EECS 120 **Fall 2004**

Homework 4 Solutions

(1) Problem 4.7 of OWN:

It is easy to verify the following properties equation:

1. $X^*(j\omega) = X(-j\omega) \Leftrightarrow x(t)$ is real.
2. $X^*(j\omega) = -X(-j\omega) \Leftrightarrow x(t)$ is pure imaginary.
3. $X(j\omega) = X(-j\omega) \Leftrightarrow x(t)$ is even.
4. $X(j\omega) = -X(-j\omega) \Leftrightarrow x(t)$ is odd.

We use the above mentioned properties to solve this problem.

(a) $X_1(j\omega) = u(\omega) - u(\omega - 2)$. Here neither (1) nor (2) hold. Therefore, $x_1(t)$ is **neither** real nor pure imaginary. Also, neither (3) nor (4) hold. Hence $x_1(t)$ is **neither** even nor odd.

(b)$X_2(j\omega) = \cos(2\omega)\sin(\frac{\omega}{2})$. Here $X_2(j\omega)$ is a real and odd signal. Therefore, by the above mentioned properties $x_2(t)$ is a **pure imaginary** and **odd** signal.

(c)$X_3(j\omega) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = \frac{\sin 2\omega}{\omega}$ and $B(\omega) = 2\omega + \frac{\pi}{2}$. Here $X_3(j\omega)$ is conjugate anti-symmetric, i.e., $X(-j\omega) = -X(j\omega)^*$ and hence its inverse Fourier transform $x_3(t)$ is **purely imaginary**. Further, $X_3(j\omega)$ is neither even nor odd, therefore $x(t)$ is also **neither** even nor odd.

(d) $X_4(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} \delta(\omega - \frac{k\pi}{4})$. $X_4(j\omega)$ is real and even and hence $x(t)$ must also be **real** and **even**.

(2) Problem 4.21 of OWN:
(a)

$$\begin{aligned} x(t) &= [e^{-\alpha t}\cos\omega_0 t]u(t) \\ &= [e^{-\alpha t}u(t)]\cos\omega_0 t \end{aligned}$$

Taking the Fourier transform we get,

$$X(j\omega) = \frac{1}{2\pi}\left[\mathfrak{F}(e^{-\alpha t}u(t)) * \mathfrak{F}(\cos\omega_0 t)\right]$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left[\frac{1}{\alpha+j\omega} * \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] \\
&= \frac{1}{2}\left[\frac{1}{\alpha+j\omega} * \delta(\omega-\omega_0) + \frac{1}{\alpha+j\omega} * \delta(\omega+\omega_0)\right] \\
&= \frac{1}{2}\left[\frac{1}{\alpha+j(\omega-\omega_0)} + \frac{1}{\alpha+j(\omega+\omega_0)}\right]
\end{aligned}
$$

(b)

$$
\begin{aligned}
x(t) &= e^{-3|t|}\sin 2t \\
&= [e^{-3t}u(t) + e^{3t}u(-t)]\sin 2t \\
&= [y(t) + y(-t)]\sin 2t
\end{aligned}
$$

where $y(t) = e^{-3t}u(t)$. Taking the Fourier transform of the above equation we get,

$$
\begin{aligned}
X(j\omega) &= \frac{1}{2\pi}[Y(j\omega) + Y(-j\omega)] * [\frac{\pi}{j}(\delta(\omega-2) - \delta(\omega+2))] \\
&= \frac{1}{2j}\left[\frac{1}{3+j\omega} + \frac{1}{3-j\omega}\right] * [(\delta(\omega-2) - \delta(\omega+2))] \\
&= \frac{1}{2j}\left[\frac{6}{9+\omega^2}\right] * [(\delta(\omega-2) - \delta(\omega+2))] \\
&= \frac{1}{2j}\left[\frac{6}{9+(\omega-2)^2} - \frac{6}{9+(\omega+2)^2}\right]
\end{aligned}
$$

