Practice Midterm 2

Problem 1 *(Wireless Downlink.)*

A base station transmits simultaneously to three mobiles. It needs to send a signal $s_1(t)$ to Mobile 1, a signal $s_2(t)$ to Mobile 2, and a signal $s_3(t)$ to Mobile 3. Since $s_1(t), s_2(t)$ and $s_3(t)$ are speech signals, they are real-valued and band-limited:

$$S_i(j\omega) \quad = \quad 0, \text{ for } |\omega| > W, \tag{1}$$

for $i = 1, 2, 3$. The base station needs to produce a real-valued output signal $y(t)$ to be transmitted out of the antenna. The FCC allows you to use the frequency band $\omega_0 \leq |\omega| \leq \omega_0 + 3W$.

- Draw the block diagram of the base station, with inputs $s_1(t), s_2(t)$ and $s_3(t)$ and output $y(t)$, where $y(t)$ must comply with FCC regulations *and* permit perfect recovery of $s_1(t), s_2(t)$ and $s_3(t)$. *Hint:* There are multiple solutions; only one is required.
- Draw the block diagram of the demodulation system at Mobile 1, with input $y(t)$ and output $s_1(t)$.

You may use arbitrary components, but carefully specify all involved parameters, such as cut-off frequencies of filters.

Problem 2 *(Amplitude modulation.)*

(a) For the discrete-time signal $x[n]$ it is known that $X(e^{j\omega}) = 0$, for $|\omega| > \pi/4$. Determine the range of ω for which the DTFT of $y[n] = \cos(\frac{5\pi}{4}n)x[n]$ must be zero. *Hint:* Select an example spectrum $X(e^{j\omega})$ and sketch the resulting DTFT of $y[n]$.

(b) The real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$. Consider the block diagram of Figure 1, where

$$H_1(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad H_2(j\omega) = \begin{cases} 1, & \text{for } |\omega| \geq 2\omega_c \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

- Pick an arbitrary (bandlimited) example spectrum for $x(t)$, and sketch the corresponding spectrum of the signal $y(t)$.
- For what values of the parameters W and ω_c is it possible to recover $x(t)$ from $y(t)$?
- Provide the block diagram of a system that recovers $x(t)$, given $y(t)$, carefully specifying all involved parameters.

(c) The real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$. The goal is to perform standard (i.e., double-sideband) AM with carrier frequency $\omega_c > 5W$. Unfortunately, the only type of modulator available is multiplication by $\cos(\frac{\omega_c}{4}t)$. Otherwise, addition, scalar multiplication, and filters can be used. Draw the block diagram of the system that achieves our goal, and if your system uses a filter, specify the desired frequency response. *Hint:* Pick an example spectrum for $x(t)$ and sketch the spectra of intermediate signals to maximize your chances for partial credit.

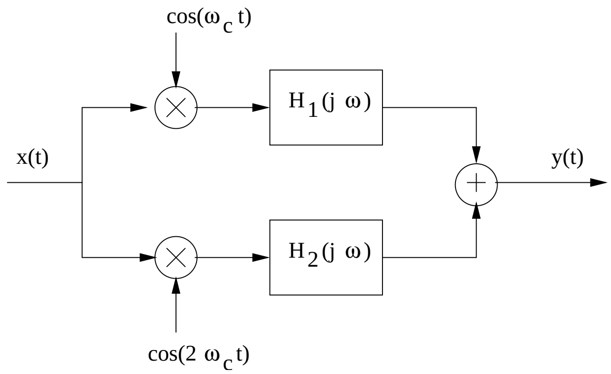


Figure 1: Block diagram for Part *(b)*.

(d) The real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$. The goal is to perform single-sideband AM with only the *lower* sideband, with carrier frequency $\omega_c > 5W$. Again, you can use addition, scalar multiplication, and multiplication by $\cos(\omega_m t)$, for arbitrary ω_m. However, this time, you only have *fixed* ideal low-pass filters with the following frequency response:

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \le \omega_c/2 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Draw the block diagram of a system that achieves the goal, clearly specifying all involved parameters, such as the frequencies of the modulators, etc. *Hint:* Pick an example spectrum for $x(t)$ and sketch the spectra of intermediate signals to maximize your chances for partial credit.

Problem 3 *(PAM.)*

Two pulses are suggested for a PAM system:

$$p_1(t) = ae^{-t}u(t), \quad \text{and} \quad p_2(t) = be^{-10t}u(t), \tag{4}$$

where a and b are positive real numbers that will be selected appropriately, leading to

$$y_i(t) = \sum_{k=-\infty}^{\infty} x[k]p_i(t - kT), \text{ for } i = 1, 2, \tag{5}$$

where we choose $T = 1$. We suppose that the data signal is bounded to $|x[n]| \le 1$. In this problem, we want to compare the two pulses $p_1(t)$ and $p_2(t)$.

