

Practice Final

Problem 1 (Sampling.)

A real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$.

(a) Suppose the signal $x(t)$ is sampled non-uniformly using the impulse train $q_1(t)$ shown in Figure 1. Show that the spectrum of the sampled signal $y_1(t) = x(t)q_1(t)$ is

$$Y_1(j\omega) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} (1 + e^{-j\frac{\pi k}{3}}) X(j(\omega - \frac{\pi k}{T})). \quad (1)$$

Carefully justify every step in your derivation, including references to results from the tables.

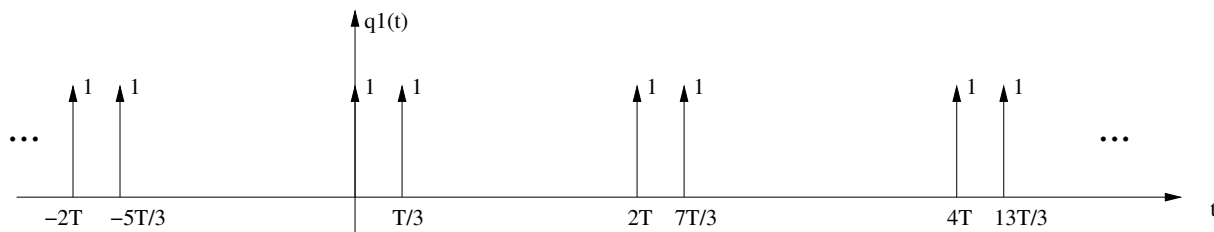


Figure 1: The sampling impulse train $q_1(t)$, where $T = \pi/W$.

(b) Is it possible to low-pass filter the signal $y_1(t)$ to get back the signal $x(t)$?

Answer: yes / no. (circle one)

Explanation: (*Hint: Pick an example band-limited spectrum for $x(t)$, and sketch the resulting spectrum of $y_1(t)$. Based on your plot, explain.*)

(c) Suppose the signal is sampled non-uniformly using the impulse train $q_2(t)$ shown in Figure 2. The impulses are in the same locations as in Figure 1, but they have different weights a and b (both real numbers). Find an expression for the spectrum $Y_2(j\omega)$ of the sampled signal $y_2(t) = x(t)q_2(t)$.

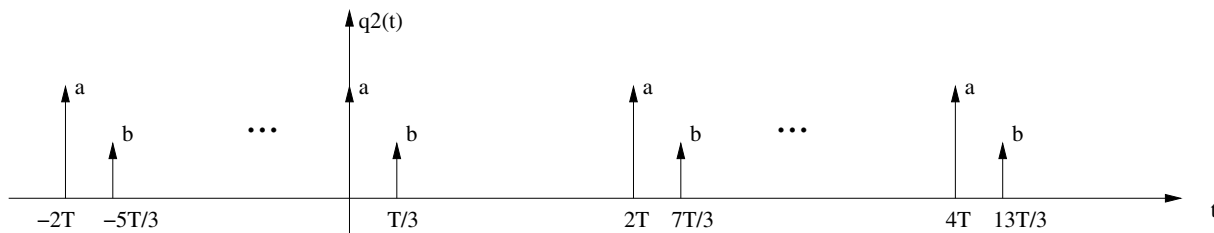


Figure 2: The sampling impulse train $q_2(t)$, where $T = \pi/W$.

(d) For $|\omega| < W$, write out the spectrum

$$Y(j\omega) = \begin{cases} Y_1(j\omega) + jY_2(j\omega), & \text{if } \omega \geq 0, \\ Y_1(j\omega) - jY_2(j\omega), & \text{if } \omega < 0. \end{cases} \quad (2)$$

Select the real numbers a and b such that for $0 < \omega < W$, it is true that $Y(j\omega) = \frac{1}{2T}(2+j(a+b))X(j\omega)$.
Hint: A complex number is zero if and only if its real part is zero *and* its imaginary part is zero.