(c)

$$x(t) = (1+\cos\pi t)h(t)$$

where $h(t) = 1$ for $|t| \leq 1$ and 0 otherwise. Therefore,

$$
\begin{aligned}
X(j\omega) &= \frac{1}{2\pi}\left[\mathfrak{F}[1+\cos\pi t] * H(j\omega)\right] \\
&= \frac{1}{2}[2\delta(\omega) + \delta(w-\pi) + \delta(w+\pi)] * H(j\omega) \\
&= \frac{1}{2}[2H(j\omega) + H(j(w-\pi)) + H(j(w+\pi))]
\end{aligned}
$$

where $H(j\omega) = \mathfrak{F}[h(t)] = 2\frac{\sin\omega}{\omega}$. Substituting this in the above equation we get,

$$X(j\omega) = \left[2\frac{\sin\omega}{\omega} + \frac{\sin(\omega-\pi)}{(\omega-\pi)} + \frac{\sin(\omega+\pi)}{(\omega+\pi)}\right]$$

(j) $x(t) = \sum_{n=-\infty}^{\infty} e^{-|t-2n|}$. We can write this function as,

$$x(t) = \sum_{n=-\infty}^{\infty} f(t-2n)$$

where $f(t) = e^{-|t|}$. Therefore,

$$\begin{aligned} X(j\omega) = \mathfrak{F}[x(t)] &= \mathfrak{F}\left[\sum_{n=-\infty}^{\infty} f(t-2n)\right] \\ &= \mathfrak{F}\left[f(t) * \sum_{n=-\infty}^{\infty} \delta(t-2n)\right] \\ &= \mathfrak{F}[f(t)] \cdot \mathfrak{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-2n)\right] \\ &= \frac{2}{1+\omega^2}\left[\frac{2\pi}{2}\sum_{n=-\infty}^{\infty} \delta(\omega-\pi n)\right] \\ &= \frac{2\pi}{1+\omega^2}\left[\sum_{n=-\infty}^{\infty} \delta(\omega-\pi n)\right] \end{aligned}$$

(3) Problem 4.36 of OWN:
(a)

$$\begin{aligned} x(t) &= [e^{-t} + e^{-3t}]u(t) \\ y(t) &= [2e^{-t} - 2e^{-4t}]u(t) \end{aligned}$$

Taking the Fourier transform of $x(t)$ and $y(t)$, we have

$$\begin{aligned} X(j\omega) &= \frac{1}{1+j\omega} + \frac{1}{3+j\omega} \\ Y(j\omega) &= 2\frac{1}{1+j\omega} - 2\frac{1}{4+j\omega} \end{aligned}$$

Therefore, the frequency response of the system is given by

$$\begin{aligned} H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\ &= \frac{3(3+j\omega)}{(2+j\omega)(4+j\omega)} \\ &= \frac{3}{2}\left[\frac{1}{(2+j\omega)} + \frac{1}{(4+j\omega)}\right] \end{aligned}$$

(b) Taking the inverse Fourier transform of the frequency response we get,

$$h(t) = \frac{3}{2}[e^{-2t} + e^{-4t}]u(t)$$

(c) We know that,

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ (2+j\omega)(4+j\omega)Y(j\omega) &= 3(3+j\omega)X(j\omega) \\ 8Y(j\omega) + 6j\omega Y(j\omega) - \omega^2 Y(j\omega) &= 9X(j\omega) + 3j\omega X(j\omega) \end{aligned}$$

Taking the inverse Fourier transform of the above equation, we get

$$\begin{aligned} 8y(t) + 6\frac{d(y(t))}{dt} + \frac{d^2(y(t))}{dt^2} &= 9x(t) + 3\frac{d(x(t))}{dt} \\ \frac{d^2(y(t))}{dt^2} + 6\frac{d(y(t))}{dt} + 8y(t) &= 3\frac{d(x(t))}{dt} + 9x(t) \end{aligned}$$

(4) Problem 4.46 of OWN:
From Fig. $P4.46$, we can derive the following input-output relationship.

