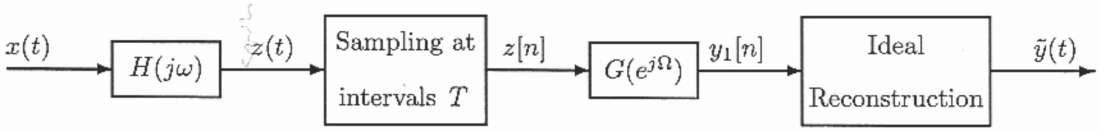
Problem 4 *(Discrete-time Differentiator.)*

We would like to construct a system D that implements a derivative, that is, for an input $x(t)$, the system should give an output $y(t)$ given by

$$y(t) \quad = \quad D\{x(t)\} = \frac{dx(t)}{dt}. \tag{7}$$

It is suggested to use the following system:



where

$$G(e^{j\Omega}) \quad = \quad j\frac{\Omega}{T}, \text{ for } |\Omega| \leq \pi. \tag{8}$$

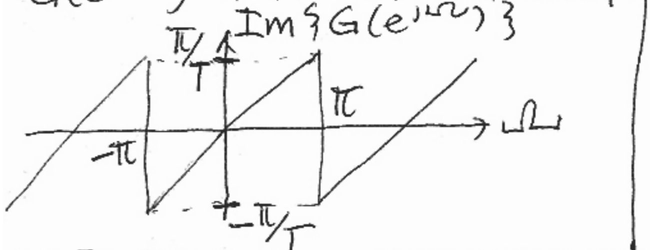
This system does not exactly implement the desired system D. Instead, it produces an output $\tilde{y}(t)$ which is, in general, not equal to the desired output $y(t)$.

(a) Suppose that

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \pi/T \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

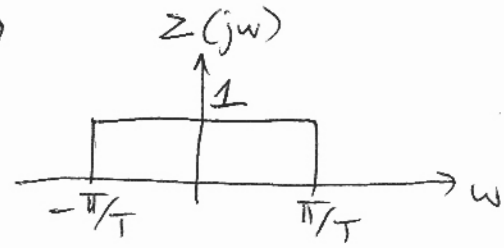
- Determine and sketch the overall frequency response (magnitude and phase) of the system with input $x(t)$ and output $\tilde{y}(t)$.

$G(e^{j\Omega})$ IS PURELY IMAGINARY:

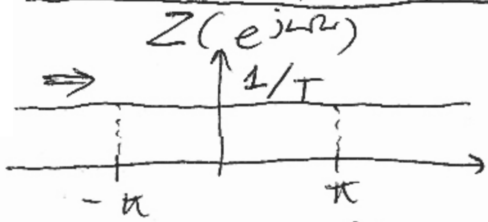


NOW, LET $\boxed{x(t)=\delta(t)}$

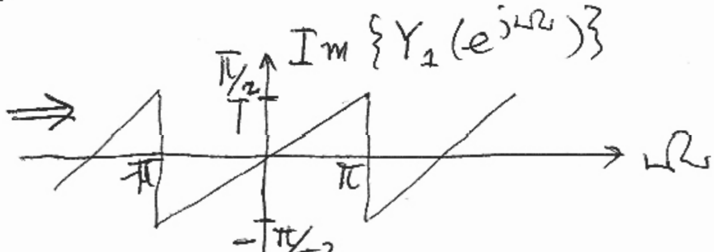
$\Rightarrow$



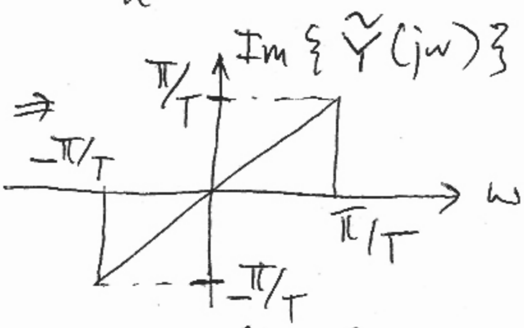
$\Rightarrow$



$\Rightarrow$

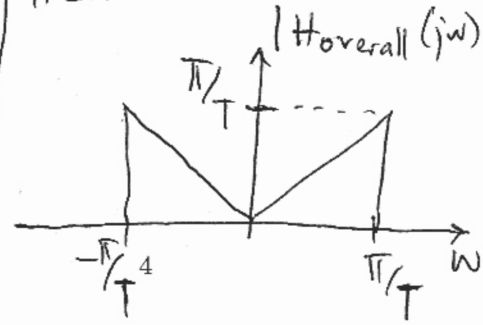


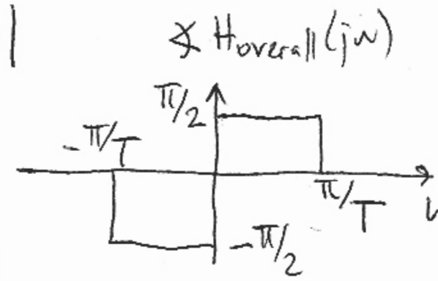
$\Rightarrow$



AND $Re\{\tilde{Y}(j\omega)\}=0$.