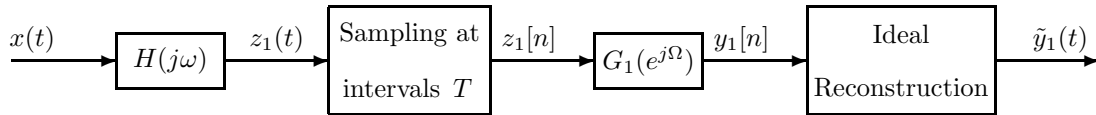
(e) It can also be shown that with the choice of a and b as in Part (d), it is true that $Y(j\omega) = \frac{1}{2T}(2-j(a+b))X(j\omega)$ for $-W < \omega < 0$. Hence, from $Y(j\omega)$, one can determine $X(j\omega)$ using a simple filter. This means that non-uniform sampling using the sampling intervals shown in Figure 1 (and in Figure 2) permits to perfectly reconstruct $x(t)$ from samples. Give an intuitive explanation why this makes sense.

Problem 2 (*Discrete-time Processing of Continuous-time Signals.*)

We would like to construct a system D that implements a derivative, that is, for an input $x(t)$, the system should give an output $y(t)$ given by

$$y(t) = D\{x(t)\} = \frac{dx(t)}{dt}. \quad (3)$$

(a) It is suggested to use the following system:



where

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \pi/T \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

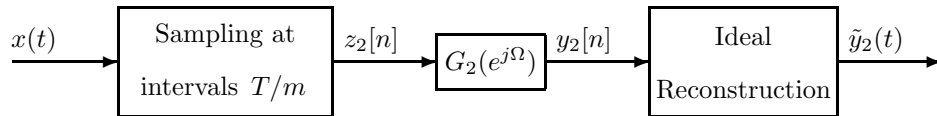
and

$$G_1(e^{j\Omega}) = j\frac{\Omega}{T}, \text{ for } |\Omega| \leq \pi. \quad (5)$$

This system does not exactly implement the desired system D . Instead, it produces an output $\tilde{y}_1(t)$ which is, in general, not equal to the desired output $y(t)$.

Determine and sketch the overall frequency response (magnitude and phase) of the system with input $x(t)$ and output $\tilde{y}_1(t)$.

(b) A second system is also proposed as follows



Note that in the second system, the Ideal Reconstruction block works with the sampler that samples at intervals of T/m , so it is not necessarily identical to the Ideal Reconstruction block in the first system.

Considering that we would like to implement a first derivative with a digital system and given the sampler in the second system, what should we choose for the filter $G_2(e^{j\Omega})$?

(c) If $m = 4$ and the input signal $x(t)$ has a Fourier transform

$$X(j\omega) = \begin{cases} 10, & \text{for } 0 \leq |\omega| \leq \frac{\pi}{T} \\ 1, & \text{for } \frac{\pi}{T} \leq |\omega| \leq \frac{5\pi}{T} \\ 0, & \text{elsewhere} \end{cases}, \quad (6)$$

Determine the error between the desired signal, $y(t)$, and the outputs of the two systems, $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$, given by

$$E_i = \int_{-\infty}^{\infty} |y(t) - \tilde{y}_i(t)|^2 dt, \quad i = 1, 2 \quad (7)$$

If $T = 10^{-3}$, which of the systems is a better choice to implement the first derivative?

Problem 3 (*PAM with fading.*)

Consider a PAM system as discussed in class: A discrete-time data signal $s[n]$ is used to produce the following modulated signal:

$$x(t) = \sum_{n=-\infty}^{\infty} s[n]p(t - nT), \quad (8)$$

where $p(t)$ is a box function that starts at time zero and ends at time $\Delta < T$, of height one.

The signal $x(t)$ is passed through a fading communication channel with frequency response

$$H(j\omega) = \frac{1}{(e^{a\Delta} - 1)(1 + j\omega/a)}, \quad (9)$$

where a is *positive and real-valued*. Call the channel output signal $y(t)$.

(a) For $a = T$ and

$$s[n] = \begin{cases} 1, & n = 0, 2 \\ 2, & n = 1, 3, \\ 0, & \text{otherwise,} \end{cases} \quad (10)$$

sketch the signal $y(t)$ at the receiver. Carefully label the time axis. *Hint:* Consider a single pulse first, and use linearity.

(b) The channel output signal $y(t)$ is sampled at times nT to yield the signal $y[n]$. Write $y[n]$ in terms of $s[n]$ (for general $s[n]$, not merely for the $s[n]$ of Part (a)). Can the relationship between $s[n]$ and $y[n]$ be understood as a discrete-time linear time-invariant (LTI) system? If so, determine its impulse response $h[n]$.