$$y(t) = [(x(t)\cos\omega_c t) * h_1(t)]\cos\omega_c t + [(x(t)\sin\omega_c t) * h_1(t)]\sin\omega_c t \qquad (1)$$

where $\mathfrak{F}[h_1(t)] = H_1(j\omega)$. Taking the Fourier transform of the first term in (1) we get,

$$\begin{aligned} & \mathfrak{F}\left[\{(x(t)\cos\omega_c t) * h_1(t)\}\cos\omega_c t\right] \\ = \; & \{\mathfrak{F}[(x(t)\cos\omega_c t)]H_1(j\omega)\} * \mathfrak{F}[\cos\omega_c t] \\ = \; & \{(X(j\omega) * \pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)])\, H_1(j\omega)\} * \pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)] \\ = \; & \{\pi[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))]H_1(j\omega)\} * \pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)] \\ = \; & \pi^2[X(j(\omega-2\omega_c))H_1(j(\omega-\omega_c)) + X(j\omega)H_1(j(\omega-\omega_c)) \\ & + X(j\omega)H_1(j(\omega+\omega_c)) + X(j(\omega+2\omega_c))H_1(j(\omega+\omega_c))] \end{aligned}$$

In an analogous fashion we can obtain the Fourier transform of the second term in (1) also. Adding up these two expressions, we get

$$Y(j\omega) = 2\pi^2 X(j\omega)\left[H_1(j(\omega-\omega_c)) + H_1(j(\omega+\omega_c))\right]$$

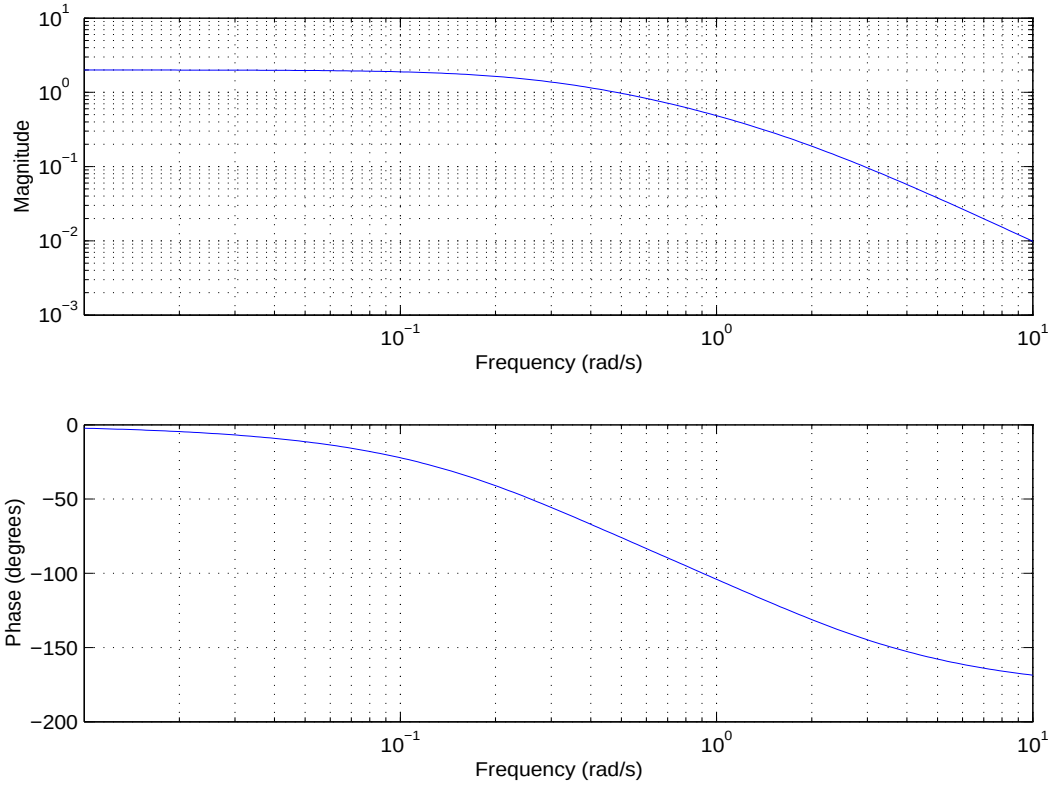
Working out the details for the system in Fig. 4.26 as done above, we get

