**Lecture 6 notes**

*Square matrices*

A square $d \times d$ matrix (real or complex) maps $\mathbf{C}^d$ to $\mathbf{C}^d$.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix}$$

where

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1d}x_d \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2d}x_d \\ \vdots \\ a_{d1}x_1 + a_{d2}x_2 + \ldots + a_{dd}x_d \end{bmatrix}$$

The matrix $A$ is linear, i.e

$$Ax_{(1)} = y_{(1)} \text{ and } Ax_{(2)} = y_{(2)} \Longrightarrow A(ax_{(1)} + bx_{(2)}) = ay_{(1)} + by_{(2)}$$

for all $a, b \in \mathbf{C}$.

Scalars ( $1 \times 1$ square matrices) commute

$d \times d$ matrices in general do not commute.

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

while

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

*Eigenvalues and Eigenvectors*

Let $A$ be a $d \times d$ matrix (real or complex).

$det(sI - A)$ is called the *characteristic polynomial* of $A$.

Any root of the characteristic polynomial is called an *eigenvalue* of $A$.

Every square matrix has at least one eigenvalue, possibly complex (from the fundamental theorem of algebra).

Let $\lambda$ be an eigenvalue of $A$.

Then $\lambda I - A$ is *singular*, so there is at least one nonzero vector $v \in \mathbf{C}^d$ with

$$(\lambda I - A)v = 0$$

Such a vector is called an *eigenvector* of $A$ corresponding to the eigenvalue $\lambda$.

Note that

$$Av = \lambda v$$

*A basis of vectors*
A set of vectors $v_1, \ldots, v_k \in \mathbf{C}^d$ is called *linearly independent* if

$$a_1 v_1 + \ldots + a_k v_k = 0 \Longrightarrow a_1 = \ldots = a_k = 0$$

If a set of vectors $v_1, \ldots, v_k \in \mathbf{C}^d$ is linearly independent then $k \leq d$.

A set of vectors $v_1, \ldots, v_k \in \mathbf{C}^d$ is said to *span* $\mathbf{C}^d$ if every vector $v \in \mathbf{C}^d$ can be written as

$$v = a_1 v_1 + \ldots + a_k v_k$$

for some coefficients $a_1, \ldots, a_k \in \mathbf{C}$.

If a set of vectors $v_1, \ldots, v_k \in \mathbf{C}^d$ spans $\mathbf{C}^d$ then $k \geq d$.

*A basis of vectors*
A *basis* for $\mathbf{C}^d$ is a set of vectors $v_1, \ldots, v_d$ that is both linearly independent and spans $\mathbf{C}^d$.

*A point to remember*
Let $A$ and $B$ be square $d \times d$ matrices.

Suppose there is a basis $v_1, \ldots, v_d$ for $C^d$ comprised of eigenvectors of $A$, with respective eigenvalues $\lambda_1, \ldots, \lambda_d$.

Suppose the same $v_1, \ldots, v_d$ for $C^d$ are *also* eigenvectors of $B$ with respective eigenvalues $\mu_1, \ldots, \mu_d$.

Then $A$ and $B$ commute.

Given $v \in \mathbf{C}^d$ write $v = a_1 v_1 + \ldots + a_k v_k$ for some coefficients $a_1, \ldots, a_k \in \mathbf{C}$.

$$\begin{aligned} ABv &= AB(\sum_{i=1}^{k} a_i v_i) \\ &= A(\sum_{i=1}^{k} a_i \mu_i v_i) \\ &= \sum_{i=1}^{k} a_i \mu_i \lambda_i v_i \end{aligned}$$

$$\begin{aligned} BAv &= BA(\sum_{i=1}^{k} a_i v_i) \\ &= B(\sum_{i=1}^{k} a_i \lambda_i v_i) \\ &= \sum_{i=1}^{k} a_i \mu_i \lambda_i v_i \end{aligned}$$

*Orthonormal basis*

• Let $v_1, \ldots, v_d$ be a basis for $\mathbf{C}^d$.

