EECS 120 **Fall 2004**

Sample Midterm 2 Solutions

Problem 1:

(a) **False**. FM is a non-linear modulation scheme. For example, suppose $m_1(t) = m_2(t) = 1$ for all t, $a = 1$, and $b = -1$. Then $am_1(t) + bm_2(t) = 0$ for all t. The transmitted signal resulting from the modulating signal $am_1(t) + bm_2(t)$ is then $u(t) := A_c \cos(2\pi f_c t)$. However $au_1(t) + bu_2(t) = A_c \cos(2\pi f_c t + 2\pi k_f t) - A_c \cos(2\pi f_c t + 2\pi k_f t) = 0$, where k_f denotes the FM modulation index, and this does not, in general, equal $u(t)$.

(b) **True**. The sampled signal is given by $x_p(t) = x(t)p(t)$, where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ is the standard impulse train signal. This implies

$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi} \left[X(j\omega) * P(j\omega) \right] \\ &= \frac{1}{T} \left[\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_c)) \right] \end{aligned}$$

where $\omega_c = \frac{2\pi}{T}$. For $\omega_c = 3\omega_M$ the various copies of $X(j\omega)$ in $X_p(j\omega)$ don't overlap and hence there is no aliasing. Thus, we can reconstruct $x(t)$ by passing $x_p(t)$ through an ideal band pass filter given by,

$$H(jw) = T \qquad \text{for } 2\omega_M < |\omega| < 3\omega_M$$

and $H(j\omega) = 0$ otherwise.

Problem 2:

Let the DSB-SC modulated signal onto the carrier $A_c \cos(\omega_c t)$ by the message signal $m(t)$ be denote by $y(t)$. We can write it as,

$$\begin{aligned} y(t) &= m(t) A_c \cos(\omega_c t) \\ &= \left[\cos(\omega_0 t) - 0.7 \sin(2\omega_0 t) \right] A_c \cos(\omega_c t) \end{aligned}$$

Taking the Fourier transform of the above equation we get,

$$
\begin{aligned}
Y(j\omega) &= \frac{A_c}{2}\left[M(j(\omega-\omega_c)) + M(j(\omega+\omega_c))\right] \\
&= \frac{A_c}{2}[\pi\left(\delta(\omega-\omega_0-\omega_c) + \delta(\omega+\omega_0-\omega_c)\right) \\
&\quad -\frac{0.7\pi}{j}\left(\delta(\omega-2\omega_0-\omega_c) - \delta(\omega+2\omega_0-\omega_c)\right) \\
&\quad +\pi\left(\delta(\omega-\omega_0+\omega_c) + \delta(\omega+\omega_0+\omega_c)\right) \\
&\quad -\frac{0.7\pi}{j}\left(\delta(\omega-2\omega_0+\omega_c) - \delta(\omega+2\omega_0+\omega_c)\right)]
\end{aligned}
$$

Now, to obtain the USSB modulated signal from the DSB-SC modulated signal we must remove the lower side band, i.e., we remove all the terms with $|\omega| < \omega_c$ from $Y(j\omega)$. By doing this we get,

$$
\begin{aligned}
U(j\omega) &= \frac{A_c}{2}[\pi\left(\delta(\omega-\omega_0-\omega_c) + \delta(\omega+\omega_0+\omega_c)\right)] \\
&\quad -\frac{A_c}{2}[\frac{0.7\pi}{j}\left(\delta(\omega-2\omega_0-\omega_c) - \delta(\omega+2\omega_0+\omega_c)\right)]
\end{aligned}
$$

Taking the inverse Fourier transform of the previous equation we get

$$
u(t) = \frac{A_c}{2}\left[\cos(\omega_0+\omega_c) - 0.7\sin(2\omega_0+\omega_c)\right]
$$

Problem 3:

Let the DSB-SC modulated signal of the message signal $m(t)$ onto the carrier $A_c\cos(\omega_c t)$ be denoted by $y(t)$. Also, let the DSB-SC modulated signal of $y(t)$ onto the carrier $A_d\cos(\omega_d t)$ be denoted by $u(t)$. Then, we have

$$
y(t) = m(t)A_c\cos(\omega_c t)
$$

and

$$
\begin{aligned}
u(t) &= y(t)A_d\cos(\omega_d t) \\
&= m(t)A_cA_d\cos(\omega_c t)\cos(\omega_d t) \\
&= \frac{m(t)}{2}A_cA_d\left[\cos((\omega_d+\omega_c)t) + \cos((\omega_d-\omega_c)t)\right] \qquad (1)
\end{aligned}
$$

Now, it is given that $x(t)$ is the USSB amplitude modulated signal of $y(t)$ onto the carrier $A_d\cos(\omega_d t)$. Since, the USSB modulated signal is obtained from the DSB-SC modulated signal by removing the lower side band, we can easily see from (1) that

$$x(t) = \frac{m(t)}{2} A_c A_d \cos((\omega_d + \omega_c)t)$$

Hence, we have

$$X(j\omega) = \frac{A_c A_d}{4} \left[M(j(\omega - \omega_c - \omega_d)) + M(j(\omega + \omega_c + \omega_d))\right]$$

Refer, to Fig. 1 for the magnitude and phase plot of $X(j\omega)$. In this figure it is assumed that $A_c A_d = 1$

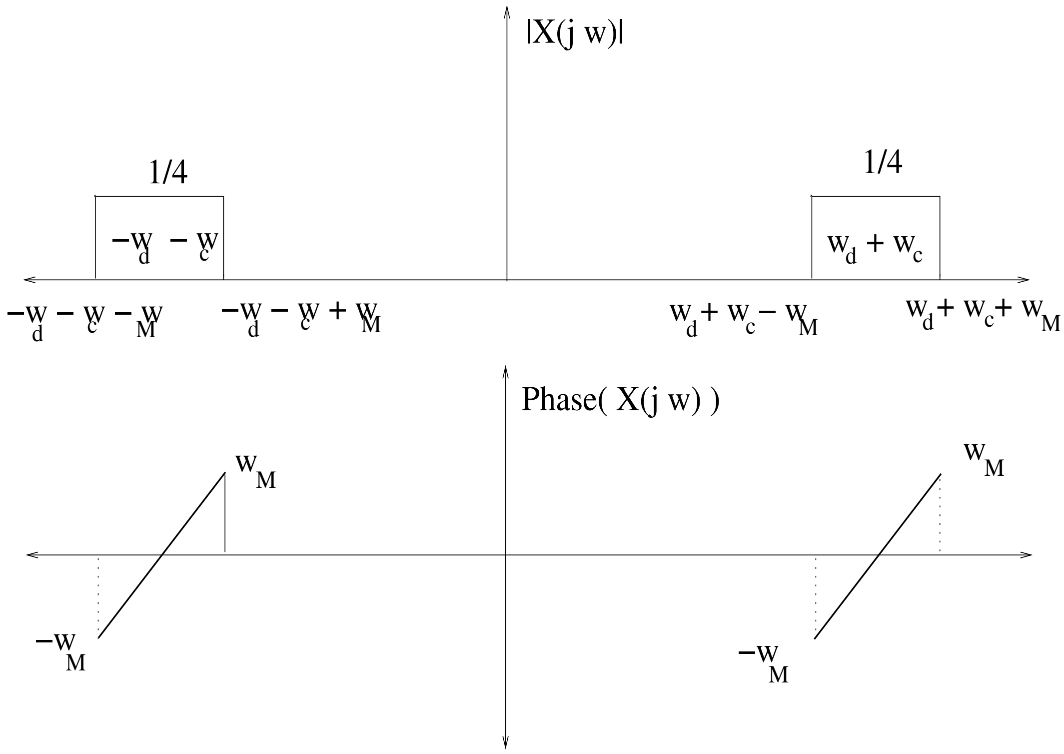

Figure 1: Magnitude and phase plot of $X(j\omega)$. Here the phase is linear in ω. One can also view this modulo 2π. In this plot, one can assume that $\omega_M < \pi$. In general depending on the value of ω_M, the phase periodically repeats.