$$\mathfrak{F}[f(t)] = F(j\omega) = 4\pi^2 X(j\omega)H_1(j(\omega+\omega_c))$$

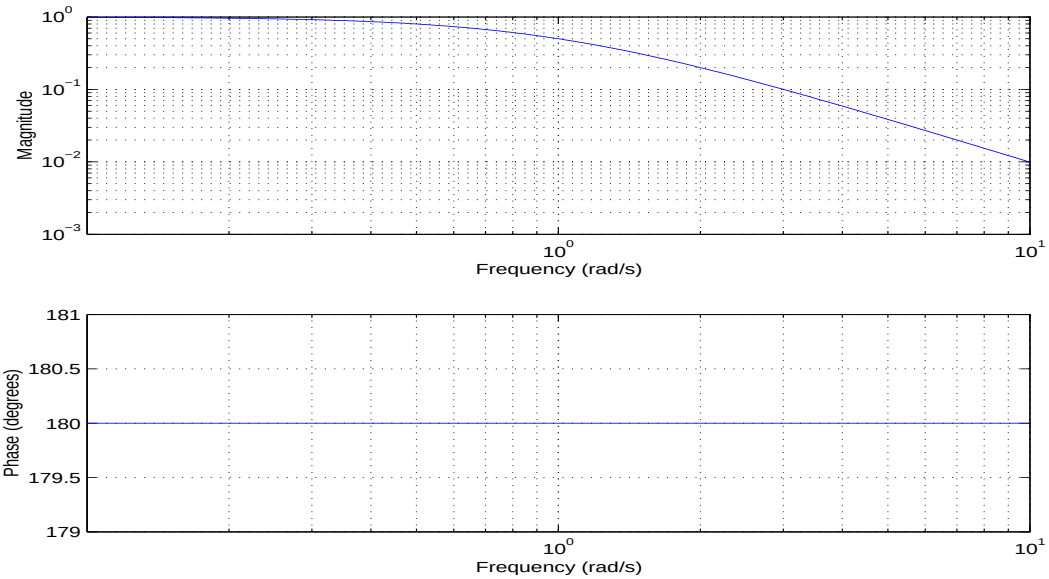
We also know that $\mathfrak{F}[\Re e\{f(t)\}] = \left[\frac{F(j\omega)+F^*(-j\omega)}{2}\right]$. Therefore,

$$\begin{aligned}
\mathfrak{F}[\Re e\{f(t)\}] &= 2\pi^2[X(j\omega)H_1(j(\omega+\omega_c)) + X^*(-j\omega)H_1^*(j(-\omega+\omega_c))] \\
&= 2\pi^2[X(j\omega)H_1(j(\omega+\omega_c)) + X(j\omega)H_1^*(j(-\omega+\omega_c))] \quad \text{(Since, } x(t) \text{ is real)} \\
&= 2\pi^2 X(j\omega)[H_1(j(\omega+\omega_c)) + H_1^*(j(-\omega+\omega_c))] \\
&= 2\pi^2 X(j\omega)[H_1(j(\omega+\omega_c)) + \{H_1(-j\omega) * \delta(-\omega+\omega_c)\}^*] \\
&= 2\pi^2 X(j\omega)[H_1(j(\omega+\omega_c)) + \{H_1(-j\omega) * \delta(\omega-\omega_c)\}^*] \\
&= 2\pi^2 X(j\omega)[H_1(j(\omega+\omega_c)) + \{H_1^*(-j\omega) * \delta^*(\omega-\omega_c)\}] \\
&= 2\pi^2 X(j\omega)[H_1(j(\omega+\omega_c)) + \{H_1(j\omega) * \delta(\omega-\omega_c)\}] \quad \text{(Since, } h_1(t) \text{ is real)} \\
&= 2\pi^2 X(j\omega)[H_1(j(\omega+\omega_c)) + H_1(j(\omega-\omega_c))] \\
&= Y(j\omega)
\end{aligned}$$

