Problem 1

(i)

$\boxed{\text{True}}$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} \sin^2\left(\frac{t}{1000}\right) dt \to \infty \qquad \text{infinite energy}$$

$\Rightarrow$ It is a power signal. $P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sin^2\left(\frac{t}{1000}\right) dt = \frac{1}{2}$

(ii)

$\boxed{\text{True}}$

$$H\{a_1 x_1(t) + a_2 x_2(t)\} = H_2\{H_1\{a_1 x_1(t) + a_2 x_2(t)\}\}$$

(by linearity of H_1) $= H_2\{a_1 H_1\{x_1(t)\} + a_2 H_1\{x_2(t)\}\}$

(by linearity of H_2) $= a_1 H_2\{H_1\{x_1(t)\}\} + a_2 H_2\{H_1\{x_2(t)\}\}$

(iii)

$\boxed{\text{False}}$

$s(t)$ must satisfy so called Dirichlet condition.

1. $s(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |s(t)|\, dt < \infty$
2. $s(t)$ has bounded variation in any finite interval.

counter examples :

e^t does not satisfy eq. 1. $\sin\left(\frac{1}{t}\right)$ does not satisfy eq 2.

(iv)

$\boxed{\text{False}}$

counterexample : $y[n] = (x[n])^2$ is time-invariant, memoryless, causa

but it is not linear

(v)

$\boxed{\text{False}}$

$$X(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow X(j\cdot 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t)\, dt$$

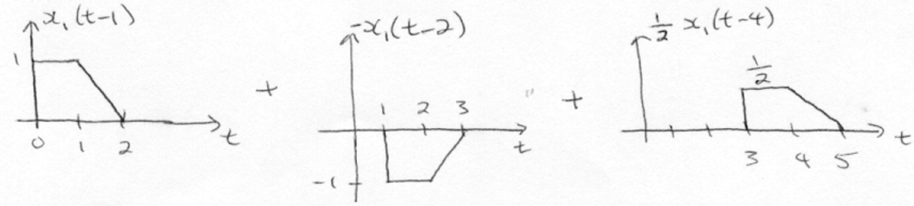
but $0 = X(j\cdot 0) \neq \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t)\, dt > 0$

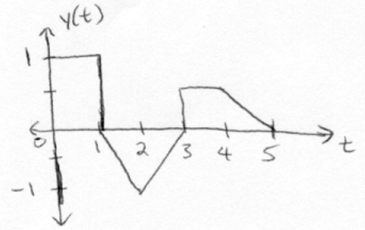
Problem 2

(i)

$$y(t) = x_1(t) * x_2(t) = x_1(t) * \left(\delta(t-1) - \delta(t-2) + \tfrac{1}{2}\delta(t-4)\right)$$

$$= x_1(t-1) - x_1(t-2) + \tfrac{1}{2} x_1(t-4)$$





(ii)

$$X(j\omega) = \begin{cases} |\omega|\, e^{-j\omega}, & \text{if } |\omega| < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-1}^{1} |\omega|\, e^{-j\omega} e^{j\omega t}\, d\omega \qquad \text{inverse FT}$$

$$= \frac{1}{2\pi}\int_{-1}^{0} -\omega\, e^{+j\omega(t-1)}\, d\omega + \frac{1}{2\pi}\int_{0}^{1} \omega\, e^{j\omega(t-1)}\, d\omega$$

integration by parts:
$dv = e^{j\omega(t-1)} \rightarrow u = \omega$
$v = \dfrac{e^{j\omega(t-1)}}{j(t-1)} \leftarrow du = 1$

$$= \frac{1}{2\pi}\left[-\omega\frac{e^{j\omega(t-1)}}{j(t-1)} - \frac{e^{j\omega(t-1)}}{(t-1)^2}\right]_{-1}^{0}$$

$$+ \frac{1}{2\pi}\left[-\omega\frac{e^{j\omega(t-1)}}{j(t-1)} + \frac{e^{j\omega(t-1)}}{(t-1)^2}\right]_{0}^{1}$$