Problem 4:

(a)

$$\begin{aligned} H(s) &= \frac{6(s-4)}{(s+3)(s^2-4s+13)} \\ &= \frac{6(s-4)}{(s+3)(s-2-3j)(s-2+3j)} \end{aligned}$$

Hence the poles of this system are at $s = -3, 2+3j, 2-3j$, and the zero of the system is at $s = 4$. Refer to Fig. 2 for the pole-zero plot of the system.

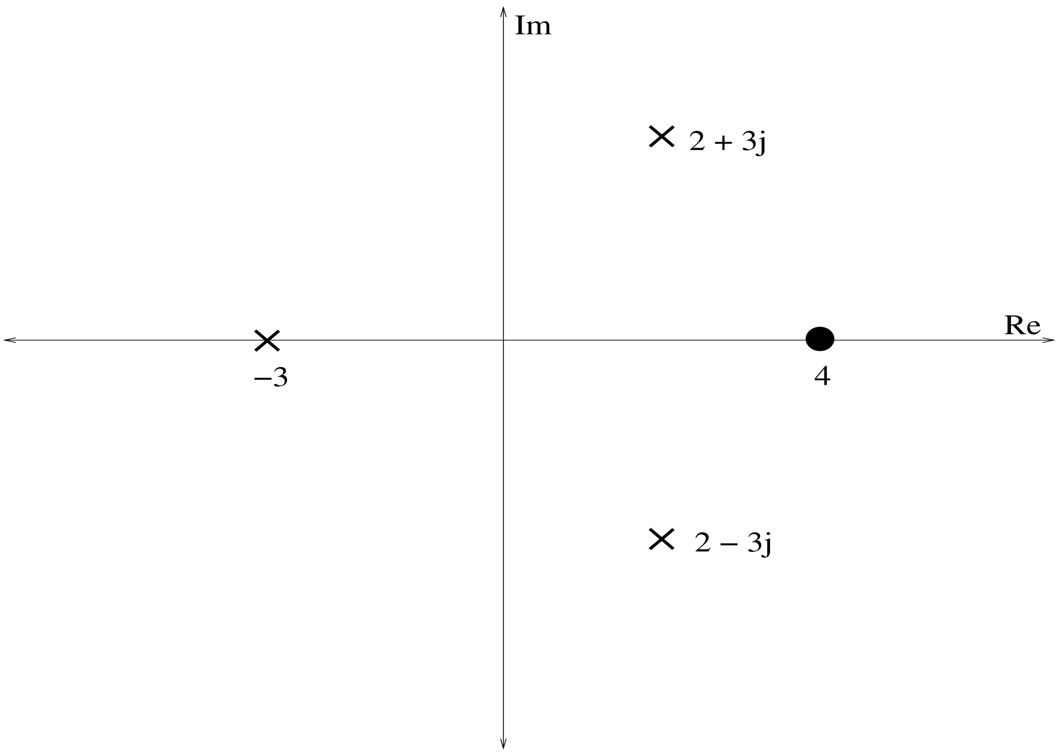

Figure 2: Pole-zero plot for the system given in 4(a)

(b) Using partial fraction expansion we can write $H(s)$ as

$$H(s) = \frac{A_1}{s+3} + \frac{A_2}{s-2-3j} + \frac{A_3}{s-2+3j}$$

where A_i are constants given by $A_1 = -\frac{21}{17}$, $A_2 = \frac{2-3j}{3-5j}$ and $A_3 = \frac{2+3j}{3+5j}$. Now, if the system is assumed to be causal then the impulse response will be given by

$$h(t) = \left[A_1 e^{-3t} + A_2 e^{(2+3j)t} + A_3 e^{(2-3j)t}\right] u(t)$$

(c) If the system is causal, then the ROC must be a right hand plane and hence must be $\Re e\{s\} > 2$. Since the ROC does not include the imaginary axis, the system is **not stable**.