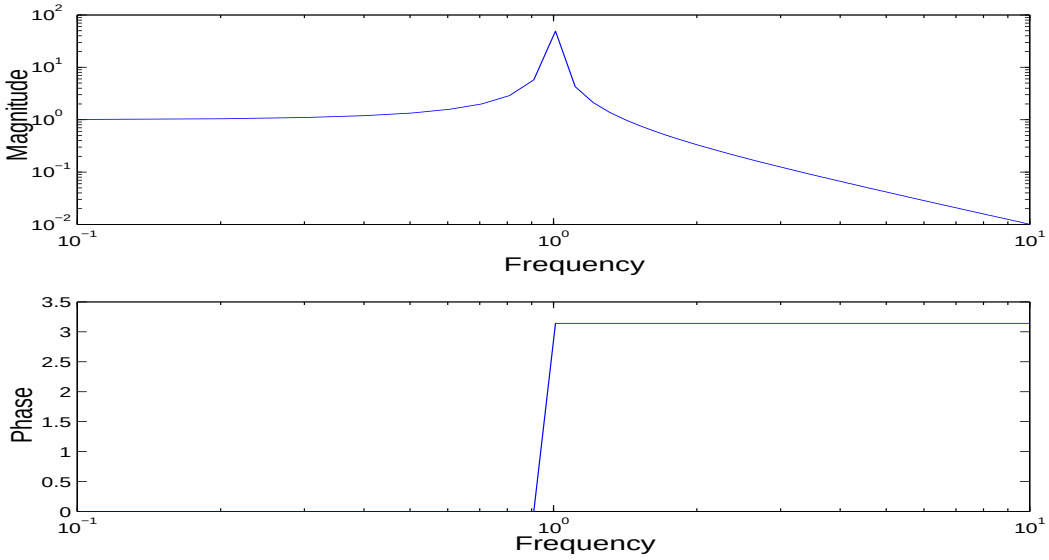
Problem 5(a):



Problem 5(b):



Problem 5(c):
(i)