The basis is called *orthonormal* if :

$$\sum_{i=1}^{d} v_{li} v_{mi}^* = \begin{cases} 1 & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases}$$

where $1 \leq l, m \leq d$.

We sometimes write this as

$$< v_l, v_m >= \delta_{lm}$$

for all $1 \leq l, m \leq d$.

• Let $v_1, \ldots, v_d$ be an orthonormal basis for $\mathbf{C}^d$. and let $v$ be an arbitrary vector in $\mathbf{C}^d$.

We can write

$$v = \sum_{i=1}^{d} a_i v_i$$

where

$$a_i = < v, v_i >$$

because, for any $1 \leq j \leq d$, we have :

$$\begin{aligned} < v, v_j > &= < \sum_{i=1}^{d} a_i v_i, v_j > \\ &= \sum_{i=1}^{d} a_i < v_i, v_j > \\ &= \sum_{i=1}^{d} a_i \delta_{ij} \\ &= a_j \end{aligned}$$

*Eigenfunctions of a discrete time LTI system*

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Consider the complex exponential input

$$x[n] = Az^n$$

The associated output :

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] A z^{n-k} \\ &= Az^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\ &= H(z) A z^n \end{aligned}$$

is a fixed constant multiple of the input.

The signal $Az^n$ is an *eigenfunction* of the system corresponding to the *eigenvalue* $H(z)$.

$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ is called the *transfer function* of the LTI system.

Consider the input

$$x[n] = \sum_r A_r z_r^n$$

The corresponding output is

$$y[n] = \sum_r A_r H(z_r) z_r^n$$

If we could express an arbitrary input as a linear combination of complex exponentials, we would be able to figure out the corresponding output from knowledge of the transfer function $H(z)$.

*Eigenfunctions of a continuous time LTI system*

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \ .$$

Consider the complex exponential input

$$x(t) = Ae^{st}$$

The associated output :

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)Ae^{s(t-\tau)}d\tau \\ &= Ae^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= H(s)Ae^{st} \end{aligned}$$

is a fixed constant multiple of the input.

The signal $Ae^{st}$ is an *eigenfunction* of the system corresponding to the *eigenvalue* $H(s)$.

$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ is called the *transfer function* of the LTI system.

Consider the input

$$x(t) = \sum_r A_r e^{s_r t}$$

The corresponding output is

$$y(t) = \sum_r A_r H(s_r) e^{s_r t}$$

If we could express an arbitrary input as a linear combination of complex exponentials, we would be able to figure out the corresponding output from knowledge of the transfer function $H(s)$.

*Fourier series expansion for a continuous time periodic signal*

Fix $T > 0$. Let $\omega_0$ denote $\frac{2\pi}{T}$.

The complex exponentials that are periodic with period $T$ are the signals $e^{jk\omega_0 t}$ for $k \in \mathbf{Z}$.

Given a periodic signal $x(t)$ with period $T$ we might hope to be able to write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

If a periodic signal $x(t)$ with period $T$ satisfies

$$\int_0^T \mid x(t) \mid^2 dt < \infty$$

and we define its *Fourier series coefficients*

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

then we have

$$\lim_{K\to\infty}\int_0^T \mid x(t) - \sum_{k=-K}^{K} a_k e^{jk\omega_0 t} \mid^2 dt = 0 \ .$$

*Fourier Series examples*

• $x(t) = \sin(\omega_0 t)$, where $\omega_0 = \frac{2\pi}{T}$.

has

$a_{-1} = a_1 = \frac{1}{2j}$ and $a_k = 0$ for $k \neq \pm 1$.

• $x(t)$ periodic with period $T$ and

$$x(t) = \begin{cases} 1 & \text{if } \mid t \mid < T_1 \\ 0 & \text{if } T_1 < \mid t \mid < \frac{T}{2} \end{cases} \ .$$

has

$a_0 = \frac{2T_1}{T}$ and $a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$ for $k \neq 0$
The special case of the above when $T_1 = \frac{T}{4}$
has
$a_0 = \frac{1}{2}$ and $a_k = 0$ for $k$ even and $a_k = \frac{1}{|k|\pi}$ for $k$ odd.