$$= \frac{1}{2\pi}\left[-\frac{1}{(t-1)^2} - \frac{e^{-j(t-1)}}{j(t-1)} + \frac{e^{-j(t-1)}}{(t-1)^2} + \frac{e^{j(t-1)}}{j(t-1)} + \frac{e^{j(t-1)}}{(t-1)^2} - \frac{1}{(t-1)^2}\right]$$

$$= \frac{1}{j2\pi(t-1)}\left(e^{j(t-1)} - e^{-j(t-1)}\right) + \frac{1}{2\pi(t-1)^2}\left(e^{j(t-1)} + e^{-j(t-1)}\right) - \frac{1}{\pi(t-1)^2}$$

$$= \frac{\sin(t-1)}{\pi(t-1)} + \frac{\cos(t-1) - 1}{\pi(t-1)^2}$$

Problem 3

(i)

$$a^2 y(t) + 2a \frac{d}{dt} y(t) + \frac{d^2}{dt^2} y(t) = x(t)$$

Taking the Fourier transform of both sides

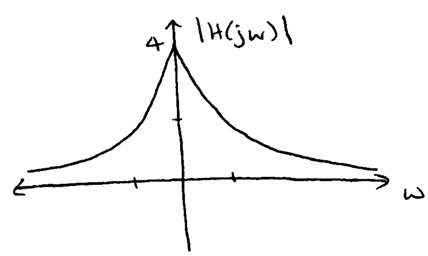
$$a^2 Y(j\omega) + 2a(j\omega)Y(j\omega) + (j\omega)^2 Y(j\omega) = X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a^2 + 2a(j\omega) + (j\omega)^2} = \frac{1}{(a+j\omega)^2}$$

(ii)

For $a = \frac{1}{2}$, $H(j\omega) = \dfrac{1}{(\frac{1}{2} + j\omega)^2}$

$$\Rightarrow |H(j\omega)| = \frac{1}{|\frac{1}{2} + j\omega|^2} = \frac{1}{\frac{1}{4} + \omega^2}$$



$\Rightarrow$ This system is a lowpass filter

as $|\omega| \to \infty$, $|H(j\omega)| \to 0$

Problem 3

(iii)

The system is stable iff $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$\boxed{\text{case 1: } a > 0}$

$$H(j\omega) = \frac{1}{(a+j\omega)^2} \xleftrightarrow{\text{FT table}} h(t) = t e^{-at} u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |t e^{-at} u(t)|\, dt = \int_{0}^{\infty} t e^{-at}\, dt$$

$$= \left[t \frac{e^{-at}}{-a} - \frac{e^{-at}}{a^2} \right]_0^{\infty}$$ integration by parts

$$= \frac{1}{a^2}$$

$\Rightarrow$ system is $\boxed{\text{Stable}}$

$\boxed{\text{Case 2: } a = 0}$

$$H(j\omega) = \frac{1}{(j\omega)^2}$$

$$u(t) - \frac{1}{2} \xleftrightarrow{\text{FT table}} \frac{1}{j\omega}$$

By the differentiation in frequency property of FT

$$t\left(u(t) - \frac{1}{2}\right) \xleftrightarrow{\text{FT prop}} \frac{1}{(j\omega)^2}$$

$$\int_{-\infty}^{\infty} \left| t\left(u(t) - \frac{1}{2}\right)\right| dt = \infty$$

$\Rightarrow$ system is $\boxed{\text{unstable}}$

$\boxed{\text{case 3: } a < 0}$

$$h(t) = -t e^{-at} u(-t) \xleftrightarrow{\text{FT}} H(j\omega) = \frac{1}{(a+j\omega)^2}$$