(c) Find a discrete-time LTI system with impulse response $g[n]$ and frequency response $G(e^{j\omega})$ that takes the received signal $y[n]$ and outputs the original transmitted signal $s[n]$.

(d) Generally, at the receiver, you do not exactly know the constant a that governs the channel behavior, but you can estimate it, giving you some \hat{a} which is hopefully close to the true a . Moreover, since the decoder that you designed in Part (c) will operate in digital logic, you have to round \hat{a} to the closest number \tilde{a} that can be represented on your DSP chip. But if the decoder of Part (c) uses \tilde{a} instead of a , it will recover a signal $\tilde{s}[n]$ that is not exactly equal to $s[n]$.

For the special case where $s[n] = 1$, for all n , find the error, defined as

$$E = \sum_{n=-\infty}^{\infty} |s[n] - \tilde{s}[n]|^2, \quad (11)$$

as a function of the a and \tilde{a} .

Problem 4 (*Continuous-time LTI systems.*)

Consider the function

$$G(s) = \frac{1 - (s/10)^6}{1 - s^6}. \quad (12)$$

(a) Draw the pole-zero diagram for $G(s)$.

(b) There are multiple linear time-invariant systems whose transfer function $H(s)$ satisfies $H(s)H(-s) = G(s)$. Among them, determine the one $H(s)$ that

1. corresponds to a stable and causal system
2. with real-valued impulse response function,
3. and for which there exists an inverse system that is also stable and causal.

Each requirement in this list corresponds to a requirement on the pole/zero plot of $H(s)$. Explicitly state these three requirements.

(c) Determine the differential equation describing the system that has the transfer function that you have found in Part (b).

(d) Draw the Bode diagram for the magnitude of the frequency response, $|H(j\omega)|$, for the system described by the differential equation that you have found in Part (c). Describe in words the operation performed by this system.

(e) How many different transfer functions $H(s)$ can be identified that satisfy $H(s)H(-s) = G(s)$ and represent stable systems (not necessarily with a real-valued impulse response)? Justify your answer.

Problem 5 (*Discrete-time LTI systems.*)

(i) The causal discrete-time LTI system G has transfer function

$$G(z) = \frac{(1 - \frac{1}{16}z^4)(1 - az^4)(1 - bz^4)}{z^9(1 - \frac{1}{8}z^3)}, \quad (13)$$

where a and b are *complex* numbers.

- Determine the poles and zeros of the system $G(z)$, and sketch them in the z plane. *Hint:* Write a and b in polar form, i.e., as $xe^{-j\alpha}$, where x is real-valued and positive, and $0 \leq \alpha < 2\pi$.
- Choose a and b such that the system with transfer function $G(z)$ is stable.
- For your choice of a and b , can the system be implemented as a real-valued LTI system, i.e., only involving multiplication by *real* numbers, delay elements, and summations? Give a brief argument.

(ii) The causal discrete-time LTI system G has transfer function

$$G(z) = \frac{1}{d + \frac{1}{2d} - z^3}, \quad (14)$$

where d is a *real* number. (This is a simple version of the $G(z)$ of Part (i).)

- For $d = 3/4$, show that the system is unstable.
- Draw a block diagram of the system, using only multiplication by real-valued constants, delay elements, and summations. Use as few delay elements as possible.

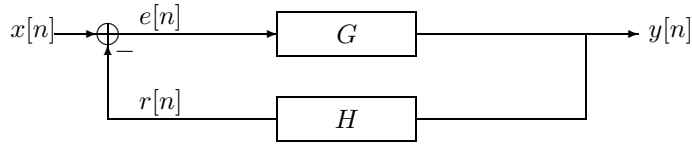


Figure 3: Feedback system.

(iii) As we have seen in class, the feedback system of Figure 3 has the following transfer function:

$$T(z) = \frac{G(z)}{1 + G(z)H(z)}. \quad (15)$$

We take $G(z)$ as in Part (ii), and $H(z)$ as follows:

$$G(z) = \frac{1}{d + \frac{1}{2d} - z^3}, \quad \text{and} \quad H(z) = -\frac{1}{2}z^{-3}. \quad (16)$$

- Determine the range of real numbers d such that the (causal) overall system with transfer function $T(z)$ is stable. *Hint:* Substitute $x = z^3$ and recall that $x^2 - (a + b)x + ab = (x - a)(x - b)$.
- Is the overall system now stable for $d = 3/4$?