2d_0 + \left\{ d_0 + 2 \sum_{k=1}^3 d_k \cos \left( \frac{2\pi kn}{7} \right) \right\} + 2 \sum_{k=1}^3 f_k \sin \left( \frac{2\pi kn}{7} \right) - 2 \sum_{k=1}^3 c_k \sin \left( \frac{2\pi kn}{7} \right) \]
\[ = a_0 - 2d_0 + \text{Ev}\{z[n]\} - \text{Od}\{z[n]\} + \text{Od}\{x[n]\} \]
\[ = a_0 - 2d_0 + z[-n] + \text{Od}\{x[n]\} \]

From the plot of the functions we can easily calculate \( a_0 \) and \( d_0 \), since they are the average of the function over one period. Therefore, \( a_0 = 1 \), and \( d_0 = 1 \). Also, we can calculate the even and odd parts of both \( x[n] \) and \( z[n] \) from its plot, and hence we have the following plot of \( y[n] \).
Figure 1: The stem plot of $y[n]$. 

(4) Problem 3.65 of OWN:

(a-i)
$$\int_0^4 u(t)v^*(t)dt = \int_0^1 1 \cdot 1 dt + \int_1^2 (-1) \cdot 1 dt + \int_2^3 (-1) \cdot (-1) dt + \int_3^4 (1) \cdot (-1) dt$$
$$= 1 - 1 + 1 - 1$$
$$= 0$$

Hence the above signals are orthogonal.

(a-ii)
$$\int_0^4 u(t)v^*(t)dt = \int_0^1 (-3e^{-t})(-3e^{-t})dt + \int_1^2 (3e^{-(t-1)})(-3e^{-(t-1)})dt$$
$$+ \int_2^3 (3e^{-(t-2)})(3e^{-(t-2)})dt + \int_3^4 (3e^{-(t-3)})(-3e^{-(t-3)})dt$$
$$= \int_0^1 (-3e^{-t})(-3e^{-t})dt + \int_0^1 (3e^{-t})(-3e^{-t})dt$$
\[ + \int_0^1 (3e^{-t})(3e^{-t})dt + \int_0^1 (3e^{-t})(-3e^{-t})dt = 0 \] (7)

Thus in this case also the two signals are orthogonal.

(a-iii)
\[ \int_0^4 u(t)v^*(t)dt = \int_0^4 \sin \left( \frac{\pi t}{2} \right) \sin \left( \frac{\pi t}{2} + \frac{\pi}{4} \right) dt = \frac{1}{2} \int_0^4 \left[ \cos \left( \frac{\pi}{4} \right) - \cos \left( \pi t + \frac{\pi}{4} \right) \right] dt = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \int_0^4 dt = 0 \]
\[ \Rightarrow \sqrt{2} > 0 \]
Therefore, the two signals are not orthogonal.

(a-iv)
\[ \int_0^4 u(t)v^*(t)dt = \int_1^2 \pi \cdot \pi dt = \pi^2 > 0 \]
Therefore in this case also the signals are not orthogonal.

(b) Let \( u(t) = \sin m\omega_0 t \), and \( v(t) = \sin n\omega_0 t \). We have,
\[ \int_0^T u(t)v^*(t)dt = \int_0^T \sin (m\omega_0 t) \sin (n\omega_0 t)dt = \frac{1}{2} \left[ \int_0^T \cos (m-n)\omega_0 t dt - \int_0^T \cos (m+n)\omega_0 t dt \right] dt \]
If \( m = n \), then the first integral is \( \frac{T}{2} \) and the second integral is 0. Hence in this case, the two signals are not orthogonal. When \( m \neq n \), then both \( \cos (m-n)\omega_0 t \) and \( \cos (m+n)\omega_0 t \) are periodic with period \( T \) and hence both the integrals are 0. Therefore, in this case the two signals are orthogonal. Further
\[ \int_0^T \sin^2 m\omega_0 t dt = \frac{1}{2} \int_0^T [1 - \cos 2\omega_0 t]dt = \frac{T}{2} \]
Thus, the two signals are not orthonormal unless \( T = 2 \).

(c) Consider the two signals,

\[
\begin{align*}
\phi_n(t) &= \frac{1}{T} [\cos n\omega_0 t + \sin n\omega_0 t] \\
\phi_m(t) &= \frac{1}{T} [\cos m\omega_0 t + \sin m\omega_0 t]
\end{align*}
\]

As, in the previous part if \( m = n \) then the \( \phi_m(t) = \phi_n(t) \) and hence the signals are not orthogonal. Now, suppose \( m \neq n \). In this case,

\[
\begin{align*}
\int_0^T \phi_m(t)\phi_n^*(t) dt &= \int_0^T \frac{1}{T^2} [\cos m\omega_0 t + \sin m\omega_0 t][\cos n\omega_0 t + \sin n\omega_0 t] dt \\
&= \frac{1}{T^2} \left[ \int_0^T \cos m\omega_0 t \cos n\omega_0 t dt + \int_0^T \cos m\omega_0 t \sin n\omega_0 t dt \right] \\
&\quad + \frac{1}{T^2} \left[ \int_0^T \sin m\omega_0 t \cos n\omega_0 t dt + \int_0^T \sin m\omega_0 t \sin n\omega_0 t dt \right] \\
&= 0
\end{align*}
\]

since each of the integrals are 0. Also,

\[
\int_0^T \phi_n^2(t) dt = \frac{1}{T^2} \int_0^T [\cos n\omega_0 t + \sin n\omega_0 t]^2 dt \\
= \frac{1}{T^2} \int_0^T [1 + \sin 2n\omega_0 t] dt \\
= \frac{1}{T}
\]

Therefore the signals \( \phi_n(t), \phi_m(t) \) are orthonormal iff \( T = 1 \).