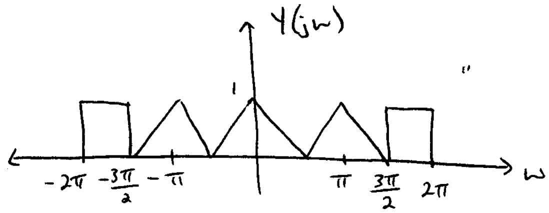
$$\int_{-\infty}^{\infty} |-t e^{-at} u(-t)|\, dt = \int_{-\infty}^{0} t e^{-at}\, dt = \frac{1}{a^2}$$ same as case 1

$\Rightarrow$ system is $\boxed{\text{stable}}$

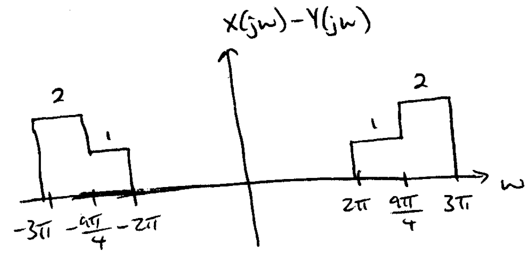
Problem 4

$$h(t) = \frac{\sin(2\pi t)}{\pi t} \xleftrightarrow{FT} H(j\omega) = \begin{cases} 1, & |\omega| < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$



$$X(j\omega) - Y(j\omega) = \begin{cases} 2, & \frac{9\pi}{4} < |\omega| < 3\pi \\ 1, & 2\pi < |\omega| < \frac{9\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$



By Parseval's theorem

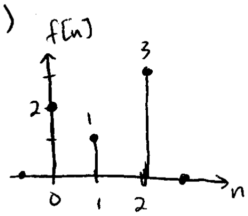
$$\int_{-\infty}^{\infty} |x(t)-y(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)-Y(j\omega)|^2\,d\omega$$

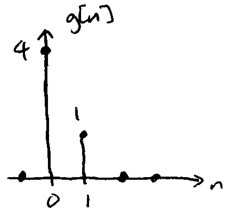
$$= \frac{2}{2\pi}\left[\int_{2\pi}^{9\pi/4} (1)^2\,d\omega + \int_{9\pi/4}^{3\pi} (2)^2\,d\omega\right]$$

$$= \frac{13}{4}$$

Problem 5

(i)





$$h[n] = f[n] * g[n]$$

$$= 4\left(2\delta[n] + \delta[n-1] + 3\delta[n-2]\right) + 1\left(2\delta[n-1] + \delta[n-2] + 3\delta[n-3]\right)$$

$$= 8\delta[n] + 6\delta[n-1] + 13\delta[n-2] + 3\delta[n-3]$$

$$h(x) = 8 + 6x + 13x^2 + 3x^3$$

(ii)

$$a[n] = \left(\tfrac{1}{2}\right)^n u[n] \xleftrightarrow{\text{FT pairs table}} A(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$b[n] = \left(\tfrac{1}{3}\right)^n u[n] \xleftrightarrow{\text{FT}} B(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$c(x) = a(x)\,b(x) \Rightarrow c[n] = a[n] * b[n] \Rightarrow C(e^{j\omega}) = A(e^{j\omega})\,B(e^{j\omega})$$

$$C(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})} = \frac{c_1}{1-\frac{1}{2}e^{-j\omega}} + \frac{c_2}{1-\frac{1}{3}e^{-j\omega}}$$

plug in $\omega = 0$ and $\omega = \infty$ $\Rightarrow$ $c_1 = 3$, $c_2 = -2$

$$C(e^{j\omega}) = \frac{3}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

$$c[n] = 3\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{3}\right)^n u[n]$$

$$c(x) = \sum_{n=0}^{\infty}\left(3\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{3}\right)^n\right)x^n$$

Problem 6

(i)

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$\boxed{x(t) = \sum_{k=-2}^{2} a_k e^{jkt} = a_{-2} e^{-j2t} + a_{-1} e^{-jt} + a_0 + a_1 e^{jt} + a_2 e^{j2t}}$$

(ii)