SINCE $X(j\omega)=1$, $H_{overall}(j\omega)=\tilde{Y}(j\omega)$,
HENCE





- For the test signal $x(t)$ with Fourier transform

$$X(j\omega) = e^{-|\omega|}, \tag{10}$$

determine the error between the desired signal, $y(t)$, and the actual system output, $\tilde{y}(t)$, given by

$$E = \int_{-\infty}^{\infty} |y(t) - \tilde{y}(t)|^2 dt, \tag{11}$$

as a function of the sampling interval T. What happens as we increase the sampling frequency?

$$y(t) = \frac{d}{dt} x(t) \Rightarrow Y(jw) = jwX(jw) = jwe^{-|w|}.$$

$$\hat{Y}(jw) = \tilde{H}(jw) X(jw) = \begin{cases} jwe^{-|w|}, & |w| \leq \pi/T \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow E = \int_{-\infty}^{\infty} |y(t) - \tilde{y}(t)|^2 dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(jw) - \tilde{Y}(jw)|^2 dw$$

$$= \frac{1}{2\pi} \left[\int_{\pi/T}^{\infty} |jwe^{-|w|}|^2 dw + \int_{-\infty}^{-\pi/T} |jwe^{-|w|}|^2 dw \right]$$

$$= \frac{1}{2\pi} \cdot 2 \cdot \int_{\pi/T}^{\infty} w^2 e^{-2w} dw$$

This integral was actually provided in the exam.

$$= \frac{1}{2\pi} \frac{e^{-\frac{2\pi}{T}}}{4} \left(4 \left(\frac{\pi}{T}\right)^2 + 4 \frac{\pi}{T} + 2 \right)$$

As we increase the sampling frequency, $T \to 0 \Rightarrow E \to 0$, which is clear: ~~The ... ~~ We capture more and more of the signal spectrum.

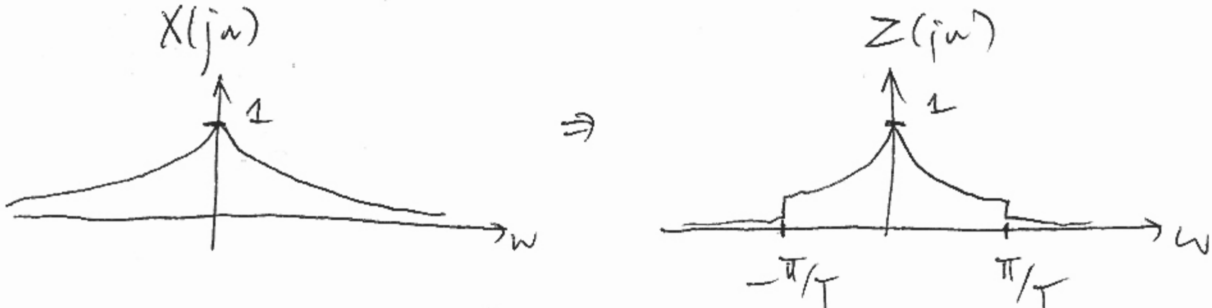
(b) Unfortunately, it is quite difficult to exactly implement ideal frequency filters like $H(j\omega)$ in Part *(a)*. As a simple model of this imperfection, suppose now that

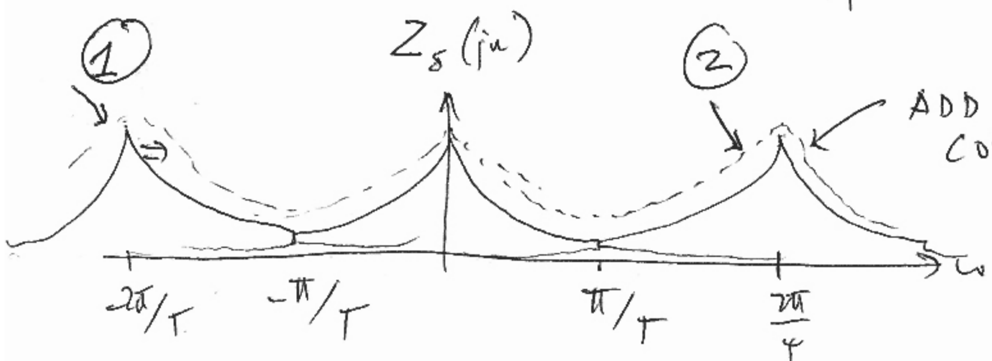
$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \pi/T \\ \epsilon & \text{otherwise.} \end{cases} \tag{12}$$

For the same test signal as in Part *(a)*, that is, $X(j\omega) = e^{-|\omega|}$, determine the spectrum $Z_\delta(j\omega)$ of the sampled signal,

$$z_\delta(t) \quad = \quad z(t) \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{13}$$

- Start with a sketch of $Z_\delta(j\omega)$, carefully labeling the *frequency* axis.
- Which adverse effect corrupts the signal $z_\delta(t)$?
- Then, write out a formula for $Z_\delta(j\omega)$. The simpler your formula, the better.





ADD UP THE SHIFTED COPIES OF $Z(jw)$.

$\Rightarrow$ $\boxed{\text{ALIASING}}$

$$Z_\delta(jw) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Z\left(j\left(w - k\frac{2\pi}{T}\right)\right).$$

CONSIDER ω BETWEEN 0 AND π/T!

$$Z_\delta(jw) = \frac{1}{T}\Big(e^{-w} + \underbrace{\varepsilon e^{-\left(\frac{2\pi}{T} + w\right)} + \dots}_{\text{contribution from (1)}} + \underbrace{\varepsilon e^{-\left(\frac{2\pi}{T} - w\right)} + \dots}_{\text{contribution from (2)}} \Big)$$

$$= \frac{1}{T}\Big(e^{-w} + \underbrace{\varepsilon \sum_{n=1}^{\infty} e^{-\left(n\frac{2\pi}{T} + w\right)}} + \underbrace{\varepsilon \sum_{n=1}^{\infty} e^{-\left(n\frac{2\pi}{T} - w\right)}} \Big)$$

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