(d) If the system is given to be stable, then the ROC must include the imaginary axis and hence must be $-3 < \Re e\{s\} < 2$. Therefore, taking the inverse Laplace transform of $H(s)$ for this ROC, we get

$$h(t) = A_1 e^{-3t} u(t) - \left[A_2 e^{(2+3j)t} + A_3 e^{(2-3j)t}\right] u(-t)$$

(e) If the system is stable, then the ROC must include the imaginary axis. The only possible ROC which satisfies this criteria for the given system function $H(s)$ is $-3 < \Re e\{s\} < 2$. Since this ROC is not a right hand plane, the system cannot be causal.

Problem 5:

The signal $v(t)$ can be written as

$$\begin{aligned} v(t) &= x(t)z(t) \\ &= x(t)[\cos(a\omega_M t) + y(t)] \\ &= x(t)\cos(a\omega_M t) + x(t)y(t) \end{aligned}$$

Since, both $x(t)$ and $y(t)$ are bandlimited to $(-\omega_M, \omega_M)$, the signal $x(t)y(t)$ is bandlimited to $(-2\omega_M, 2\omega_M)$. Similarly, the signal $x(t)\cos(a\omega_M t)$ is band limited to $(-\omega_M - a\omega_M, \omega_M + a\omega_M)$. Since it is given that $a > 1$, we have $\omega_M + a\omega_M > 2\omega_M$. Therefore the signal $v(t)$ is also band limited in $(-(1+a)\omega_M, (1+a)\omega_M)$. Now, if we were to reconstruct $x(t)$ and $y(t)$, we would be able to reconstruct $v(t)$ from its samples $v(nT)$. By the Nyquist sampling theorem, this is possible only if $\frac{2\pi}{T} > 2(1+a)\omega_M$, for all values of $a > 1$. Now, the remaining question to be answered is: When can we reconstruct, $x(t)$ and $y(t)$ from $v(t)$?

As observed before, $x(t)y(t)$ is bandlimited to $(-2\omega_M, 2\omega_M)$ and the signal $x(t)\cos(a\omega_M t)$ is band limited to

$$(-(a+1)\omega_M, -(a-1)\omega_M) \ \cup \ ((a-1)\omega_M, (a+1)\omega_M)$$

Therefore, in order to reconstruct $x(t)$ from $v(t)$, the spectrum of $x(t)y(t)$ and $x(t)\cos(a\omega_M t)$ must not overlap. This is possible only if $2\omega_M < (a-1)\omega_M$, i.e., $a > 3$. If $a > 3$, then we can reconstruct $x(t)$ from $v(t)$ by multiplying $v(t)$ by $\cos(a\omega_M t)$ and then passing the resulting signal through a low pass filter with cut-off frequency ω_M and gain 2. Once, we have reconstructed $x(t)$ then $y(t)$ can be obtained by

$$y(t) = \frac{v(t)}{x(t)} - \cos(a\omega_M t)$$

where the last step is valid only under the assumption that $x(t) \neq 0$. Thus, the condition for reconstructing both $x(t)$ and $y(t)$ from samples of $v(t)$ is, $a > 3$ and $\frac{2\pi}{T} > 2(1+a)\omega_M$.