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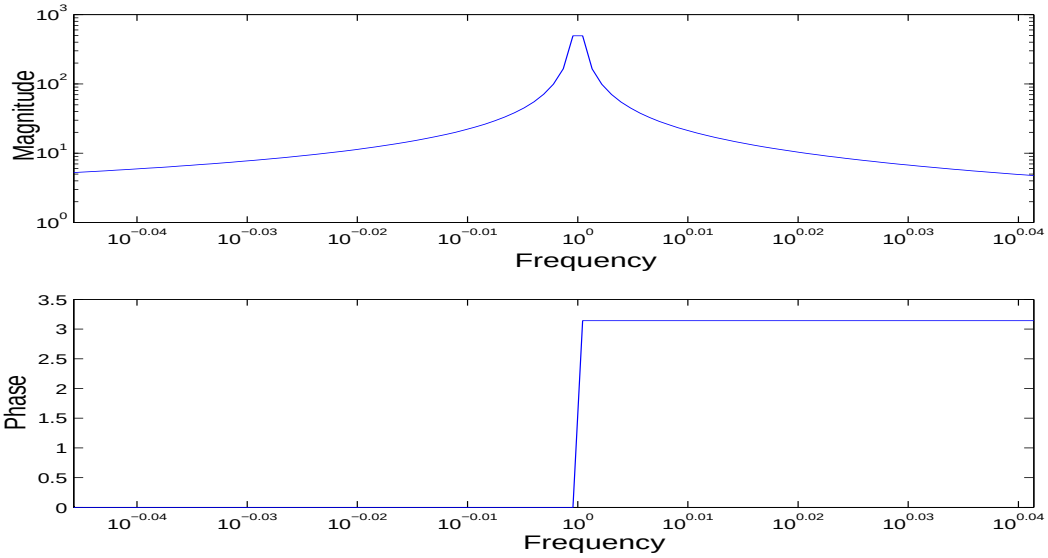
Magnitude
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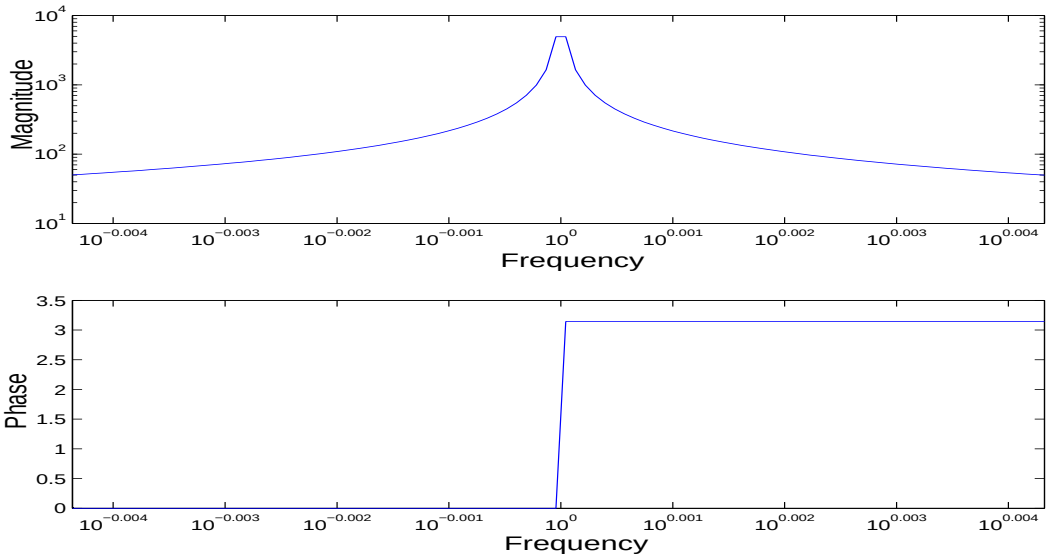
(iii)


Magnitude
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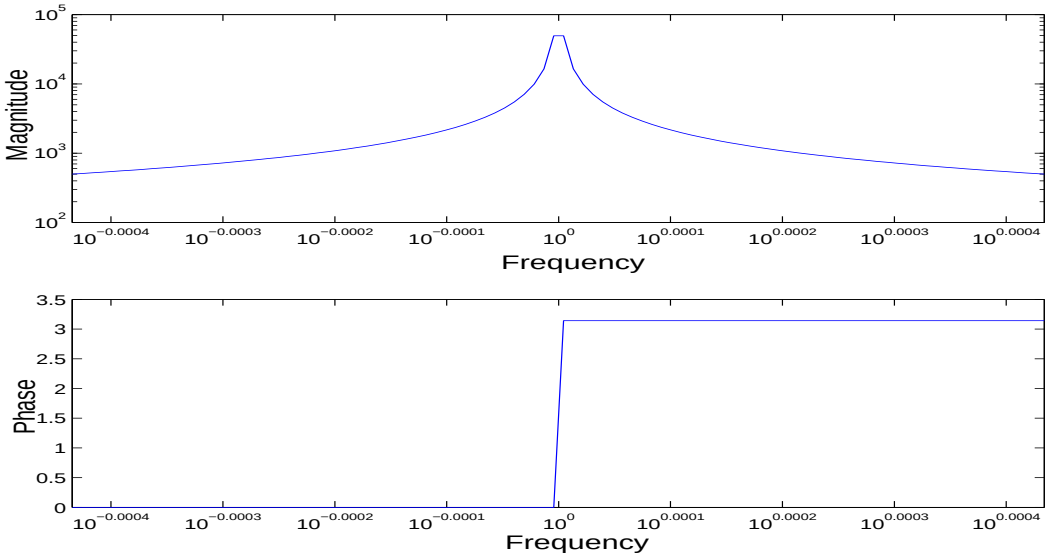
(iv)