By Fourier series properties

$x(t)$ real and odd $\xleftrightarrow{FS}$ a_k imaginary and odd

a_k odd $\Rightarrow \boxed{a_0 = 0}$

$$x(t) = a_{-1} e^{-jt} + a_1 e^{jt} + a_{-2} e^{-j2t} + a_2 e^{j2t}$$

$$x(t-\pi) = a_{-1} e^{-j(t-\pi)} + a_1 e^{j(t-\pi)} + a_{-2} e^{-j2(t-\pi)} + a_2 e^{j2(t-\pi)}$$

$$= -a_{-1} e^{-jt} - a_1 e^{jt} + a_{-2} e^{-j2t} + a_2 e^{j2t}$$

$$0 = x(t-\pi) + x(t)$$

$$= 2a_{-2} e^{-j2t} + 2a_2 e^{j2t}$$

$$= 2a_2 (-e^{-j2t} + e^{j2t})$$

$$= j2a_2 \sin(2t)$$

$$\Rightarrow \boxed{a_2 = 0}$$

$$\Rightarrow a_{-2} = 0$$

(iii)

By Parseval's equation

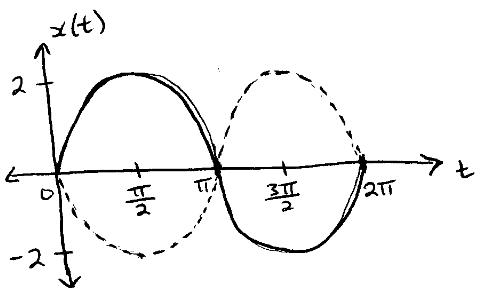
$$\frac{1}{2\pi}\int_0^{2\pi} |x(t)|^2\, dt = \sum |a_k|^2$$

$$\Rightarrow \quad 2 = |a_{-1}|^2 + |a_1|^2 = 2\,|a_1|^2$$

$$\Rightarrow \quad a_1 = \pm j$$

(iv)

$$x(t) = \pm j\, e^{-jt} \mp j\, e^{jt} = \pm\, 2 \sin(t)$$



Problem 7

$$y(t) = x\left(t - \left(\int_t^{t+1} x(u)\,du\right)^2\right)$$

(i)

The system is $\boxed{\text{not linear}}$.

Counter example :

input $x_1(t) = \underbrace{1}_{}\!{}^{\forall t}$ has corresponding output $y_1(t) = 1 \;\; \forall t$

input $x_2(t) = u(t)$ has output $y_2(t) = u(t-1)$

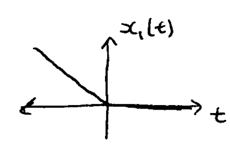
however, input $x(t) = x_1(t) + x_2(t) = 1 + u(t)$

has output $y(t) = 1 + u(t-4) \neq y_1(t) + y_2(t)$

(ii)

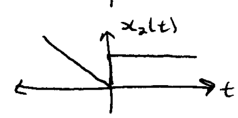
The system is $\boxed{\text{not causal}}$.

Counter example :



for inputs $x_1(t) = |t|\, u(-t)$

$x_2(t) = |t|\, u(-t) + u(t)$



we have $x_1(t) = x_2(t)$ for $t < 0$

however the corresponding outputs $y_1(t)$ and $y_2(t)$

need not satisfy $y_1(t) = y_2(t)$ for $t < 0$

$$y_1\left(-\tfrac{1}{2}\right) = x_1\left(-\tfrac{1}{2} - \left(\tfrac{1}{8}\right)^2\right) = \frac{33}{64}$$

$$y_2\left(-\tfrac{1}{2}\right) = x_2\left(-\tfrac{1}{2} - \left(\tfrac{5}{8}\right)^2\right) = \frac{57}{64}$$

(iii)

The system is $\boxed{\text{stable}}$.

If the input is bounded, i.e. $\exists\, M$ s.t. $|x(t)| < M < \infty \;\; \forall t$

then the output is bounded, $|y(t)| = \left| x\left(t - \left[\int_t^{t+1} x(u)\,du\right]^2\right)\right| < M \;\; \forall t$.