Problem 6:

We claim that $\omega_M \leq \frac{\omega_0}{2}$. Suppose this is not true, i.e., assume that if possible $\omega_M > \frac{\omega_0}{2}$. Let $\omega_a = \min(\omega_M, \omega_0)$. Thus by our assumption $\omega_a > \frac{\omega}{2}$. By the assertion of the problem $z[n] = y_b[n]$. This implies that these two signals must have the same Fourier transform also, i.e., $Z(e^{j\omega}) = Y_b(e^{j\omega})$. Now, $z[n]$ is obtained by passing $x_b[n]$ through a low-pass filter with cut-off frequency ω_0. Therefore, the spectrum of $z[n]$ is bandlimited in $(-\omega_0, \omega_0)$. Further, since $y_b[n]$ is the decimation of $y[n]$ and since the spectrum of $y[n]$ is bandlimited in $(-\omega_a, \omega_a)$, the spectrum of $y_b[n]$ is bandlimited in $(-2\omega_a, 2\omega_a)$. But since we are allowed $2\omega_a > \omega_0$, this would imply that $z[n] \neq y_b[n]$ in general. Therefore, our assumption is wrong and hence $\omega_M \leq \frac{\omega_0}{2}$.

Now, we have to see if this condition is sufficient. If $\omega_M \leq \frac{\omega_0}{2}$, then $y[n]$ must be equal to $x[n]$. This, implies that $x_b[n] = y_b[n]$ (since these both these signals are obtained by decimation of the same signal). Furthermore, $x_b[n]$ is bandlimited in $(-2\omega_M, 2\omega_M) \in (-\omega_0, \omega_0)$, and $z[n]$ is the output when $x_b[n]$ is passed through a low-pass filter with cut-off frequency ω_0. This implies that $z[n] = x_b[n]$.

From the above argument we can conclude that $z[n] = y_b[n]$ and hence $\omega_M \leq \frac{\omega_0}{2}$ is sufficient.

Problem 7:

$x_1(t)$ has the Laplace transform

$$X_1(s) = \frac{s-1}{s+2} \qquad \text{ROC: } \Re e\{s\} > -2. \tag{2}$$

and $x_2(t)$ has the Laplace transform

$$X_2(s) = \frac{s+2}{s^2+1} \qquad \text{ROC: } \Re e\{s\} < 0. \tag{3}$$

Let $x_3(t) = x_1(t) * x_2(t)$. Then, the ROC of the Laplace transform of $x_3(t)$ must include the intersection of the ROC's of $X_1(s)$ and $X_2(s)$. Therefore, the ROC of $X_3(s)$ must be $\mathfrak{Re}\{s\} < 0$. For, this ROC the Laplace transform is given by

$$\begin{aligned} X_3(s) &= X_1(s)X_2(s) \\ &= \frac{s-1}{s+2} \cdot \frac{s+2}{s^2+1} \\ &= \frac{s-1}{s^2+1} \qquad \text{ROC: } \mathfrak{Re}\{s\} < 0. \end{aligned}$$

Problem 8:

(a) The magnitude response plot for the given pole-zero plot must be "(5)". Here the pole and zero are symmetric with respect to the imaginary axis and hence the distance to the pole is equal to the distance to the zero from any point on the imaginary axis. Thus the magnitude response must be constant.

(b) Here there is a zero at $s = 0$. Therefore, the magnitude response must be zero at $\omega = 0$. The only plot which satisfies this property is "(2)" and hence is the right answer.

(c)This system has two poles and no zeros. Therefore, the magnitude response must tend to zero as $\omega \to \pm\infty$. Further, the two poles are close to the imaginary axis and hence there should be two peaks in the magnitude response when ω is equal to the imaginary part of the poles. Thus the answer in this case is "(1)".

(e) This system has two zeros and no poles. Furthermore, the zeros are symmetric with respect to the real axis. Thus, the magnitude response must be quadratic and must be an even function. Also, the magnitude response must go to ∞ as $\omega \to \pm\infty$. Thus the answer in this case is "(4)".

(d) Finally, the only remaining answer for this system is "(3)". One can easily conclude this because, the given system has two poles with imaginary parts close to zero and same real parts. Thus, the magnitude response must look like the one in (3).