(d) Let \( \phi_m(t) = e^{jm\omega_0 t} \) and \( \phi_n(t) = e^{jn\omega_0 t} \). We now show that these functions are orthogonal on any interval of length \( T = \frac{2\pi}{\omega_0} \) if \( m \neq n \). We have,

\[
\begin{align*}
\int_\alpha^{\alpha+T} \phi_m(t)\phi_n^*(t) dt &= \int_\alpha^{\alpha+T} e^{jm\omega_0 t} e^{-jn\omega_0 t} dt \\
&= \int_\alpha^{\alpha+T} e^{j(m-n)\omega_0 t} dt \\
&= \frac{e^{j(m-n)\omega_0 T} - e^{j(m-n)\omega_0 \alpha}}{j(m-n)\omega_0}
\end{align*}
\]
\[
\begin{align*}
&= \frac{e^{j(m-n)\omega_0\alpha} - e^{j(m-n)\omega_0\alpha}}{(m-n)\omega_0} \\
&= \frac{e^{j(m-n)\omega_0\alpha\alpha} [e^{j(m-n)\omega_0\alpha_T} - 1]}{(m-n)\omega_0} \\
&= \frac{e^{j(m-n)\omega_0\alpha\alpha\alpha\alpha} [e^{j(m-n)2\pi} - 1]}{(m-n)\omega_0} \\
&= 0 \quad (8)
\end{align*}
\]

Now consider the case when \( m = n \). In this case also we evaluate the integral
\[
\int_{\alpha}^{\alpha+T} \phi_m(t)\phi_n^*(t) dt = \int_{\alpha}^{\alpha+T} e^{jm\omega_0t} e^{-jm\omega_0t} dt \\
= \int_{\alpha}^{\alpha+T} 1 dt \\
= T
\]

Thus the function \( \phi_k(t) \) are orthogonal but not orthonormal, except for the case when \( T = 1 \).

(e)
\[
\int_{-T}^{T} x_0(t)x^*_e(t)dt = \int_{-T}^{T} \left[ x(t) + x(-t) \right] \left[ \frac{x(t) - x(-t)}{2} \right]^* dt \\
= \frac{1}{4} \int_{-T}^{T} x(t)x^*(t) - x(-t)x^*(-t)dt + \int_{-T}^{T} x(-t)x^*(t) - x(t)x^*(-t)dt \\
= 0
\]

The last step follows from the fact that \( \int_{-T}^{T} x(t)x^*(t)dt = \int_{-T}^{T} x(-t)x^*(-t)dt \)
and \( \int_{-T}^{T} x(-t)x^*(t)dt = \int_{-T}^{T} x(t)x^*(-t)dt \) which can be obtained by changing the variable of integration from \( t \) to \( -t \). Hence the odd and even parts of \( x(t) \) are orthogonal over the interval \((-T, T)\).

(f) Let \( \phi_k(t) \) be an orthogonal set of signals over the interval \((a, b)\). Now, consider the new set of signals \( \frac{1}{\sqrt{A_k}} \phi_k(t) \), where \( A_k = \int_a^b |\phi_k(t)|^2 dt \). For \( m \neq n \),
\[
\int_a^b \frac{1}{\sqrt{A_m}}\phi_m(t)\frac{1}{\sqrt{A_n}}\phi_n^*(t)dt = \frac{1}{\sqrt{A_m}} \frac{1}{\sqrt{A_n}} \int_a^b \phi_m(t)\phi_n^*(t) = 0
\]
since \( \phi_m(t) \) and \( \phi_n(t) \) are orthogonal to start with. Now, for \( m = n \), we have
\[
\int_a^b \frac{1}{\sqrt{A_m}}\phi_m(t)(\frac{1}{\sqrt{A_m}})^*\phi_m^*(t)dt = \frac{1}{\sqrt{A_m}} (\frac{1}{\sqrt{A_m}})^* \int_a^b \phi_m(t)\phi_m^*(t)
\]
\[
\int_a^b |x(t)|^2 dt = \int_a^b x(t)x^*(t) dt
\]

Thus, the new set of signals are orthonormal.

\[
\phi_1(t), \phi_2(t), \ldots, \phi_N(t) \text{ are nonzero only in the time interval } 0 \leq t \leq T \text{ and are orthonormal over this interval. The output at time } T \text{ when } \phi_j(t) \text{ is applied to a LTI system with system response } h_i(t) \text{ is given by }
\]

\[
y(T) = \int_{-\infty}^T \phi_j(\tau) h_i(T - \tau) d\tau
\]

\[
= \int_0^T \phi_j(\tau) \phi_i(T - (T - \tau)) d\tau
\]

\[
= \int_0^T \phi_j(\tau) \phi_i(\tau) d\tau
\]

\[
= \delta(i, j)
\]

Hence proved.
where $\delta(i, j) = 1$ iff $i = j$. The last step follows from the fact that $\phi_i(t)$ and $\phi_j(t)$ are orthogonal in the time interval $[0, T]$. 