(a) Select $a = 2$ and $b = 2\sqrt{10}$. For this choice, it can be shown that the pulse energy is the same for $p_1(t)$ and for $p_2(t)$. (You *don't* have to show this!) Now consider the transmission of $p_1(t)$ and $p_2(t)$, respectively, across a communication channel with impulse response $h(t)$ and corresponding frequency response

$$H(j\omega) = \frac{1}{6 + j\omega}. \tag{6}$$

This yields an output signal $z_i(t) = (p_i * h)(t)$, for $i = 1, 2$.

- Evaluate the energy of the received signals, $z_1(t)$ and $z_2(t)$, respectively.
- Which received signal has the larger energy?
- How is it possible that even though the two pulses have the same transmitted energy, their received energies differ?

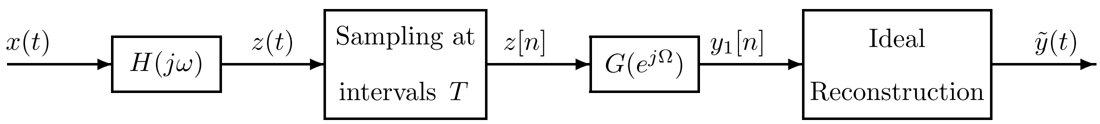
(b) (Hard problem) To have a fair comparison, we have to make sure that the powers of the transmitted signals $y_1(t)$ and $y_2(t)$, respectively, are equal. To adjust the power, assume that $x[n] = 1$ for all n, i.e., for $-\infty < n < \infty$. Determine the relationship between a and b such that for this particular $x[n]$, the signals $y_1(t)$ and $y_2(t)$ have the same power. (As seen in class, this provides a worst case analysis.) *Hint:* By contrast to Part *(a)*, this question studies the *power* of the entire signal, rather than the energy of a single pulse.

Problem 4 *(Discrete-time Differentiator.)*

We would like to construct a system D that implements a derivative, that is, for an input $x(t)$, the system should give an output $y(t)$ given by

$$y(t) \quad = \quad D\{x(t)\} = \frac{dx(t)}{dt}. \tag{7}$$

It is suggested to use the following system:



where

$$G(e^{j\Omega}) \quad = \quad j\frac{\Omega}{T}, \text{ for } |\Omega| \leq \pi. \tag{8}$$

This system does not exactly implement the desired system D. Instead, it produces an output $\tilde{y}(t)$ which is, in general, not equal to the desired output $y(t)$.

(a) Suppose that

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \pi/T \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

- Determine and sketch the overall frequency response (magnitude and phase) of the system with input $x(t)$ and output $\tilde{y}(t)$.
- For the test signal $x(t)$ with Fourier transform

  $$X(j\omega) \quad = \quad e^{-|\omega|}, \tag{10}$$

  determine the error between the desired signal, $y(t)$, and the actual system output, $\tilde{y}(t)$, given by

  $$E \quad = \quad \int_{-\infty}^{\infty} |y(t) - \tilde{y}(t)|^2 dt, \tag{11}$$

  as a function of the sampling interval T. What happens as we increase the sampling frequency?

(b) Unfortunately, it is quite difficult to exactly implement ideal frequency filters like $H(j\omega)$ in Part *(a)*. As a simple model of this imperfection, suppose now that

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \pi/T \\ \epsilon & \text{otherwise.} \end{cases} \tag{12}$$

For the same test signal as in Part *(a)*, that is, $X(j\omega) = e^{-|\omega|}$, determine the spectrum $Z_\delta(j\omega)$ of the sampled signal,

$$z_\delta(t) \quad = \quad z(t) \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{13}$$

- Start with a sketch of $Z_\delta(j\omega)$, carefully labeling the *frequency* axis.
- Which adverse effect corrupts the signal $z_\delta(t)$?
- Then, write out a formula for $Z_\delta(j\omega)$. The simpler your formula, the better.

Problem 5 *(LTI System Analysis.)*

A causal LTI system has a transfer function

$$H(s) \quad = \quad \frac{(s+4)(s^2+5s+6)}{(s+1)(s^2-2s+3)}. \tag{14}$$

Determine the differential equation that describes this system. Find the impulse response $h(t)$. Is the system stable? Does this system have a stable and causal inverse system?